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EFFECTIVE COMPUTATION OF NONLINEAR H-INFINITY CONTROL LAWS

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Abstract. *Robustness is an essential pre-requisite to any control system. Furthermore, there exist situations where a fixed linear controller can not be used. The nonlinear H-infinity control theory allows the design of nonlinear robust controllers. However, there is not, until now, an effective procedure to compute control laws based on this theory. The difficulty to find a closed form solution which satisfies the Hamilton-Jacobi-Isaacs inequality (equation), that arises in the nonlinear H-infinity control theory, is recognized as the main limitation to use the results from that theory. In this work, some theoretic results which contribute to find a solution to this problem are presented. These results can also provide an estimate for the controller validity region, a subject that apparently has been largely neglected in the open scientific literature. The systematic implementation of these results is done via the formulation and the solution of some optimization problems. As an illustration the proposed approach is partially applied to control a chemical engineering system.*

Keywords: *Nonlinear control, Robust control, H-infinity control, Inequalities*

1. INTRODUCTION

Robustness is a fundamental pre-requisite to any control system [Zhou *et al.*, 1996]. Furthermore, there exist situations where a fixed linear controller can not be used to control a process [Secchi *et al.*, 1999]. The nonlinear H-infinity control theory allows the design of nonlinear robust controllers. However, there is not, until now, an effective procedure to compute control laws based on this theory. In fact, the difficulty to find a closed form solution to the Hamilton-Jacobi-Issacs (HJI) inequality (equation), that arises in the nonlinear H-infinity control theory similarly to the Riccati inequality (equation) in the corresponding linear problem, is recognized as the main limitation to use the results from that theory [Isidori and Lin, 1998] [Zhan and Wang, 1996].

During the 1990's, several methods to solve the HJI inequality (equation) arose in the literature [Van der Schaft, 1992] [Isidori and Kang, 1995] [Beard and McLain, 1998] [Tsiotras *et al.*, 1998] [Yang *et al.*, 1997] [Huang, 1998]. Most of these methods are based on the application of hard numerical methods to approximate the HJI equation. Following the suggestion of the first theoretical works [Van der Schaft, 1992] [Isidori and Kang, 1995], the main approach of these methods is to search for a solution via polynomial expansions around the origin, according to Lukes's work (1969). Probably, the only exception which tries to explore the extra degrees of freedom allowed by solving the inequality instead of the equation is the work of Isidori and Kang (1995). Unfortunately, their procedure to find a solution is not much practical.

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Besides the absence of an effective method to compute the control laws, another remarkable problem of the nonlinear H-infinity control theory is that its theoretical results do not provide any a priori information concerning the size of the local attracting region for the closed-loop system (an exception to this rule is the work of Beard and McLain (1998)). So, the quantification of how better is the nonlinear controller over the linear one remains unanswered.

From the experience acquired in solving that control problem using some of these methods from the literature, it seems evident that none produce a satisfactory solution to our problem[1]. Maybe for this reason there are no reports of experimental applications of the nonlinear H-infinity control theory[2].

The objective of this work is to present the results obtained in our research concerning the searching and development of effective and simple methods to find closed form solutions to the HJI inequality. Our intent is to show that, with the help of some new theoretic results, a very satisfactory solution, which includes an estimate of the controller validity region for some classes of mathematical functions, can be obtained.

The sequence of this work presents, in section 2, the formulation of the nonlinear H-infinity control problem. In section 3, the theoretical results that will be used to solve the problem are presented. In section 4, the systematic use of these results is formalized by enunciating and solving three optimization problems. Section 5 illustrates the application of the developed methodology designing a control law for a chemical engineering system. Finally, section 6 summarizes the main conclusions of the work.

