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On the combinatorial rank of quantum groups

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Porto Alegre, 30 de junho de 2022.

PhD thesis presented by Vanusa Moreira Dylewski [1] in partial fulfillment of the requirements for the degree of Doctor in Mathematics at Universidade Federal do Rio Grande do Sul.

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Agradecimentos

Primeiramente agradeço a Deus e dedico essa conquista aos meus pais, Paulo e Branca, e aos meus irmãos, Jardel e Itamar, que me incentivaram e sempre me ajudaram. Obrigada por acreditarem em mim.

Agradeço a minha orientadora, Bárbara, por ter aceito me orientar e por todo apoio e suporte que me deu durante esses 6 anos de estudo. Obrigada por todos os ensinamentos e paciência que teve comigo. Agradeço, também, a Carol por toda ajuda e contribuições para a tese.

Agradeço aos professores da banca, em especial ao João Matheus que fez muitas contribuições para aperfeiçoar o nosso trabalho.

Agradeço ao meu amigo Guilherme, que se tornou um irmão ao longo desses anos. Obrigada por me segurar nos momentos difíceis e estar ao meu lado em mais uma conquista.

Por último, agradeço ao meu fiel amigo Bud, meu companheirinho de todas as horas.

Resumo

Seja $\mathfrak{g}$ uma álgebra de Lie simples de tipo G_2 ou F_4. Nesta tese calculamos o posto combinatório da parte positiva da versão multiparâmetro do pequeno grupo quântico de Lusztig $u_q^+(\mathfrak{g})$.

Abstract

Let $\mathfrak{g}$ be a simple Lie algebra of type G_2 or F_4. In this thesis we calculate the combinatorial rank of the positive part of the multiparameter version of the small Lusztig quantum group $u_q^+(\mathfrak{g})$.

Contents

Chapter 1

Introduction

Let H be a character Hopf algebra. We notice that by a corollary of the Heyneman-Radford Theorem [8, Proposition 2.4.2] every nonzero bi-ideal of a character Hopf algebra has a nonzero skew-primitive element. We also have that skew-primitive elements generate a Hopf ideal and, unlike the classical case of universal enveloping algebras, in the quantum case a Hopf ideal is not necessarily generated by its skew-primitives. In this sense, the concept of a combinatorial rank is introduced in Section 2.7 "measuring" how distant an specific Hopf ideal is from being generated by its skew primitive elements.

We consider J a Hopf ideal of H and we construct the sequence $0 = J_0 \subsetneq J_1 \subsetneq J_2 \subsetneq \ldots \subsetneq J_i \subsetneq \ldots \subsetneq J$ of Hopf ideals. The construction of this sequence is given as follows:

- We define J_1 as the Hopf ideal generated by skew-primitive elements of J.If $J_1 \neq J$, then $\frac{J}{J_1} \neq 0$ is a Hopf ideal and has a skew-primitive element.
- We define $\frac{J_2}{J_1}$ as the ideal generated by skew-primitive elements of $\frac{G\langle X\rangle}{J_1}$, where $J_2 = \pi^{-1}(\frac{J_2}{J_1})$ with $\pi : G\langle X\rangle \to \frac{G\langle X\rangle}{J_1}$.
- If $J_2 \neq J$ then define $\frac{J_3}{J_2}$ as the ideal generated by skew-primitive elements of $\frac{G\langle X\rangle}{J_2}$, where $J_3 = \pi^{-1}(\frac{J_3}{J_2})$ with $\pi : G\langle X\rangle \to \frac{G\langle X\rangle}{J_2}$.
- Following this process until the sequence stabilizes, that is, $J_\kappa = J$ for some κ.

If $J = \ker \varphi$, where $\varphi : G\langle X\rangle \to H$, the length κ of this sequence is called the *combinatorial rank* of H.

The definition of combinatorial rank was proposed by V. Kharchenko and A. Alvarez in [14], where they proved that $\kappa(u_q^+(\mathfrak{g})) = \lfloor \log_2 n \rfloor + 1$ in the case that

$\mathfrak{g}$ is a simple Lie algebra of type A_n. Later, V. Kharchenko and M. L. Díaz Sosa showed similar results for the Frobenius-Lusztig kernel of type B_n, C_n and D_n (see [15] and [16]). They proved that $\kappa(u_q^+(\mathfrak{g})) = \lfloor \log_2(n-1) \rfloor + 2$ for the cases B_n and C_n, and $\kappa(u_q^+(\mathfrak{g})) = \lfloor \log_2(2n-3) \rfloor + 1$ for the case D_n. However, Ardizzoni [6] also investigated conditions under which some particular graded braided bialgebras have finite combinatorial rank. We still have, trivially, that $\kappa(U_q^+(\mathfrak{g})) = 1$, for any simple Lie algebra $\mathfrak{g}$.

The quantum groups $U_q^+(\mathfrak{g})$ and $u_q^+(\mathfrak{g})$, where $\mathfrak{g}$ is a simple Lie algebra, are important examples of quantum algebras. The cases where $\mathfrak{g}$ is a Lie algebra of types A_n, B_n, C_n and D_n were extensively studied. We also have a good amount of results on G_2. However there are few studies specifically on F_4. In this thesis we calculate the combinatorial rank of the algebra $u_q^+(\mathfrak{g})$, where $\mathfrak{g}$ is a simple Lie algebra of types G_2 and F_4, continuing the investigation for "small" quantum groups $u_q^+(\mathfrak{g})$.

In the first chapter we introduce the general notation, definitions and basic results necessary for this work. In the second chapter we list existing results about $u_q^+(G_2)$ and we proved that $\kappa(u_q^+(G_2)) = 3$, describing the complete chain of Hopf ideals J_i, $i \in \{1, 2, 3\}$. Finally, in the third chapter we go deeper into the case that $\mathfrak{g}$ is a simple Lie algebra of type F_4 and we develop results to prove that the combinatorial rank of $u_q^+(F_4)$ equals 4.

Chapter 2

Preliminaries

Let $\mathbf{k}$ be an algebraically closed field of characteristic zero. In this chapter we will state definitions and basic results used in this work. These results are already known and can be found in the references [4], [13] and [17].

2.1 Character Hopf algebras

In this section we will define character Hopf algebras and present some properties.

Definition 2.1.1. A Hopf algebra H is a *character Hopf algebra* if the group G of all group-like elements is commutative and H is generated over $\mathbf{k}[G]$ by skew-primitive semi-invariants a_i, $i \in I$:

$$\Delta(a_i) = a_i \otimes 1 + g_i \otimes a_i, \quad g^{-1}a_i g = \chi^i(g)a_i, \quad g, g_i \in G,$$

where χ^i, $i \in I$, are characters of the group G.

Definition 2.1.2. A variable x is called a *quantum variable* if a group-like element $g_x \in G$ and a character $\chi^x \in G^*$ are associated with x.

Let x_i be the quantum variable associated with a_i. For each word u in $X = \{x_i | i \in I\}$ we denote by g_u an element of G that appears from u by replacing each x_i with g_i. Similarly we denote by χ^u a character that appears from u by replacing each x_i with χ^i. Now we define a bilinear skew-commutator on homogeneous linear combinations of words using the formula

$$[u, v] = uv - \chi^u(g_v)vu, \tag{2.1}$$

where we use the notation $\chi^u(g_v) = p_{uv} = p(u,v)$. These brackets satisfy the following Jacobi and skew-differential identities

$$[u \cdot v, w] = p_{vw}[u,w] \cdot v + u \cdot [v,w], \tag{2.2}$$

$$[u, v \cdot w] = [u,v] \cdot w + p_{uv} v \cdot [u,w]. \tag{2.3}$$

$$[[u,v],w] = [u,[v,w]] + p_{wv}^{-1}[[u,w],v] + (p_{vw} - p_{wv}^{-1})[u,w] \cdot v \tag{2.4}$$

$$[[u,v],w] = [u,[v,w]] + p_{vw}[[u,w],v] + p_{uv}(p_{vw}p_{wv} - 1)v \cdot [u,w] \tag{2.5}$$

If p_{vv} is a primitive t-th root of the unit then we also have the restricted identities

$$[u, v^t] = [\ldots[[u,v],v],\ldots,v], \tag{2.6}$$

$$[v^t, u] = [v,[v,\ldots[v,u]\ldots]]. \tag{2.7}$$

The group G acts on the free algebra $\mathbf{k}\langle X\rangle$ by $g^{-1}ug = \chi^u(g)u$, where u is an arbitrary monomial in X. The skew group algebra $G\langle X\rangle$ has the natural Hopf algebra structure

$$\Delta(x_i) = x_i \otimes 1 + g_i \otimes x_i, \quad i \in I, \quad \Delta(g) = g \otimes g.$$

2.2 Hard hyper-letters

Let H be a character Hopf algebra. In particular, we can consider $H = G\langle X\rangle$, where $X = \{x_i | i \in I\}$, or H to be the image of $G\langle X\rangle$ by an homomorphism of Hopf algebras.

Let us fix a Hopf algebra homomorphism

$$\xi : G\langle X\rangle \to H, \quad \xi(x_i) = a_i, \quad \xi(g) = g, \quad i \in I, \quad g \in G.$$

Definition 2.2.1. A *constitution* of a word u in $G \cup X$ is a family of non-negative integers $\{m_x, x \in X\}$ such that u has m_x occurrences of x. Certainly almost all m_x in the constitution are zero.