2. NONLINEAR H-INFINITY CONTROL PROBLEM

The nonlinear H-infinity control theory deals with the nonlinear extension of the H-infinity problem[3]. The results of this theory started to arise in the literature after the publication of the famous paper by Doyle, Glover, Khargonekar and Francis (1989). For simplicity, in this work, only systems described by input-affine models will be regarded. Consider Eq. (1),

$$\begin{aligned} \dot{x} &= f(x) + g(x).u + k(x).w \\ y &= x \\ z &= \begin{bmatrix} h(x) \\ u \end{bmatrix} \end{aligned} \qquad (1)$$

where $x \in M$ ($M \subseteq \Re^n$) is the vector of state variables, $w \in \Re^q$ is the vector of exogenous inputs, $u \in \Re^m$ is the vector of control actions, $y \in \Re^p$ is the vector of measured variables and

[1] An ideal approximation to the nonlinear H-infinity control law must satisfy the following criteria [Longhi, 1998]:

A. It must be explicit, to have an easy implementation and it must be simple to tune.
B. It should not require a prohibitive computational effort (both time for computation and memory requirements).
C. The validity region can be defined or at least it must be known.
D. The augment of the controller complexity must imply the improvement of its performance. This improvement can be measured in two ways:
   D.1 Some distance between the approximation and the true solution and;
   D.2 The size of the closed-loop attracting region.

[2] The authors do not know any report concerning experimental applications of the theory to real systems

[3] The H-infinity control was originally conceived for linear systems. It must be remarked that the H-infinity norm does not make sense for nonlinear systems. The correct would be to say the induced L_2 gain.

z is the vector of exogenous outputs which characterizes the control objective; f(x) and g(x) are nonlinear smooth functions, i.e., belonging to the C^∞ class, with $f(x_0)=h(x_0)=0$.

It will be considered, also for simplicity, that the origin is the solution of interest (i.e., f(0)=0 and h(0)=0 for w=u=0) and that f is observable by h.

Given the nonlinear state feedback (y=x) description of Eq. (1), the nonlinear H-infinity control objectives are two:

1. to stabilize the plant (in closed-loop) and
2. to attenuate the influence of the exogenous inputs, w, in the objective variable, z.

The influence from w(t) on z(t) is measured as the finite L_2-gain between these variables. Here, this gain is defined as in Van der Schaft (1992).

Definition 1 (Finite L_2-gain). Given any $\gamma > 0$, the mapping from w(t) to z(t) is said to have L_2-gain less than or equal to γ if, under the zero initial condition $x(0)=0$,

$$\int_0^T \|z(t)\|^2 dt \le \gamma^2 \int_0^T \|w(t)\|^2 dt \qquad (2)$$

for all $T \ge 0$ and all $w(\cdot) \in L_2(0,T)$, where $\|\bullet\|$ denotes the Euclidean norm.

The solution for this L2-gain attenuation problem for a system described by Eq. (1) can be given by theorem 1 [Van der Schaft, 1992].

Theorem 1 (Local solution to L2-gain attenuation problem). *Consider the nonlinear system of Eq. (1) and a real parameter, $\gamma > 0$. Suppose that exists a smooth solution, $V(x) \ge 0$, to the Hamilton-Jacobi equation (Eq. (3)):*

$$\begin{aligned}&\frac{\partial V}{\partial x}(x).f(x) - \frac{1}{2}\frac{\partial V}{\partial x}(x)\left(g(x).g^T(x) - \frac{1}{\gamma^2}.k(x).k^T(x)\right)\frac{\partial^T V}{\partial x}(x) + \frac{1}{2}.h^T(x).h(x) = 0\\ &V(0)=0\end{aligned} \qquad (3)$$

or to the Hamilton-Jacobi inequality (Eq. (4)):

$$\begin{aligned}&\frac{\partial V}{\partial x}(x).f(x) - \frac{1}{2}\frac{\partial V}{\partial x}(x)\left(g(x).g^T(x) - \frac{1}{\gamma^2}.k(x).k^T(x)\right)\frac{\partial^T V}{\partial x}(x) + \frac{1}{2}.h^T(x).h(x) \le 0\\ &V(0)=0\end{aligned} \qquad (4)$$

so, the closed-loop system with the feedback of Eq. (5):

$$u = -g^T(x).\frac{\partial^T V}{\partial x}(x) \qquad (5)$$

has locally a L_2 gain (from w *to* $z = \begin{bmatrix} h(x) \\ u \end{bmatrix}$*) less than or equal to γ.*