Let us fix an arbitrary complete order $<$ on the set X, and let Γ^+ be the free additive (commutative) monoid generated by X. The monoid Γ^+ is a completely ordered monoid with respect to the following order:

$$m_1x_{i_1}+m_2x_{i_2}+\ldots+m_kx_{i_k}>m'_1x_{i_1}+m'_2x_{i_2}+\ldots+m'_kx_{i_k} \tag{2.8}$$

if the first from the left nonzero number in $(m_1-m'_1, m_2-m'_2, \ldots, m_k-m'_k)$ is positive, where $x_{i_1}>x_{i_2}>\ldots>x_{i_k}$ in X. We associate a formal degree $D(u)=\sum_{x\in X} m_x x\in\Gamma^+$ to a word u in $G\cup X$, where $\{m_x|x\in X\}$ is the constitution of u. Respectively, if $f=\sum\alpha_i u_i\in G\langle X\rangle$, $0\neq\alpha_i\in\mathbf{k}$ then

$$D(f)=max_i\{D(u_i)\}. \tag{2.9}$$

On the set of all words in X we fix the lexicographical order with the priority from the left to the right, where a proper beginning of a word is considered to be greater than the word itself.

Definition 2.2.2. A non-empty word u is called a *standard word* (or *Lyndon word*, or *Lyndon-Shirshov word*) if $vw>wv$ for each decomposition $u=vw$ with non-empty v,w.

Definition 2.2.3. A *non-associative word* is a word where brackets $[,]$ are somehow arranged to show how multiplication applies.

If $[u]$ denotes a non-associative word, then by u we denote an associative word obtained from $[u]$ by removing the brackets. Of course, $[u]$ is not uniquely defined by u in general.

Definition 2.2.4. The set of *standard non-associative words* is the biggest set SL that contains all variables x_i and satisfies the following properties:

1. If $[u]=\big[[v],[w]\big]\in SL$ then $[v],[w]\in SL$, and $v>w$ are standard.
2. If $[u]=\big[\big[[v_1],[v_2]\big],[w]\big]\in SL$ then $v_2\leq w$.

Theorem 2.2.5. (Shirshov's Theorem) [23, Lemma 2] *Every standard word u has only one alignment of brackets such that the defined non-associative word $[u]$ is standard.*

In order to find this alignment we use the following procedure: the factors v,w of the non-associative decomposition $[u]=\big[[v],[w]\big]$ are standard words such that $u=vw$ and v has the minimal length.

Definition 2.2.6. An *hyper-letter* is a polynomial that equals a non-associative standard word where the brackets mean (2.1). An *hyper-word* is a word in hyper-letters.

The hyper-letters were first invented and named super-letters by Kharchenko. However, not to make confusion with the same terminology used for super Lie algebras, Angiono renamed them hyper-letters.

By Shirshov's Theorem, every standard word u defines only one hyper-letter that will be denoted by $[u]$. The order on the hyper-letters is defined in the natural way: $[u] > [v] \Leftrightarrow u > v$.

Since quantum Borel algebras $U_q^+(\mathfrak{g})$ and $u_q^+(\mathfrak{g})$, which will be defined in 2.5.1 and 2.5.3, are homogeneous in each variable, in what follows we suppose that H is a Γ^+-graded character Hopf algebra, that is, H is homogeneous in each of the generators a_i.

Definition 2.2.7. An hyper-letter $[u]$ is called *hard in H* if its value in H is not a linear combination of hyper-words of the same degree (2.9) in hyper-letters smaller than $[u]$.

Proposition 2.2.8. [11, Corollary 2] *An hyper-letter $[u]$ is hard in H if and only if the value in H of the standard word u is not a linear combination of values of smaller words of the same degree (2.9).*

Proposition 2.2.9. [12, Lemma 4.8] *Let B be a set of hyper-letters containing $x_1, \ldots, x_n$. If each pair $[u], [v] \in B$, $u > v$ satisfies one of the following conditions*

1) $\big[[u], [v]\big]$ *is not a standard non-associative word;*

2) *the hyper-letter* $\big[[u], [v]\big]$ *is not hard in H;*

3) $\big[[u], [v]\big] \in B$*;*

then the set B includes all hard in H hyper-letters.

Definition 2.2.10. We say that the *height* of a hard in H hyper-letter $[u]$ equals $h = h([u])$ if h is the smallest number such that

1. p_{uu} is a primitive t-th root of 1 and either $h = t$ or $h = tl^r$, where $l = char(\mathbf{k})$,

2. the value of $[u]^h$ in H is a linear combination of hyper-words of the same degree (2.9) in hyper-letters smaller than $[u]$.

If there exists no such number then the height equals infinity.

Lemma 2.2.11. [12, Lemma 4.9] *If $\boldsymbol{T} \in H$ is an homogeneous skew-primitive element then*

$$\boldsymbol{T} = \alpha[u]^h + \sum \alpha_i W_i, \quad \alpha \neq 0, \tag{2.10}$$

where $[u]$ is a hard hyper-letter, W_i are basis words in hyper-letters smaller than $[u]$. Here if p_{uu} is not a root of unity then $h = 1$; if p_{uu} is a primitive $t-$th root of unity then $h = 1$, or $h = t$, or $h = tl^k$, where l is the characteristic.

Definition 2.2.12. An element u is said to be *skew-central* if for every homogeneous v we have $uv = \alpha vu$, $\alpha = \alpha(v) \in \mathbf{k}$. Certainly it is equivalent to a system of n equalities $ux_i = \alpha_i x_i u$, $1 \leq i \leq n$, $\alpha_i \in \mathbf{k}$.

Example 2.2.13. For example, all group-like elements in $G\langle X\rangle$ are skew-central since $x_i g_j = p_{ij} g_j x_i$, where $i, j \in \{1, 2, \cdots, n\}$.

2.3 PBW-generators

In this section we will define PBW-basis.

Definition 2.3.1. Let S be an algebra over $\mathbf{k}$ and A be a subalgebra of S with a fixed basis $\{a_j | j \in J\}$. A linearly ordered subset $W \subseteq S$ is said to be a set of *PBW-generators of S over A* if there exists a function $h : W \to \mathbb{Z}^+ \cup \infty$, called the height function, such that the set of all products

$$a_j w_1^{n_1} w_2^{n_2} \dots w_k^{n_k}, \tag{2.11}$$

where $j \in J$, $w_1 < w_2 < \dots < w_k \in W$, $0 \leq n_i < h(w_i)$, $1 \leq i \leq k$ is a basis of S. The value $h(w)$ is referred to as the *height* of w in W. If $A = \mathbf{k}$ is the ground field, then we shall call W simply as a set of PBW-generators of S.

Definition 2.3.2. Let W be a set of PBW-generators of S over a subalgebra A. Suppose that the set of all words in W as a free monoid has its own order $\prec$ (that is, $a \prec b$ implies $cad \prec cbd$ for all words $a, b, c, d \in W$). The *leading word* of $s \in S$ is the maximal word $m = w_1^{n_1} w_2^{n_2} \dots w_k^{n_k}$ that appears in the decomposition of s in the basis (2.11). The *leading term* of s is the sum am of all terms $\alpha_i a_i m$, $\alpha_i \in \mathbf{k}$, that appear in the decomposition of s in the basis (2.11), where m is the leading word of s.

Theorem 2.3.3. [11, Theorem 2] *The values of all hard in H hyper-letters with the height function definided in 2.2.10 form a set of PBW-generators for H over $\mathbf{k}[G]$.*

2.4 Convex order

Let (V, c) be a braided vector space of diagonal type, with $dim\,V = \theta$. In other words, there is a basis $(x_i)_{i\in\mathbb{I}_\theta}$, $I_\theta = \{1, 2, \dots, \theta\}$, and a braiding matrix $\mathbf{p} = (p_{ij})_{i,j\in\mathbb{I}_\theta}$ such that

$$c(x_i \otimes x_j) = p_{ij} x_j \otimes x_i.$$

Let $\Delta^{\mathbf{p}}$ be the generalized root system associated to $\mathbf{p}$ and $\Delta^{\mathbf{p}}_+ = \{\beta_1, \cdots, \beta_M\}$ the subset of positive roots. Let α_i, $i \in \mathbb{I}_\theta$, be the simple roots. We denote $x_{\alpha_i} = x_i, i \in \mathbb{I}_\theta$.

Definition 2.4.1. Consider a root system $\Delta^{\mathbf{p}}_+$ with a fixed total order $<$. We say that the order is

- *convex* if for any $\alpha, \beta \in \Delta^{\mathbf{p}}_+$ such that $\alpha < \beta$ and $\alpha + \beta \in \Delta^{\mathbf{p}}_+$ we have

$$\alpha < \alpha + \beta < \beta;$$

- *subconvex* if for any $\alpha, \beta \in \Delta^{\mathbf{p}}_+$ such that $\alpha < \beta$ and $\alpha + \beta \in \Delta^{\mathbf{p}}_+$ we have

$$\alpha < \alpha + \beta;$$

- *strongly convex* if for each ordered subset $\alpha_1 \leq \cdots \leq \alpha_k \in \Delta^{\mathbf{p}}_+$ with $\alpha := \sum \alpha_i \in \Delta^{\mathbf{p}}_+$ we have

$$\alpha_1 < \alpha < \alpha_k.$$

Theorem 2.4.2. [4, Theorem 2.11] *Given an order on $\Delta^{\mathbf{p}}_+$, the following statements are equivalent:*

(1) the order is associated with a reduced expression of the longest element,

(2) the order is strongly convex,

(3) the order is convex.

Each simple root α_i is associated to the quantum variable x_i, $i \in \{1, \cdots, \theta\}$. Moreover, each positive root β_j is associated to a PBW-generator of the Hopf algebra, see [4, Theorem 3.9].

Definition 2.4.3. We say that a PBW-basis is *convex basis* if the order of the associated roots is convex.

We notice that a quantum algebra may have more than one convex set of PBW-generators, even if we fix the order of the simple roots. However, if we suppose that the elements are hyper-letters, we have only one possible convex basis, as stated in the next proposition.