If we desire that the origin also presents asymptotic stability, some additional hypotheses must be done:

1. V(x) must be positive definite (V(x) > 0) and satisfy 2.1 or 2.2;

2.1 V(x) must satisfy, instead of Eqs. (3) or (4), the strict inequality of Eq. (6):

$$\frac{\partial V}{\partial x}(x).f(x)-\frac{1}{2}\frac{\partial V}{\partial x}(x).\left(g(x).g^T(x)-\frac{1}{\gamma^2}.k(x).k^T(x)\right)\frac{\partial^T V}{\partial x}(x)^T+\frac{1}{2}.h(x).h^T(x)<0$$
$$V(0)=0 \quad (6)$$

2.2 Every limited trajectory x(t) of the non-disturbed system (with w=0) $\dot{x}=f(x)+g(x).u$ which satisfies $z=[h(x) \quad u]^T=0$ for all t > 0 goes to the origin when time approaches to infinity.

If conditions 1 and 2.1 (or 2.2) are satisfied, then it can be shown that V(x) is a Lyapunov function for the non-disturbed system. If V(x) satisfies theorem 1 and it is a Lyapunov function, the problem of disturbance attenuation with internal stability (i.e., the nonlinear H-infinity control problem) via state feedback is locally solved. It must be noted that theorem 1 does not say anything about the size of the controller validity region. So, before to go ahead we must define the validity region for the nonlinear H-infinity controller and its estimates.

Definition 2 (Nonlinear H-infinity controller validity region). The region of the state space of Eq. (1) that, subject to the nonlinear state feedback law from theorem 1, simultaneously satisfies the HJI inequality and guarantees asymptotic stability for the non-disturbed closed-loop system, is referred to as the validity region corresponding to the controller of Eq. (5). Any subset of this state space region is referred to as an estimate of the controller validity region.

The region where the HJI inequality is satisfied is simply the intersection between its negativeness region and the positiveness region of V(x). The region of asymptotic stability can be estimated using the results from the Lyapunov stability theory. Then, to determine the controller validity region, it must be obtained a solution that furnishes not only a mathematical form to V(x) but also an estimate of the state space region where V(x) is positive definite (V(x) > 0) and the HJI inequality is negative definite ($H_*(x) < 0$). Some useful results to estimate these regions are presented in the next section.

3. PRELIMINARY THEORETIC RESULTS

The problem of finding a solution to the nonlinear H-infinity control problem can be viewed as the problem of finding a positive scalar function

$$V(x)>0 \quad (7)$$

which satisfies a strictly (or non-strictly added by a detectability condition) inequality known as HJI inequality

$$H_*(x)<0 \quad (8)$$

both in some neighborhood of the desired solution (for simplicity, the origin is that solution).

Therefore, independently on the method used to solve the nonlinear H-infinity control problem it would be of great value a method which allows the local sign evaluation of multivariable scalar functions. If such a method is available, it should be used in two ways: (a) as a synthesis tool, forcing V(x) to satisfy Eqs. (7) and (8); (b) as an analysis tool, just verifying if the Eqs. (7) and (8) are satisfied in some neighborhood of x = 0.

The acquired experience in solving problems of this kind shows that V(x) and H*(x) can be rewritten as quadratic forms in accordance with Eq. (9)[4].

$$\begin{cases} V(x) = y^T(x).P.y(x) \\ H_*(x) = z^T(x).H.z(x) \end{cases} \quad (9)$$

The basis of these quadratic forms are nonlinear functions of the states, x:

$$\begin{cases} y(x) = [y_1(x) \;\; y_2(x) \; \ldots \; y_n(x)]^T \\ z(x) = [z_1(x) \;\; z_2(x) \; \ldots \; z_m(x)]^T \end{cases} \quad (10)$$