Remark 2.4.4. Notice that by [4, Lemma 4.5] a PBW-basis of hyper-letters being convex implies that, for all $[u]$, $[v]$ ($[u] > [v]$) in the basis, we have $\big[[u],[v]\big]$ is a linear combination of super-words $[w] = [w_1]\cdots[w_k]$, where $[u] > [w_i] > [v]$, $i = 1, \cdots, k \in \mathbb{N}$, $[w_i]$ belongs to the PBW-basis and $[w]$ has the same degree of $\big[[u],[v]\big]$.

Proposition 2.4.5. *Let B be a convex set of PBW-generators formed by hyper-letters. Then B is constituted by the hard hyper-letters.*

Proof. Let B be a convex PBW-basis of hyper-letters. By Remark 2.4.4 and Definition 2.2.7, for every pair $[u],[v] \in B$, such that $[u] > [v]$, we have that $\big[[u],[v]\big] \in B$ or $\big[[u],[v]\big]$ is not hard. Then it satisfies conditions 2 or 3 of Proposition 2.2.9. Therefore B is constituted by hard hyper-letters. □

2.5 Quantum algebras

In this section we define the algebras $U_q^+(\mathfrak{g})$ and $u_q^+(\mathfrak{g})$, where $\mathfrak{g}$ is a simple Lie algebra.

Definition 2.5.1. Let $C = ||a_{ij}||$ be a generalized Cartan matrix symmetrizable by $D = diag(d_1, \ldots, d_n)$, $d_i a_{ij} = d_j a_{ji}$. Denote by $\mathfrak{g}$ a Kac-Moody algebra defined by C (see [9]). Suppose that the quantification parameters $p_{ij} = p(x_i, x_j) = \chi^i(g_j)$ are related by

$$p_{ii} = q^{d_i}, \quad p_{ij}p_{ji} = q^{d_i a_{ij}}, \quad 1 \leq i, j \leq n. \tag{2.12}$$

The *multiparameter quantization* $U_q^+(\mathfrak{g})$ of the Borel subalgebra $\mathfrak{g}^+$ is a character Hopf algebra generated by $x_1, \ldots, x_n, g_1, \ldots, g_n$ and defined by Serre relations with the skew brackets (2.1) in place of the Lie operation:

$$[[\ldots[[x_i, x_j], x_j], \ldots], x_j] = 0, \quad 1 \leq i \neq j \leq n, \tag{2.13}$$

where x_j appears $1 - a_{ji}$ times.

Remark 2.5.2. By [10, Theorem 6.1] the left side of each of these relations is skew-primitive in $G\langle X\rangle$. So the ideal generated by these elements is a Hopf ideal.

Definition 2.5.3. If the multiplicative order t of q is finite, then we define $u_q^+(\mathfrak{g})$ as $G\langle X\rangle/\Lambda$, where Λ is the biggest Hopf ideal in $G\langle X\rangle^{(2)}$, which is the set (an ideal) of noncommutative polynomials without free and linear terms. From [14, Lemma 2.2] this is a Γ^+-homogeneous ideal. Certainly Λ contains all skew-primitive elements of $G\langle X\rangle^{(2)}$ (each one of them generates a Hopf ideal). Hence, by [10, Theorem 6.1], relations (2.13) are still valid in $u_q^+(\mathfrak{g})$.

Notice that the subalgebra A generated by $x_1, \dots, x_n$ over $\mathbf{k}$ in $U_q^+(\mathfrak{g})$ is a Nichols algebra of Cartan type if q is not a root of 1, see [2]. In the same way, if $q^t = 1$ for an integer t, the same thing is valid for $A \subseteq u_q^+(\mathfrak{g})$. This is particularly useful since in [3] there are many results for the Nichols algebra A. However, if q is a root of 1, then the subalgebra generated by $x_1, \dots, x_n$ in $U_q^+(\mathfrak{g})$ is not a Nichols algebra.

2.6 Differential calculus

In this section we list important results for calculating the height of the PBW-generators of $u_q^+(G_2)$ and $u_q^+(F_4)$ in the chapters 3 and 4.

Definition 2.6.1. The subalgebra A generated by $x_1, ..., x_n$ over $\mathbf{k}$ in $U_q^+(\mathfrak{g})$ (respectively, $u_q^+(\mathfrak{g})$) has a differential calculus defined by

$$\partial_i(x_j) = \delta_i^j, \qquad \partial_i(uv) = \partial_i(u)v + p(u, x_i)u\partial_i(v). \tag{2.14}$$

Lemma 2.6.2. ([18, Lemma 2.10]) *Let $u \in \mathbf{k}\langle X\rangle$ be an homogeneous in each x_i element. If p_{uu} is a t-th primitive root of 1, then*

$$\partial_i(u^t) = p(u, x_i)^{t-1} \underbrace{[u, [u, \cdots, [u}_{t-1}, \partial_i(u)] \cdots]]. \tag{2.15}$$

Lemma 2.6.3. (Milinski-Schneider criterion, see [21]) *If a polynomial $f \in \mathbf{k}\langle X\rangle$ with no one free terms is such that $\partial_i(f) = 0$ in $u_q^+(\mathfrak{g})$ for every $x_i \in X$, then $f = 0$ in $u_q^+(\mathfrak{g})$.*

2.7 Combinatorial rank

We notice that by [13, Proposition 1.7] each ideal generated by skew-primitive elements is a Hopf ideal, but a Hopf ideal is not always generated by its skew-primitive elements. However, the skew-primitive relations play an important role in the construction of character Hopf algebras due to the following result.

Theorem 2.7.1. [19, Corollary 5.3] *Every nonzero bi-ideal of a character Hopf algebra has a nonzero skew-primitive element.*

Let H be a character Hopf algebra and J a hopf ideal of H. We construct the sequence $0 = J_0 \subsetneq J_1 \subsetneq J_2 \subsetneq \ldots \subsetneq J_i \subsetneq \ldots \subsetneq J$ of Hopf ideals in the following way. We define J_1 as the Hopf ideal generated by skew-primitive elements of J. If $J_1 \neq J$, then $\frac{J}{J_1} \neq 0$ is a Hopf ideal and has a skew-primitive element. We define $\frac{J_2}{J_1}$ as the ideal generated by skew-primitive elements of $\frac{J}{J_1}$, where $J_2 = \pi^{-1}(\frac{J_2}{J_1})$ with $\pi : G\langle X\rangle \to \frac{G\langle X\rangle}{J_1}$. If $J_2 \neq J$ then define $\frac{J_3}{J_2}$ as the ideal generated by skew-primitive elements of $\frac{J}{J_2}$. Following this process, this sequence of Hopf ideals stabilizes if $J_\kappa = J$ for some κ.

Lemma 2.7.2. [13, Lemma 1.24]

$$\bigcup_{i=1}^{\infty} J_i = J.$$

Definition 2.7.3. If G is an abelian set of group-like elements, X is a set of skew-primitive elements and a combinatorial representation of H by means of generators and relations $\varphi : G\langle X\rangle \to H$ is given with $J = \ker\varphi$. We say that the *combinatorial rank* of H is the lenght κ of the above sequence, or infinite if the sequence does not stabilizes.

Consider the projections $\psi_1 : G\langle X\rangle \to u_q^+(\mathfrak{g})$ and $\psi_2 : G\langle X\rangle \to U_q^+(\mathfrak{g})$ the extensions of $x_i \mapsto a_i$. We know that $\ker\psi_1 = \Lambda$ is the biggest Hopf ideal in $G\langle X\rangle^{(2)}$ and $\ker\psi_2$ is generated by the Serre relations (2.13). In order to calculate the combinatorial rank $\kappa(u_q^+(\mathfrak{g}))$ we should consider $J = \Lambda$. However, we have that $\ker\psi_2 \subseteq \ker\psi_1 = \Lambda$ and the defining relations for $U_q^+(\mathfrak{g})$ are skew-primitive. We also have from Proposition 3.3.1 and Theorem 4.3.2 that the only homogeneous skew-primitive elements in $U_q^+(\mathfrak{g})$ are $x_1, \cdots, x_n$ and $x_1^{h_1}, \cdots, x_n^{h_n}$ in the considered cases, where h_i is the height of x_i. This implies that the only skew-primitive elements in $G\langle X\rangle$ belong to the ideal generated by these elements and the Serre relations. This way, instead of the morphism $\psi_1 : G\langle X\rangle \to u_q^+(\mathfrak{g})$ we may use the induced one $\varphi : U_q^+(\mathfrak{g}) \to u_q^+(\mathfrak{g})$. In the next chapters we consider $J = \ker\varphi$ a Hopf ideal of $U_q^+(\mathfrak{g})$.

Chapter 3

Combinatorial rank of the quantum groups of type G_2

3.1 Quantum groups of type G_2

In this section we are going to explicit a set of PBW-generators for $U_q^+(G_2)$ (respectively, $u_q^+(G_2)$, if $q^t = 1$ for $t > 3$).

Let us first remember that the algebra $U_q^+(G_2)$ is defined by two generators x_1, x_2 and two relations

$$[[x_1, x_2], x_2] = 0, \quad [x_1, [x_1, [x_1, [x_1, x_2]]]] = 0, \tag{3.1}$$

where the brackets mean the skew commutator (2.1). Relations (2.12) take up the form $p_{11}^3 = p_{22}$, $p_{12}p_{21} = p_{22}^{-1}$, and $p_{11} = q$. In what follows we shall suppose that $q^2 \neq 1$ and $q^3 \neq 1$. We notice that we do not follow exactly the notation in [22]. Minor adaptations were made in order to directly use results from [1] and [7].

In the following theorems we present the PBW-bases of $U_q^+(G_2)$ and $u_q^+(G_2)$.

Theorem 3.1.1. [22, Theorem 3.4] *If q is not a root of 1, then the values in $U_q^+(G_2)$*

of the elements

$$\begin{aligned} [A] &= x_1, \\ [B] &= [x_1, [x_1, [x_1, x_2]]], \\ [C] &= [x_1, [x_1, x_2]], \\ [D] &= [[x_1, [x_1, x_2]], [x_1, x_2]], \\ [E] &= [x_1, x_2], \\ [F] &= x_2. \end{aligned} \tag{3.2}$$

form a set of PBW-generators for $U_q^+(G_2)$ *over* $\mathbf{k}[G]$*, and each element has infinite height. If we suppose that* $x_1 > x_2$*, then* $A > B > C > D > E > F$*.*

Remark 3.1.2. If q is a root of 1 then the elements $[u]$ from list (3.2) also have infinite height in $U_q^+(G_2)$. Indeed, if $[u]$ has a finite height then the value of $[u]^h$ in $U_q^+(G_2)$ is a linear combination of words in hard hyper-letters that are smaller than $[u]$. But no element from the list (3.2) can be written as this linear combination. For example, if $[u] = [A] = x_1$, $[u]^h = x_1^h$ has degree $(h, 0)$ and all the other smaller elements of list have a degree (M, N), where $M \in \{0, 1, 2, 3\}$ and $N \in \{1, 2\}$. Therefore $[u]^h = 0$, which is a contradiction.

We note that $U_q^+(G_2)$ and $u_q^+(G_2)$ have the same PBW-generators but its elements have different heights. The following results are used to find the height of the elements in $u_q^+(G_2)$.

Theorem 3.1.3. [22, Theorem 3.6] *If* q *has finite multiplicative order* t*,* $t > 3$*, then the values in* $u_q^+(G_2)$ *of the elements from list* (3.2) *form a set of PBW-generators for* $u_q^+(G_2)$ *over* $\mathbf{k}[G]$*. The height* h *of* $[u] \in \{[A], [C], [E]\}$ *equals* t*. For* $[u] \in \{[B], [D], [F]\}$ *we have* $h = t$ *if* 3 *is not a divisor of* t *and* $h = \frac{t}{3}$ *otherwise. In all cases* $[u]^h = 0$ *in* $u_q^+(G_2)$*.*

We notice that the basis obtained in the previous results is not just a PBW-basis, but the unique PBW-basis constituted by the hard hyper-letters (see [11]). It is also a convex basis [4]. In addition we observe that, altough the second result is proved for $t > 4$ and $t \neq 6$ in the listed reference, it actually can be obtained for every $t > 3$ using a different proof, as in [3]. However, the cases where $t = 2$ or $t = 3$ do not generate the same algebra. In fact, using the Milinski-Schneider criterion [21], if $t = 2$, then we have $[B] = [C] = [D] = 0$. In this case the generated algebra with PBW-generators $\{x_1, [x_1, x_2], x_2\}$ is isomorphic to A_2 so [14] provides

$\kappa = 2$. Similarly, if $t = 3$, $[B] = [C] = [D] = [E] = 0$ and the only remaining PBW-generators are x_1 and x_2. In this case $\kappa = 1$ as $x_1^{h_1}$ and $x_2^{h_2}$ are skew-primitive.

3.2 The coproduct formula of quantum groups of type G_2

In this section we present the explicit coproduct formula for the elements $[u]^{h_u}$ where $[u]$ is a PBW-generator of $u_q^+(G_2)$ and h_u is the height of $[u]$.

The following results are already known.

Proposition 3.2.1. [20, Theorem 4.2] *The coproduct formula of elements from list* (3.2) *are:*

- $\Delta(x_1) = x_1 \otimes 1 + g_1 \otimes x_1$
- $\Delta([B]) = [B] \otimes 1 + g_{1112} \otimes [B] + (1-q^{-3})q^2 x_1 g_{112} \otimes [C] + (1-q^{-3})(1-q^{-2})q^2 x_1^2 g_{12} \otimes [E] + (1-q^{-3})(1-q^{-2})(1-q^{-1}) x_1^3 g_2 \otimes x_2$
- $\Delta([C]) = [C] \otimes 1 + g_{112} \otimes [C] + (1-q^{-2})(1+q) x_1 g_{12} \otimes [E] + (1-q^{-3})(1-q^{-2}) x_1^2 g_2 \otimes x_2$
- $\Delta([D]) = [D] \otimes 1 + g_{11122} \otimes [D] + (1-q^{-3})q^2 [C] g_{12} \otimes [E] + (1-q^{-3})^2 q^2 [C] x_1 g_2 \otimes x_2 + (1-q^{-3})(q^3 - q^2 - q) p_{21} [B] g_2 \otimes x_2 + (1-q^{-3})^2 (1-q^{-2})(1-q^{-1}) p_{21} x_1^3 g_{22} \otimes x_2^2 + (1-q^{-3})^2 (1-q^{-2}) q^2 x_1^2 g_{122} \otimes x_2 [E] + (1-q^{-3})(1-q^{-2}) q^2 x_1 g_{1122} \otimes x_{12}^2$
- $\Delta([E]) = [E] \otimes 1 + g_{12} \otimes [E] + (1-q^{-3}) x_1 g_2 \otimes x_2$
- $\Delta(x_2) = x_2 \otimes 1 + g_2 \otimes x_2$

Proposition 3.2.2. [8, Proposition 4.3] *Let* $\boldsymbol{k}$ *be an algebraically closed field of characteristic zero and* $q \in \mathbf{k}$ *such that* $q^t = 1$, *with* $t > 3$. *Suppose that* 3 *is not a divisor of* t. *Then we have the following statement in* $G\langle X \rangle$*:*

- $\Delta(x_1^t) = x_1^t \otimes 1 + g_1^t \otimes x_1^t$
- $\Delta([B]^t) = [B]^t \otimes 1 + g_1^{3t} g_2^t \otimes [B]^t + 3(1-q^{-1})^t p_{21}^{\frac{t(t-1)}{2}} x_1^t g_1^{2t} g_2^t \otimes [C]^t + 3(1-q^{-2})^t (1-q^{-1})^t p_{21}^{t(t-1)} x_1^{2t} g_1^t g_2^t \otimes [E]^t + (1-q^{-3})^t (1-q^{-2})^t (1-q^{-1})^t p_{21}^{\frac{3t(t-1)}{2}} x_1^{3t} g_2^t \otimes x_2^t$
- $\Delta([C]^t) = [C]^t \otimes 1 + g_1^{2t} g_2^t \otimes [C]^t + 2(1-q^{-2})^t p_{21}^{\frac{t(t-1)}{2}} x_1^t g_1^t g_2^t \otimes [E]^t + (1-q^{-3})^t (1-q^{-2})^t p_{21}^{t(t-1)} x_1^{2t} g_2^t \otimes x_2^t$

- $\Delta([D]^t) = [D]^t \otimes 1 + g_1^{3t}g_2^{2t} \otimes [D]^t + 3(1-q^{-1})^t p_{21}^{\frac{t(t-1)}{2}}[C]^t g_1^t g_2^t \otimes [E]^t - (1-q^{-3})^t p_{21}^{\frac{t(3t-1)}{2}}[B]^t g_2^t \otimes x_2^t + 3(1-q^{-2})^t(1-q^{-1})^t p_{21}^{t(t-1)} x_1^t g_1^{2t} g_2^{2t} \otimes [E]^{2t} + 3(1-q^{-3})^t(1-q^{-2})^t(1-q^{-1})^t p_{21}^{\frac{3t(t-1)}{2}} x_1^{2t} g_1^t g_2^{2t} \otimes x_2^t[E]^t + 3(1-q^{-3})^t(1-q^{-1})^t p_{21}^{t(t-1)}[C]^t x_1^t g_2^t \otimes x_2^t + (1-q^{-3})^{2t}(1-q^{-2})^t(1-q^{-1})^t p_{21}^{t(3t-2)} x_1^{3t} g_2^{2t} \otimes x_2^{2t}$

- $\Delta([E]^t) = [E]^t \otimes 1 + g_1^t g_2^t \otimes [E]^t + (1-q^{-3})^t p_{21}^{\frac{t(t-1)}{2}} x_1^t g_2^t \otimes x_2^t$

- $\Delta(x_2^t) = x_2^t \otimes 1 + g_2^t \otimes x_2^t$

In the case that 3 *divides* t *we have:*

- $\Delta(x_1^t) = x_1^t \otimes 1 + g_1^t \otimes x_1^t$

- $\Delta([B]^{\frac{t}{3}}) = [B]^{\frac{t}{3}} \otimes 1 + g_1^t g_2^{\frac{t}{3}} \otimes [B]^{\frac{t}{3}} + (1-q^{-3})^{\frac{t}{3}}(1-q^{-2})^{\frac{t}{3}}(1-q^{-1})^{\frac{t}{3}} p_{21}^{\frac{t(t-3)}{6}} x_1^t g_2^{\frac{t}{3}} \otimes x_2^{\frac{t}{3}}$

- $\Delta([C]^t) = [C]^t \otimes 1 + g_1^{2t} g_2^t \otimes [C]^t - (1-q^{-2})^t(1-q^{-1})^t p_{21}^{\frac{t(t-1)}{2}} x_1^t g_1^t g_2^t \otimes [E]^t + 3(1-q^{-2})^{\frac{-t}{3}}(1-q^{-1})^{\frac{t}{3}} p_{21}^{\frac{t(t+1)}{6}}[B]^{\frac{t}{3}} g_1^t g_2^{\frac{2t}{3}} \otimes [D]^{\frac{t}{3}} + (1-q^{-3})^t(1-q^{-2})^t p_{21}^{t(t-1)} x_1^{2t} g_2^t \otimes x_2^t + 3(1-q^{-3})^{\frac{t}{3}}(1-q^{-2})^{\frac{t}{3}}(1-q^{-1})^{\frac{t}{3}} p_{21}^{\frac{t^2}{3}}[B]^{\frac{2t}{3}} g_2^{\frac{t}{3}} \otimes x_2^{\frac{t}{3}} + 3(1-q^{-3})^{\frac{2t}{3}}(1-q^{-2})^{\frac{2t}{3}}(1-q^{-1})^{\frac{2t}{3}} p_{21}^{\frac{t(t-1)}{2}}[B]^{\frac{t}{3}} x_1^t g_2^{\frac{2t}{3}} \otimes x_2^{\frac{2t}{3}} + 3(1-q^{-3})^{\frac{t}{3}}(1-q^{-2})^{\frac{2t}{3}}(1-q^{-1})^{\frac{2t}{3}} p_{21}^{\frac{t(t-1)}{3}} x_1^t g_1^t g_2^t \otimes x_2^{\frac{t}{3}}[D]^{\frac{t}{3}}$