With no loss of generality matrices P and H can be written as real symmetric matrices:

$$\begin{aligned} &P \in \Re^{nxn} \quad (p_{ij} = p_{ji}) \\ &H \in \Re^{mxm} \quad (H_{ij} = H_{ji}) \end{aligned} \quad (11)$$

Expressing Eqs. (7) and (8) in terms of the quadratic forms of Eq. (9) is the basis of this work. The approach used to solve the HJI inequality is based on mathematical results concerning the positiveness[5] of multivariable scalar functions written as quadratic forms. These results can be summarized in the definitions 3 and 4 and theorems 2 to 4, presented below. The proofs of these theorems can be found in [Longhi *et al.*, 2000].

Definition 3 (Global suitable function). If a scalar function of n variables assumes real values when its variables belong to the real field, and assumes the null value only when all its variables are set to zero, then this function is called a global suitable function.

Theorem 2 (Sufficient criterion for global positiveness). *Let* $V(x)$ *be a scalar function of* n *real variables. If the quadratic form representation of* $V(x)$ *is* $y^T.B.y$, *where the elements of* y *are global suitable functions of* x *and* B *is a symmetric real matrix obtained directly from the coefficients of* $V(x)$, *then a sufficient condition to* $V(x)$ *be positive (negative) is that* B *be a positive (negative) definite matrix.*

Theorem 3 (Local properties of a scalar function). *If a real multivariable scalar function has a null solution at the origin, has null gradient value at the origin, can be extended in Maclaurin Series, and rewritten as:* $V(x) = x^T.P.x + f_{NL}(x)$, *where* P *is a real symmetrical matrix and* $f_{NL}(x)$ *contains the terms with order greater than 2, then the local sign of* $V(x)$ *can be inferred by analyzing the eigenvalues of the matrix* P *in the following way:*

1. *If the eigenvalues are all positive (negative),* $V(x)$ *is locally positive (negative) definite;*
2. *If at least one eigenvalue is null and the others have the same sign, nothing can be said about the sign of* $V(x)$;
3. *If at least one eigenvalue has sign different from the others, then* $V(x)$ *is locally sign undefined.*

[4] The variables y(x) and z(x) are not the same as in Eq. (1).
[5] To prove the negativity of V(x) in theorems 2 and 4, it is sufficient to prove the positiveness of (-1).V(x).

Definition 4 (Real local region). The real local region of a multivariable scalar function y(x) is the set composed by the subsets of the real field where each element of x can assume values such that y(x) is real and $y \neq 0$ unless $x = 0$.

Theorem 4 (Local positiveness with estimate of the positiveness region size). *Let* V(x) *be a scalar function of n real variables. If the quadratic form representation of* V(x) *is* $y^T.B.y$, *where the components of* y *are functions of* x *and* B *is a symmetric real matrix obtained directly from* V(x), *then a sufficient condition for the local positive definiteness of* V(x) *is that B be a positive definite matrix.*
Furthermore, a local region with this definite sign is obtained from the intersection of the real local regions of y(x) *involved in the quadratic form description of* V(x).

4. EFFECTIVE SOLUTION TO NONLINEAR H-INFINITY CONTROL PROBLEM

The proposal of this work is to explore the theoretic results of section 3 in a systematic fashion. For this end, we combine these mathematical results with an optimization procedure. This is important because applying theorems 2 to 4 directly to solve the nonlinear H-infinity control problem based only in the algebraic talent of the control engineer may result in a very tedious work or even useless if the problem's dimension is reasonably large.

As it was seen before, finding a solution to the nonlinear H-infinity control problem can be viewed as the problem of finding a positive scalar function V(x) which satisfies the HJI inequality. So, the first step to use the results of theorems 2 to 4 is to write the HJI inequality for a certain attenuation level (γ) and verify if a local solution exists. This leads to the optimization problem 1.