- $\Delta([D]^{\frac{t}{3}}) = [D]^{\frac{t}{3}} \otimes 1 + g_1^t g_2^{\frac{2t}{3}} \otimes [D]^{\frac{t}{3}} + (1-q^{-3})^{\frac{2t}{3}}(1-q^{-2})^{\frac{t}{3}}(1-q^{-1})^{\frac{t}{3}} p_{21}^{\frac{t(t-2)}{3}} x_1^t g_2^{\frac{2t}{3}} \otimes x_2^{\frac{2t}{3}} + 2(1-q^{-3})^{\frac{t}{3}} p_{21}^{\frac{t(t-1)}{6}}[B]^{\frac{t}{3}} g_2^{\frac{t}{3}} \otimes x_2^{\frac{t}{3}}$

- $\Delta([E]^t) = [E]^t \otimes 1 + g_1^t g_2^t \otimes [E]^t + 3(1-q^{-3})^{\frac{t}{3}}(1-q^{-2})^{\frac{-t}{3}}(1-q^{-1})^{\frac{-t}{3}} p_{21}^{\frac{t(t+1)}{6}}[D]^{\frac{t}{3}} g_2^{\frac{t}{3}} \otimes x_2^{\frac{t}{3}} + 3(1-q^{-3})^{\frac{2t}{3}}(1-q^{-2})^{\frac{-t}{3}}(1-q^{-1})^{\frac{-t}{3}} p_{21}^{\frac{t^2}{3}}[B]^{\frac{t}{3}} g_2^{\frac{2t}{3}} \otimes x_2^{\frac{2t}{3}} + (1-q^{-3})^t p_{21}^{\frac{t(t-1)}{2}} x_1^t g_2^t \otimes x_2^t$

- $\Delta(x_2^{\frac{t}{3}}) = x_2^{\frac{t}{3}} \otimes 1 + g_2^{\frac{t}{3}} \otimes x_2^{\frac{t}{3}}$

Altough the above proposition can be found in [8], a very similar version of it was first presented in [1, Section 4].

3.3 The combinatorial rank of quantum groups of type G_2

In this section, we prove the necessary results to determine $\kappa(U_q^+(G_2))$.

Proposition 3.3.1. *The skew-primitive homogeneous elements of* $U_q^+(G_2)$ *of total degree greater than or equal to one are* $x_1, x_2, x_1^{h_1}$ *and* $x_2^{h_2}$*, where* h_i *is the height of* x_i*.*

Proof. From Lemma 2.2.11, if $v \in U_q^+(G_2)$ is an homogeneous skew-primitive element, then $v = \alpha[u]^h + \sum \alpha_i W_i$ where $[u]$ is an element from list (3.2) and W_i are basis words smaller than $[u]$ with the same degree as $[u]^h$. If p_{uu} is not a root of the unit we have $h = 1$. If p_{uu} is a primitive t-th root of unit, then $h = 1$ or $h = t$.

If $[u] = x_1$ or $[u] = x_2$, then clearly there are no other basis words W_i of degree $(h, 0)$ or $(0, h)$, so $v = [u]^h$. If $[u] = [E]$, then $[u]^h$ has degree (h, h) which can not be obtained by basis words $[E]^r[F]^s$ that have degree $r(1, 1) + s(0, 1)$ unless $s = 0$. Thus $v = [E]^h$. If $[u] = [D]$, simmilarly the degree $(3h, 2h)$ can not be obtained as $r(3, 2) + s(1, 1) + l(0, 1)$ with $s \neq 0$ or $l \neq 0$. The same occurs for $[u] = [C]$ and $[u] = [B]$. This provides that the possible skew-primitive elements are $[u]^h$. If $h = 1$, then the only skew-primitive PBW-generators are x_1 and x_2, what is proved by Proposition 3.2.1. If $h = t$, then Proposition 3.2.2 shows that again only x_1^h and x_2^h are skew-primitive. □

Proposition 3.3.2. *The elements $[u]^h$ are skew central in $U_q^+(G_2)$, where $[u]$ belongs to the list (3.2) and h is the height of $[u]$.*

Proof. First we notice that $x_i[u]^h = \alpha[u]^h x_i$, for $i = 1, 2$ implies that $v[u]^h = \alpha[u]^h v$, for every homogeneous $v \in U_q^+(G_2)$. If $[u] \in \{[A], [C], [E]\}$ then necessarily $p_{uu} = q$, so p_{uu} is a t-th primitive root of the unit and $h_u = t \geq 4$. In the case that $[u] \in \{[B], [D], [F]\}$ we have $p_{uu} = q^3$ providing $h_u = 2$ if $t = 6$, $h_u = 3$ if $t = 9$ and $h_u \geq 4$ otherwise.

Using that the provided basis is convex [4, Lemma 4.5] we know that the skew-commutator of two PBW-generators $[u], [v]$, with $[u] > [v]$, is a linear combination of intermediate basis elements with the same degree as $\big[[u], [v]\big]$. Consequently we have $\big[[B], [C]\big] = \big[[C], [D]\big] = \big[[D], [E]\big] = 0$, $\big[[A], [D]\big] = \alpha_1[C]^2$, $\big[[B], [D]\big] = \alpha_2[C]^3$, $\big[[B], [E]\big] = \alpha_3[C]^2$, $\big[[B], [F]\big] = \alpha_4[D] + \alpha_5[E][C]$, $\big[[C], [F]\big] = \alpha_6[E]^2$ and $\big[[D], [F]\big] = \alpha_7[E]^3$ with $\alpha_i \in \mathbf{k}$ for every i. In fact, this has been explicited in [8, Lemma 4.1] where all coefficients α_i have been calculated.

If $[u] = [A] = x_1$ then clearly $x_1 x_1^h = x_1^h x_1$. As we have $[x_1, [x_1, [x_1, [x_1, x_2]]]] = 0$, using (2.7) with $h \geq 4$ we obtain $[x_1^h, x_2] = [x_1, [x_1, \ldots [x_1, x_2] \ldots]] = 0$. Thus $x_1^h x_2 = p_{12}^h x_2 x_1^h$ and $x_1 = [A]$ is skew-central. For $[u] = [F] = x_2$, similarly $x_2 x_2^h = x_2^h x_2$ and $[[x_1, x_2], x_2]$ associated with (2.6) guarantee that $[x_1, x_2^h] = 0$ for $h \geq 2$ and $x_1 x_2^h = p_{12}^h x_2^h x_1$.

In the case that $[u] = [E] = [x_1, x_2]$ we have $\big[[E], x_2\big] = 0$ so from equation (2.7) we obtain $\big[[E]^h, x_2\big] = 0$, then $[E]^h x_2 = p_{12}^h p_{22}^h x_2 [E]^h$. On the other hand $\big[\big[[x_1, [E]\big], [E]\big], [E]\big] = \big[\big[[C], [E]\big], [E]\big] = \big[[D], [E]\big] = 0$ therefore $h \geq 4$ and (2.6) provide $\big[x_1,[E]^h\big]=0$ and $x_1[E]^h=p_{11}^h p_{12}^h[E]^h x_1$.

For $[u]=[C]$ we notice that $\big[\big[x_1,[C]\big],[C]\big]=\big[[B],[C]\big]=0$ and $\big[[C],x_2\big]=\alpha_6[E]^2$. From formula (2.3) we obtain $\big[[C],[E]^2\big]=p_{12}q^2(1+q)[E][D]=\alpha_8[E][D]$, $\big[[C],[E][D]\big]=[D]^2$ and $\big[[C],[D]^2\big]=0$ so

$$\begin{aligned}\big[[C],\big[[C],\big[[C],\big[[C],x_2\big]\big]\big]\big]&=\alpha_6\big[[C],\big[[C],\big[[C],[E]^2\big]\big]\big]\\&=\alpha_6\alpha_8\big[[C],\big[[C],[E][D]\big]\big]\\&=\alpha_6\alpha_8\big[[C],[D]^2\big]=0\end{aligned}$$

thus $h\geq 4$, (2.6) and (2.7) provide $x_1[C]^h=p_{12}^h p_{11}^{2h}[C]^h x_1$ and $[C]^h x_2=p_{12}^{2h}p_{22}^h x_2[C]^h$.

Now we suppose $[u]=[D]$. In this case we have

$$\big[\big[x_1,[D]\big],[D]\big]=\alpha_1\big[[C]^2,[D]\big]=0,$$

$$\big[[D],\big[[D],x_2\big]\big]=\alpha_7[[D],[E]^3]=0$$

so from formulas (2.6) and (2.7) we obtain $x_1[D]^h=p_{11}^{3h}p_{12}^{2h}[D]^h x_1$ and $[D]^h x_2=p_{12}^{3h}p_{22}^{2h}x_2[D]^h$ for $h\geq 2$.