Optimization problem 1 (Local nonlinear H-infinity control solution). *Choose the form of function* V(x) *and substitute in* $H_*(x)$. *Expand these two functions in McLaurin series:* $V(x) = x^T.P.x + f_{NL}(x)$, $H_*(x) = x^T.H.x + g_{NL}(x)$ *where* P *and* H *are real symmetric matrices and* $f_{NL}(x)$ *and* $g_{NL}(x)$ *contains the terms with order greater than* 2. *Choose the parameters of* V(x) *in a way to minimize the* γ *level attenuation* ($\gamma > 0$) *of the HJI inequality subject to the constraints* $P > 0$ *and* $H < 0$:

However, the solution of optimization problem 1, despite its simplicity, can only furnish a local solution without any information about the size of the local controller validity region. Its main importance is to give an answer to the question about the existence of a local solution to the problem. If its solution exists the obtained γ–value can also be interpreted as the lowest bound to $\gamma > 0$ for any nonlinear form of V(x) (Van der Schaft, 1992). This last information can be very useful in the next optimizations problems.

If it is desired quantitative informations concerning the size of the local controller validity region, it is required the solution of the optimization problems 2 or 3.

Optimization problem 2 (Global nonlinear H-infinity control solution). *Choose the form of function* V(x) *and substitute in* $H_*(x)$. *Write these two functions as quadratic form representations:* $V(x) = y^T.Py$ *and* $H_*(x) = z^T.H.z$ *where the elements of* y *and* z *are global suitable functions of* x *and* P *and* H *are symmetric real matrices obtained directly from the coefficients of* V(x) *and* $H_*(x)$, *respectively. Choose the parameters of* V(x) *in a way to minimize the* γ *level attenuation* ($\gamma > 0$) *of the HJI inequality subject to the constraints* $P > 0$ *and* $H < 0$.

Optimization problem 3 (Nonlinear H-infinity control solution with size of local region). *Choose the form of function* V(x) *and substitute in* $H_*(x)$. *Write these two functions as quadratic form representations:* $V(x) = y^T.Py$ *and* $H_*(x) = z^T.H.z$ *where* P *and* H *are symmetric real matrices obtained directly from the coefficients of* V(x) *and* $H_*(x)$, *respectively. Choose the parameters of* V(x) *in a way to minimize the* γ *level attenuation* ($\gamma > 0$) *of the HJI inequality subject to the constraints* $P > 0$ *and* $H < 0$. *The solution solves locally the problem within a region defined by the intersection of the real local regions of* y(x) *and* z(x) *involved in the quadratic form descriptions of* V(x) *and* $H_*(x)$.

Finding an effective solution to optimization problems 1 to 3 considering general nonlinear functions is a difficult task. However, if we constrain the results of section 3 only to the class of polynomials, we can arrive to an approximated procedure to solve these problems[6].

So, to effectively solve the optimization problem 1, the function (polynomial) V(x) must be chosen such that when substituted in the corresponding HJI inequality all possible combinations of second order polynomials are formed. This way, we will be solving this problem in the most broad context.

To solve problem 2, a similar approach can be used, but now we have a serious limitation. Different from problem 1, we can not select V(x) to provide all possible combinations to solve the problem. So, this problem can be solved with success if it is known that exists a certain form of V(x) that globally satisfies its HJI inequality. In this case, the optimization problem is used only to give the optimal parameters to this particular V(x) form.

If the optimization problem 1 is solved, then, at least, a local solution exists. To find an estimate to the size of this local stability region we can try to solve the optimization problem 3 somehow. At this time, the more interesting procedure to solve the optimization problem 3 is the one given by the following steps (using the terminology from theorem 4):

1. Chose the functions y(x) and z(x) such that both matrices P and H have no null diagonal elements.
2. The new expanded space resulting from the quadratic form description of V(x) and $H_*(x)$ must have enclosed at least one component of y(x) or z(x) with one parameter which regulates the size of the sign definiteness region.
3. Maximize the size of the sign definiteness region (or, alternatively, the size of the stability region) for a certain range of attenuation γ, while maintaining the positiveness of V(x) and the negativeness of $H_*(x)$ as constraints.