Finally, if $[u]=[B]$ then $\big[x_1,[B]\big]=0$ ensures $\big[x_1,[B]^h\big]=0$ for $h\geq 2$ and $x_1[B]^h=p_{11}^{3h}p_{12}^h[B]^h x_1$. For the variable x_2, using formula (2.3) we have

$$\begin{aligned}\big[[B],\big[[B],x_2\big]\big]&=\big[[B],\alpha_4[D]+\alpha_5[E][C]\big]\\&=\alpha_4\big[[B],[D]\big]+\alpha_5\big[[B],[E][C]\big]\\&=(\alpha_2\alpha_4+\alpha_5\alpha_9)[C]^3.\end{aligned}$$

In the case $h=2$, from [8, Lemma 4.1] we have

$$\alpha_2=\frac{p_{12}^2q^3(q-1)(q^3-1)}{q+1},\quad \alpha_4=p_{12}q(q^2-q-1),\quad \alpha_5=p_{12}^2q^2(q^3-1)$$

and using that $\big[[B],[E][C]\big]=\frac{p_{12}q(q^3-1)}{q+1}[C]^3$ we may explicitly calculate

$$\beta=\alpha_2\alpha_4+\alpha_5\alpha_9=\frac{p_{12}^3q^3(q-1)(q^6-1)}{q+1}$$

and see that it is zero as $h=2$ if and only if $q^6=1$. If $h\geq 3$ we have $\beta\neq 0$, however, $\big[[B],[C]^3\big]=0$ and consequently $\big[[B],\big[[B],\big[[B],x_2\big]\big]\big]=0$. Therefore $\big[[B]^h,x_2\big]=0$ for $h\geq 2$ and $[B]^h x_2=p_{12}^{3h}p_{22}^h x_2[B]^h$. $\square$

We consider $\varphi : U_q^+(G_2) \to u_q^+(G_2)$ the natural projection and we have the following result.

Proposition 3.3.3. *The set $J = \ker\varphi$ is generated by the elements $[u]^h$, where $[u]$ is an element from list* (3.2) *and h is the height of $[u]$.*

Proof. The fact that the kernel J contains the elements $[u]^h$ follows immediately from Theorem 3.1.3 as it shows that $[u]^h = 0$ in $u_q^+(\mathfrak{g})$. Now we consider $v = [F]^{n_1}[E]^{n_2}[D]^{n_3}[C]^{n_4}[B]^{n_5}[A]^{n_6}$ belonging to $\ker\varphi \subseteq U_q^+(G_2)$. If $n_i < h_i$ for every $i \in \{1, 2, \ldots, 6\}$ with h_i the height of the corresponding element, then v is a basis element of $u_q^+(G_2)$ and therefore $\varphi(v) \neq 0$, which is a contradiction. So we may assume that there is a $n_i \geq h_i$ for a fixed i, and then v is a multiple of the respective element $[u]^{h_i}$ and belongs to the ideal generated by this element. Now let $v = \alpha_1 v_1 + \alpha_2 v_2 \in \ker\varphi$. If $\varphi(v_1) = 0$ then $\varphi(v_2) = 0$ and both v_1, v_2 are multiples of elements of the form $[u]^{h_i}$. Thus v belongs to the ideal generated by these elements. If $\varphi(v_1)$ and $\varphi(v_2)$ are both not zero with $v_1 \neq \alpha v_2$ then $\varphi(v)$ is a sum of linearly independent basis elements of $u_q^+(G_2)$, so $\varphi(v) \neq 0$. Inductively we have the same result for $v = \alpha_1 v_1 + \ldots + \alpha_t v_t \in \ker\varphi$. Thus we obtain that J is generated by the elements $[u]^h$. □

As a conclusion of the previous results, the Hopf ideal J is generated by linearly independent skew-central elements $[u]^h$, with $[u] \in \{[A], [B], [C], [D], [E], [F]\}$. Now we calculate the combinatorial rank of $u_q^+(G_2)$.

Theorem 3.3.4. *The combinatorial rank $\kappa(u_q^+(G_2))$ is* 3.

Proof. Consider $J = \ker\varphi$ the Hopf ideal of $U_q^+(G_2)$. First we address the case where 3 is not a divisor of t, with $q^t = 1$, and in this case the height of all PBW-generators from list (3.2) is $h = t$. As $J \subseteq G\langle X\rangle^{(2)}$, from Proposition 3.3.1, the only skew-primitive elements in J are $[A]^t = x_1^t$ and $[F]^t = x_2^t$. We define J_1 as the Hopf ideal of J generated by x_1^t and x_2^t. Since these elements are skew-central, we may consider J_1 as a right (or left) ideal. Now we prove that $[u]^t$ is not in J_1 for $[u] \in \{[B], [C], [D], [E]\}$. Suppose that

$$[u]^t = \alpha_1 y_1 x_1^t + \alpha_2 y_2 x_2^t.$$

We may write $y_1, y_2 \in U_q^+(G_2)$ in the PBW-basis and then skew-commute x_1^t and x_2^t, writing $[u]^t$ as a linear combination of basis elements of $U_q^+(G_2)$. So, on both sides of the equality we have linear combinations of basis elements, however, on the right side we have necessarily x_1^t or x_2^t on every term. This provides that $[u]^t$ is not one of the elements on the right side, so we have a contradiction. Thus $[u]^t \notin J_1$, unless $[u] = x_1$ or $[u] = x_2$.

From Proposition 3.2.2, we see that $[B]^t$, $[C]^t$ and $[E]^t$ are skew-primitive elements in $\frac{J}{J_1}$. Thus they belong to J_2 and $J_1 \subsetneq J_2$. As $[D]^t$ is not skew-primitive in $\frac{J}{J_1}$, it remains to notice that it is not in J_2. Suppose that

$$[D]^t = \alpha_1 y_1 [A]^t + \alpha_2 y_2 [B]^t + \alpha_3 y_3 [C]^t + \alpha_4 y_4 [E]^t + \alpha_5 y_5 [F]^t.$$

Again we write y_i in the PBW-basis and appropriately skew-commute each term $[u]^t$, obtaining the inconsistency of writing the basis element $[D]^t$ as a linear combination of other basis elements. Again using Proposition 3.2.2 we see that $[D]^t$ is skew-primitive in $\frac{J}{J_2}$, so it belongs to J_3. As J_3 contains all the elements that generate J, we have that $J_3 = J$ and $\kappa = 3$.

For the case that 3 divides t, analogously Proposition 3.2.2 and the fact that $[u]^h$ is skew-central guarantees that J_1 is generated by $[A]^t$ and $[F]^{\frac{t}{3}}$, J_2 is generated by J_1, $[B]^{\frac{t}{3}}$, $[D]^{\frac{t}{3}}$ and $[E]^t$ and J_3 is generated by J_2 and $[C]^t$. So, again $\kappa = 3$. $\square$

As a final remark, we notice that, similarly to [15, Theorem 6.1], the result $\kappa(u_q^+(G_2)) = 3$ provides immediately the same combinatorial rank for the negative quantum Borel subalgebra $u_q^-(G_2)$. As a consequence, using the triangular decomposition we also obtain $\kappa(u_q(G_2)) = 3$.

Chapter 4

Combinatorial rank of the quantum groups of type F_4

In this chapter we denote by β_n the coefficient $1-q^{-n}$, where n is a natural number.

4.1 Quantum groups of type F_4

In this section we are going to explicit a set of PBW-generators for $U_q^+(F_4)$ (and $u_q^+(F_4)$).

Let us first remember that the algebra $U_q^+(F_4)$ is defined by four generators x_1, x_2, x_3, x_4 and relations

$$\begin{aligned} [x_1,[x_1,x_2]]] &= 0, \quad [[x_1,x_2],x_2]=0, \\ [x_2,[x_2,x_3]]] &= 0, \quad [[[x_2,x_3],x_3],x_3]=0, \\ [x_3,[x_3,x_4]]] &= 0, \quad [[x_3,x_4],x_4]=0, \\ [x_1,x_3] &= [x_1,x_4]=[x_2,x_4]=0, \end{aligned} \tag{4.1}$$

where the brackets mean the skew commutator (2.1). Relations (2.12) take up the form $p_{11}=p_{22}=p_{33}^2=p_{44}^2=q^2$, $p_{12}p_{21}=q^{-2}=p_{23}p_{32}$, $p_{34}p_{43}=q^{-1}$ and $p_{13}p_{31}=p_{14}p_{41}=p_{24}p_{42}=1$. In what follows we shall suppose that $q^2\neq 1$.

In the following theorem we present a PBW-basis of $U_q^+(F_4)$.

Theorem 4.1.1. *The values in $U_q^+(F_4)$ of the elements*

$$
\begin{aligned}
[A] &= x_1,\\
[B] &= [x_1, x_2],\\
[C] &= [x_1, [x_2, x_3]],\\
[D] &= [x_1, [[x_2, x_3], x_3]],\\
[E] &= [[x_1, [[x_2, x_3], x_3]], x_2],\\
[F] &= [x_1, [x_2, [x_3, x_4]]].\\
[G] &= [x_1, [[x_2, [x_3, x_4]], x_3]],\\
[H] &= [[x_1, [[x_2, [x_3, x_4]], x_3]], [[x_1, [[x_2, [x_3, x_4]], x_3]], x_2]],\\
[I] &= [[x_1, [[x_2, [x_3, x_4]], x_3]], x_2],\\
[J] &= [[x_1, [[x_2, [x_3, x_4]], x_3]], [x_2, x_3]],\\
[K] &= [x_1, [[x_2, [x_3, x_4]], [x_3, x_4]],\\
[L] &= [[x_1, [[x_2, [x_3, x_4]], [x_3, x_4]], x_2],\\
[M] &= [[x_1, [[x_2, [x_3, x_4]], [x_3, x_4]], [x_2, x_3]],\\
[N] &= [[x_1, [[x_2, [x_3, x_4]], [x_3, x_4]], [[x_2, x_3], x_3]],\\
[O] &= [[[x_1, [[x_2, [x_3, x_4]], [x_3, x_4]], [[x_2, x_3], x_3]], x_2],\\
[P] &= x_2,\\
[Q] &= [x_2, x_3],\\
[R] &= [[x_2, x_3], x_3],\\
[S] &= [x_2, [x_3, x_4]],\\
[T] &= [[x_2, [x_3, x_4]], x_3],\\
[U] &= [[x_2, [x_3, x_4]], [x_3, x_4]],\\
[V] &= x_3,\\
[W] &= [x_3, x_4],\\
[X] &= x_4,
\end{aligned}
\tag{4.2}
$$

form a convex set of PBW-generators for $U_q^+(F_4)$ over $\mathbf{k}[G]$, and each element has infinite height. If we suppose that $x_1 > x_2 > x_3 > x_4$, then $A > B > \ldots > W > X$.

Proof. This statement follows from the fact that $U_q^+(F_4)$ is a bosonization of a Nichols algebra generated by x_1, x_2, x_3, x_4 and the results from [3, Section 4B].

Now we have to see that all heights are infinite. Consider $[u]$ an element from list (4.2). With a simple calculation we obtain that $p([u],[u]) = q$ for $[u] \in \{[C],[F],[G],[I],[J],[M],[Q],[S],[T],[V],[W],[X]\}$ and $p([u],[u]) = q^2$ for $[u] \in \{[A],[B],[D],[E],[H],[K],[L],[N],[O],[P],[R],[U]\}$. If q is not a root of 1, then $p(u,u)$ is not a primitive t-th root of 1 for any t. From Definition 2.2.10 we have that $h([u])$ is infinite. If q is a root of unity we also obtain that $h([u])$ is infinite, in the same way of Remark 3.1.2. $\square$

We notice that the PBW-basis obtained in the previous results is a convex basis. From Proposition 2.4.5 it is also the unique PBW-basis constituted by the hard hyper-letters.

Now we prove results to calculate the height of the elements in the list (4.2) in $u_q^+(F_4)$ when q is a root of 1. In order to simplify calculations, in the Appendix we list all the commutators between the basis elements.

Proposition 4.1.2. *The derivatives of the elements from the list* (4.2) *are given in the following table:*

	∂_1	∂_2	∂_3	∂_4
$[A]$	1	0	0	0
$[B]$	$\beta_2 x_2$	0	0	0
$[C]$	$\beta_2[Q]$	0	0	0
$[D]$	$\beta_2[R]$	0	0	0
$[E]$	$\beta_2^2[R]x_2 - \beta_1\beta_2 p_{32}[Q]^2$	0	0	0
$[F]$	$\beta_2[S]$	0	0	0
$[G]$	$\beta_2[T]$	0	0	0
$[H]$	$\alpha[T][I] + \theta[O] + \gamma[R][L] + \omega x_2[N] +$ $\lambda[Q]^2[R] + \mu[S][J] + \rho[Q][M] + \tau[R]x_2[K]$	0	0	0
$[I]$	$-\beta_1\beta_2 p_{32}[S][Q] + \beta_2^2[T]x_2$	0	0	0
$[J]$	$\beta_1\beta_2[T][Q] - \beta_1\beta_2 p_{32}[S][R]$	0	0	0
$[K]$	$\beta_2[U]$	0	0	0
$[L]$	$\beta_2^2[U]x_2 - \beta_1\beta_2 p_{32}p_{42}[S]^2$	0	0	0
$[M]$	$\beta_2^2[U][Q] - \beta_2^2 p_{42}p_{43}[T][S]$	0	0	0
$[N]$	$\beta_2^2[U][R] - \beta_2^2 p_{42}p_{43}[T]^2$	0	0	0
$[O]$	$\beta_2^3[U][R]x_2 - \beta_2^3 p_{42}p_{43}[T]^2x_2 + \beta_2^3 p_{32}p_{42}p_{43}q[T][S][Q] -$ $\beta_1\beta_2^2 p_{32}[U][Q]^2 - \beta_1\beta_2^2 p_{32}^3 p_{42} q[S]^2[R]$	0	0	0
$[P]$	0	1	0	0
$[Q]$	0	$\beta_2 x_3$	0	0
$[R]$	0	$\beta_1\beta_2 x_3^2$	0	0
$[S]$	0	$\beta_2[W]$	0	0
$[T]$	0	$\beta_1\beta_2[W]x_3$	0	0
$[U]$	0	$\beta_1\beta_2[W]^2$	0	0
$[V]$	0	0	1	0
$[W]$	0	0	$\beta_1 x_4$	0
$[X]$	0	0	0	1

Here $\alpha = \beta_2^2 q$, $\theta = \beta_2 p_{21}p_{24}p_{31}^2 p_{34}p_{41}(1+q)^{-1}(1+q^{-1}-q^2)$, $\gamma = \beta_1\beta_2 p_{41}p_{42}p_{43}^3 q(q-q^{-1}-1)$, $\omega = \beta_1\beta_2 p_{31}^2 p_{32}^4 p_{34}p_{41}p_{42}q^4(1-q^{-1}-q^{-2})$, $\lambda = -\beta_1^2\beta_2 p_{12}p_{32}^3 p_{41}p_{42}^3 p_{43}^3 q^4$, $\mu = -\beta_2^2 p_{31}p_{32}^2 p_{34}q^3$, $\rho = \beta_1\beta_2 p_{31}p_{32}^2 p_{41}p_{42}p_{43}q^2(1-q+q^{-1})$ *and* $\tau = \beta_1\beta_2^2 p_{12}p_{32}^2 p_{41}p_{42}^3 p_{43}^3 q^4$.

Proof. Since $[A] = x_1$, $[P] = x_2$, $[V] = x_3$ and $[X] = x_4$, from the definition,

$\partial_1([A]) = 1$, $\partial_i([A]) = 0$ for $i = \{2,3,4\}$;

$\partial_2([P]) = 1$, $\partial_i([P]) = 0$ for $i = \{1,3,4\}$;

$\partial_3([V]) = 1$, $\partial_i([V]) = 0$ for $i = \{1,2,4\}$;

$\partial_4([X]) = 1$, $\partial_i([X]) = 0$ for $i = \{1, 2, 3\}$.

For $[B] = [x_1, x_2] = x_1 x_2 - p_{12} x_2 x_1$, we have

$$\begin{aligned}\partial_1([B]) =& \partial_1(x_1x_2) - p_{12}\partial_1(x_2x_1)\\ =& \partial_1(x_1)x_2 + p_{11}x_1\partial_1(x_2) - p_{12}(\partial_1(x_2)x_1 + p_{21}x_2\partial_1(x_1))\\ =& x_2 - p_{12}p_{21}x_2 = \beta_2 x_2\end{aligned}$$

and $\partial_i([B]) = 0$ for $i = \{2, 3, 4\}$.

For $[Q] = [x_2, x_3] = x_2 x_3 - p_{23} x_3 x_2$, we have

$$\begin{aligned}\partial_2([Q]) =& \partial_2(x_2x_3) - p_{23}\partial_2(x_3x_2)\\ =& \partial_2(x_2)x_3 + p_{22}x_2\partial_2(x_3) - p_{23}(\partial_2(x_3)x_2 + p_{32}x_3\partial_2(x_2))\\ =& x_3 - p_{23}p_{32}x_3 = \beta_2 x_3\end{aligned}$$

and $\partial_i([Q]) = 0$ for $i = \{1, 3, 4\}$.

For $[W] = [x_3, x_4] = x_3 x_4 - p_{34} x_4 x_3$, we have

$$\begin{aligned}\partial_3([W]) =& \partial_3(x_3x_4) - p_{34}\partial_3(x_4x_3)\\ =& \partial_3(x_3)x_4 + p_{33}x_3\partial_3(x_4) - p_{34}(\partial_3(x_4)x_3 + p_{43}x_4\partial_3(x_3))\\ =& x_4 - p_{34}p_{43}x_4 = \beta_1 x_4\end{aligned}$$

and $\partial_i([W]) = 0$ for $i = \{1, 2, 4\}$.

Now for $[R] = [[x_2, x_3], x_3] = \big[[Q], x_3\big] = [Q]x_3 - p_{23}p_{33}x_3[Q]$, we have

$$\begin{aligned}\partial_2([R]) =& \partial_2([Q])x_3 + p_{22}p_{32}[Q]\partial_2(x_3) - p_{23}p_{33}(\partial_2(x_3)[Q] + p_{32}x_3\partial_2([Q]))\\ =& \beta_2 x_3^2 - \beta_2 p_{23}p_{32}p_{33}x_3^2 = \beta_1\beta_2 x_3^2\end{aligned}$$

and $\partial_i([R]) = 0$ for $i = \{1, 3, 4\}$.

For $[S] = [x_2, [x_3, x_4]] = \big[x_2, [W]\big] = x_2[W] - p_{23}p_{24}[W]x_2$, we have

$$\begin{aligned}\partial_2([S]) =& \partial_2(x_2)[W] + p_{22}x_2\partial_2([W]) - p_{23}p_{24}(\partial_2([W])x_2 + p_{32}p_{42}[W]\partial_2(x_2))\\ =& [W] - p_{23}p_{24}p_{32}p_{42}[W] = \beta_2[W]\end{aligned}$$

and $\partial_i([S]) = 0$ for $i = \{1, 3, 4\}$.

Again, for $[T] = \big[[S], x_3\big] = [S]x_3 - p_{23}p_{33}p_{43}x_3[S]$, we have

$$\begin{aligned}\partial_2([T]) =&\partial_2([S])x_3 + p_{22}p_{32}p_{42}[S]\partial_2(x_3) - p_{23}p_{33}p_{43}(\partial_2(x_3)[S] + p_{32}x_3\partial_2([S]))\\ =&\beta_2[W]x_3 - \beta_2 p_{23}p_{32}p_{33}p_{43}x_3[W] = \beta_1\beta_2[W]x_3\end{aligned}$$

and for $i = \{1, 3, 4\}$ we have $\partial_i([T]) = 0$.