It must be noted that this last procedure only gives a sub-optimal solution because we are not minimizing γ. However, as it is well known in the control literature, arriving closer to the minimum of γ frequently reduces significantly the stability region. So, a more realistic approach is not to find the minimum but to search for a solution inside an interval: $\gamma_{min} < \gamma < \gamma_{max}$. These conditions can be easily entered in an optimization problem as two new inequalities constraints.

To end this section, it must be said that, all problems stated here are nonlinear constrained optimization problems. The solutions presented, despite the well succeeded results, are not the definitive ones. The search for more systematic solutions for optimization problems 2 and 3 is the actual focus of the research. All results of this section were implemented using the Matlab (with some functions from the optimization toolbox) and Maple softwares.

[6] It must be said that this constraint is not so restrictive because an important result from the nonlinear system theory states that all smooth functions can be represented by Volterra series (i.e., infinite polynomial series).

5. APPLICATION TO A CHEMICAL REACTOR MODEL

The CSTR with van de Vusse reaction scheme has been used as a benchmark problem for nonlinear process control algorithms [Engell and Klatt, 1993]. It consists of the following reaction scheme:

$$A \xrightarrow{k_1} B \xrightarrow{k_2} C$$

$$2A \xrightarrow{k_3} D$$

Here, B is the wanted product and C and D are the undesired byproducts. The van de Vusse reaction is carried out in an ideal isothermal continuous stirred tank reactor (CSTR) which is modeled by:

$$\frac{dC_A}{dt} = \frac{F_{in}}{V_R}(C_{Ain} - C_A) - [k_1(T)C_A + k_3(T).C_A{}^2] \qquad (12)$$

$$\frac{dC_B}{dt} = -\frac{F_{in}}{V_R}.C_B + [k_1(T)C_A + k_2(T).C_B] \qquad (13)$$

In the present study, it is used the same kinetic parameters used in Sistu and Bequette (1995), i.e., C_{Ain} = 50 gmol/liter, k_1 = 50 h^{-1}, k_2 = 100 h^{-1} and k_3 = 10 liter/(gmol.h). The manipulated and controlled variables are $f = F_{in}/V_R$ (the inverse of the residence time) and C_B (concentration of component B), respectively. The variable f is assumed to vary without constraints.

The control objective stated in this work is to maintain the production of B at a certain value despite the uncertainties in the mathematical model. To use the results from nonlinear H-infinity theory we have to rewrite the system equations such that the steady state of interest is the null solution (the origin). So, we define the new variables:

$$\begin{aligned} x &= [x_1 \quad x_2]^T = [C_A - C_{ASS} \quad C_B - C_{BSS}]^T \\ u &= f - f_{SS} \end{aligned} \qquad (14)$$

where the index SS means the steady state of interest.

To eliminate the C_B off-set, the system is augmented by a new state x_3, such that $\dot{x}_3 = -x_2$. So, the system model can be rewritten according to Eq. (1), where f(x), g(x) and h(x) are described by Eqs. (15) to (17).

$$f(x) = \begin{pmatrix} -k_1.x_1 - k_3.x_1{}^2 - (2.C_{ASS}.k_3 + f_{SS}).x_1 \\ k_1.x_1 - k_2.x_2 - f_{SS}.x_2 \\ -x_2 \end{pmatrix} \qquad (15)$$

$$g(x) = \begin{pmatrix} C_{Ain} - C_{ASS} - x_1 \\ -C_{BSS} - x_2 \\ 0 \end{pmatrix} \qquad (16)$$

$$h(x) = x_2 \qquad (17)$$

To introduce robustness to the system, the first state equation is added by an exogenous perturbation, w(t). Returning to Eq. (1), the amplitude of that perturbation is arbitrarily assumed to be described by Eq. (18). It must be noted that the correspondence between this perturbation and the realistic variations on parameters k_1, k_2, k_3 and C_{Ain} is not obvious.

$$k(x) = \begin{pmatrix} 0.1 \\ 0 \\ 0 \end{pmatrix} \tag{18}$$

Then, the HJI inequality for this system assumes the form of Eq. (19).

$$\begin{aligned} H_*(x) = & \left(\left[-k_1 - 2.k_3.C_{ASS} - f_{SS}\right].x_1 - k_3.x_1^2\right).V_{x_1} + \left(k_1.x_1 - [k_2 + f_{SS}].x_2\right).V_{x_2} + \left(-x_2\right).V_{x_3} \\ & x_2^2 + \frac{1}{4}.\left(\frac{1}{\gamma^2}.k^T(x).k(x).V_{x_1}^2 - [C_{Ain} - C_{ASS} - x_1]^2.V_{x_1}^2 - \right. \\ & \left. 2.[C_{Ain} - C_{ASS} - x_1].[-C_{BSS} - x_2].V_{x_1}.V_{x_2} - [-C_{BSS} - x_2]^2.V_{x_2}^2\right) < 0 \end{aligned} \tag{19}$$

To solve the optimization problem 1 for this system we must depart from the form of Eq. (20) as a solution to V(x):

$$V(x) = a_{11}.x_1^2 + a_{12}.x_1 x_2 + a_{13}.x_1.x_3 + a_{22}.x_2^2 + a_{23}.x_2.x_3 + a_{33}.x_3^2 \tag{20}$$

So, with the help of the software MATLAB, we arrive to the following solution[7]:

$a_{11} = 1.309534257854861.10^{1}$; $a_{12} = -9.081782877831930.10^{-5}$;
$a_{13} = 6.866826057038236.10^{-3}$; $a_{22} = 4.633988340265012.10^{-3}$;
$a_{23} = -6.803489668553544.10^{-3}$; $a_{33} = 6.639058486487478.10^{-1}$.
$\gamma = 1.500664046938835.10^{-4}$

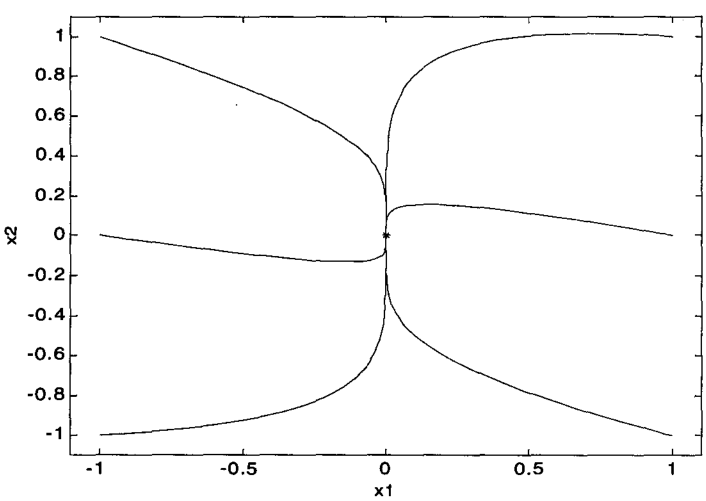

Figure 1. Some trajectories of the non-disturbed closed-loop system.

Figure 1 shows the asymptotically stable behavior of the non-disturbed system around the origin. It must be remarked that the size of the controller validity region can not be defined

[7] The values of the parameters a_{ij} need to have the precision presented in the text due to the minimization of γ.

just solving optimization problem 1. If it was desired to estimate this region we should solve the optimization problem 2 or 3.

6. CONCLUSION

The main contribution of this work is the development of some methods to effectively solve the nonlinear H-infinity control problem. These methods were developed departing from theoretic results obtained in preliminary works [Longhi *et al.*, 2000]. The results of this work are applicable to a broad class of nonlinear systems. This can be justified if it is remembered that all smooth nonlinear systems (i.e., which belongs to the C^{∞} class) can be described by infinite polynomial series (Volterra series). An example of the application of the method to a chemical engineering model was also presented.

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