For $[U] = \big[[S], [W]\big] = [S][W] - p_{23}p_{24}q[W][S]$, we have

$$\begin{aligned}\partial_2([U]) =&\partial_2([S])[W] + p_{22}p_{32}p_{42}[S]\partial_2([W]) - p_{23}p_{24}q(\partial_2([W])[S] + p_{32}p_{42}[W]\partial_2([S]))\\ =&\beta_2[W]^2 - \beta_2 p_{23}p_{24}p_{32}p_{42}q[W]^2 = \beta_1\beta_2[W]^2\end{aligned}$$

and $\partial_i([U]) = 0$ for $i = \{1, 3, 4\}$.

For $[C] = [x_1, [x_2, x_3]] = \big[x_1, [Q]\big] = x_1[Q] - p_{12}p_{13}[Q]x_1$, we have

$$\begin{aligned}\partial_1([C]) =&\partial_1(x_1)[Q] + p_{11}x_1\partial_1([Q]) - p_{12}p_{13}(\partial_1([Q])x_1 + p_{21}p_{31}[Q]\partial_1(x_1))\\ =&[Q] - p_{12}p_{13}p_{21}p_{31}[Q] = \beta_2[Q]\end{aligned}$$

and for $i = \{2, 3, 4\}$ we have $\partial_i([C]) = 0$.

Now for $[D] = \big[x_1, [R]\big] = x_1[R] - p_{12}p_{13}^2[R]x_1$, we have

$$\begin{aligned}\partial_1([E]) =&\partial_1(x_1)[R] + p_{11}x_1\partial_1([R]) - p_{12}p_{13}^2(\partial_1([R])x_1 + p_{21}p_{31}^2[R]\partial_1(x_1))\\ =&[R] - p_{12}p_{13}^2p_{21}p_{31}^2[R] = \beta_2[R]\end{aligned}$$

and for $i = \{2, 3, 4\}$ we have $\partial_i([D]) = 0$.

For $[E] = \big[[D], x_2\big] = [D]x_2 - p_{12}p_{32}^2q^2x_2[D]$, we have

$$\begin{aligned}\partial_1([E]) =&\partial_1([D])x_2 + p_{11}p_{21}p_{31}^2[D]\partial_1(x_2) - p_{12}p_{32}^2q^2(\partial_1(x_2)[D] + p_{21}x_2\partial_1([D]))\\ =&\beta_2[R]x_2 - \beta_2 p_{12}p_{21}p_{32}^2q^2x_2[R] = \beta_2^2[R]x_2 - \beta_1\beta_2 p_{32}[Q]^2\end{aligned}$$

and $\partial_i([E]) = 0$ for $i = \{2, 3, 4\}$.

Again, for $[F] = \big[x_1, [S]\big] = x_1[S] - p_{12}p_{13}p_{14}[S]x_1$, we have

$$\begin{aligned}\partial_1([F]) =&\partial_1(x_1)[S] + p_{11}x_1\partial_1([S]) - p_{12}p_{13}p_{14}(\partial_1([S])x_1 + p_{21}p_{31}p_{41}[S]\partial_1(x_1))\\ =&[S] - p_{12}p_{13}p_{14}p_{21}p_{31}p_{41}[S] = \beta_2[S]\end{aligned}$$

and $\partial_i([F]) = 0$ for $i = \{2, 3, 4\}$.

For $[G]=\big[x_1,[T]\big]=x_1[T]-p_{12}p_{13}^2p_{14}[T]x_1$, we have

$$\begin{aligned}\partial_1([G])=&\partial_1(x_1)[T]+p_{11}x_1\partial_1([T])-p_{12}p_{13}^2p_{14}(\partial_1([T])x_1+p_{21}p_{31}^2p_{41}[T]\partial_1(x_1))\\=&[T]-p_{12}p_{13}^2p_{14}p_{21}p_{31}^2p_{41}[T]=\beta_2[T]\end{aligned}$$

and $\partial_i([G])=0$ for $i=\{2,3,4\}$.

Now for $[I]=\big[[G],x_2\big]=[G]x_2-p_{12}p_{32}^2p_{42}q^2x_2[G]$, we have

$$\begin{aligned}\partial_1([I])=&\partial_1([G])x_2+p_{11}p_{21}p_{31}^2p_{41}[G]\partial_1(x_2)-p_{12}p_{32}^2p_{42}q^2(\partial_1(x_2)[G]+p_{21}x_2\partial_1([G]))\\=&\beta_2[T]x_2-\beta_2p_{12}p_{21}p_{32}^2x_2[T]p_{42}q^2=-\beta_1\beta_2p_{32}[S][Q]+\beta_2^2[T]x_2\end{aligned}$$

and $\partial_i([I])=0$ for $i=\{2,3,4\}$.

For $[H]=\big[[G],[I]\big]=[G][I]-p_{12}p_{32}^2p_{42}q^2[I][G]$, we have

$$\begin{aligned}\partial_1([H])=&\partial_1([G])[I]+p_{11}p_{21}p_{31}^2p_{41}[G]\partial_1([I])-p_{12}p_{32}^2p_{42}q^2(\partial_1([I])[G]+p_{11}p_{21}^2p_{31}^2p_{41}[I]\partial_1([G]))\\=&\beta_2[T][I]+p_{11}p_{21}p_{31}^2p_{41}(\beta_1\beta_2p_{32}[G][S][Q]+\beta_2^2[G][T]x_2)-\\&-p_{12}p_{32}^2p_{42}q^2(-\beta_1\beta_2p_{32}[S][Q][G]+\beta_2^2[T]x_2[G]+\beta_2p_{11}p_{21}^2p_{31}^2p_{41}[I][T])\end{aligned}$$

Using the appendix formulae, we have

$$\begin{aligned}\partial_1([H])=&\beta_2^2q[T][I]+\beta_2p_{21}p_{24}p_{31}^2p_{34}p_{41}(1+q)^{-1}(1+q^{-1}-q^2)[O]+\\&+\beta_1\beta_2p_{41}p_{42}p_{43}^3q(q-q^{-1}-1)[R][L]+\beta_1\beta_2p_{31}^2p_{32}^4p_{34}p_{41}p_{42}q^4(1-q^{-1}-q^{-2})x_2[N]-\\&-\beta_1^2\beta_2p_{12}p_{32}^3p_{41}p_{42}^3p_{43}^3q^4[Q]^2[R]-\beta_2^2p_{31}p_{32}^2p_{34}q^3[S][J]+\\&+\beta_1\beta_2p_{31}p_{32}^2p_{41}p_{42}p_{43}q^2(1-q+q^{-1})[Q][M]+\beta_1\beta_2^2p_{12}p_{32}^2p_{41}p_{42}^3p_{43}^3q^4[R]x_2[K]\end{aligned}$$

and $\partial_i([H])=0$ for $i=\{2,3,4\}$.

Now for $[J]=\big[[G],[Q]\big]=[G][Q]-p_{12}p_{13}p_{32}p_{42}p_{43}q^2[Q][G]$, we have

$$\begin{aligned}\partial_1([J])=&\partial_1([G])[Q]+p_{11}p_{21}p_{31}^2p_{41}[G]\partial_1([Q])-p_{12}p_{13}p_{32}p_{42}p_{43}q^2(\partial_1([Q])[G]+p_{21}p_{31}[Q]\partial_1([G]))\\=&\beta_2[T][Q]-\beta_2p_{12}p_{13}p_{21}p_{31}p_{32}p_{42}p_{43}q^2[Q][T]=\beta_1\beta_2[T][Q]-\beta_1\beta_2p_{32}[S][R]\end{aligned}$$

and $\partial_i([J])=0$ for $i=\{2,3,4\}$.

For $[K]=\big[x_1,[U]\big]=x_1[U]-p_{12}p_{13}^2p_{14}^2[U]x_1$, we have

$$\begin{aligned}\partial_1([K])=&\partial_1(x_1)[U]+p_{11}x_1\partial_1([U])-p_{12}p_{13}^2p_{14}^2(\partial_1([U])x_1+p_{21}p_{31}^2p_{41}^2[U]\partial_1(x_1))\\=&[U]-p_{12}p_{13}^2p_{14}^2p_{21}p_{31}^2p_{41}^2[U]=\beta_2[U]\end{aligned}$$

and $\partial_i([K]) = 0$ for $i = \{2,3,4\}$.

Again, for $[L] = \big[[K], x_2\big] = [K]x_2 - p_{12}p_{32}^2p_{42}^2q^2x_2[K]$, we have

$$\begin{aligned}\partial_1([L]) =& \partial_1([K])x_2 + p_{11}p_{21}p_{31}^2p_{41}^2[K]\partial_1(x_2) - p_{12}p_{32}^2p_{42}^2q^2(\partial_1(x_2)[K] + p_{21}x_2\partial_1([K]))\\ =&\beta_2[U]x_2 - \beta_2p_{12}p_{21}p_{32}^2p_{42}^2q^2x_2[U] = \beta_2^2[U]x_2 - \beta_1\beta_2p_{32}p_{42}[S]^2\end{aligned}$$

and $\partial_i([L]) = 0$ for $i = \{2,3,4\}$.

For $[M] = \big[[K], [Q]\big] = [K][Q] - p_{12}p_{13}p_{32}p_{42}^2p_{43}^2q^2[Q][K]$, we have

$$\begin{aligned}\partial_1([M]) =& \partial_1([K])[Q] + p_{11}p_{21}p_{31}^2p_{41}^2[K]\partial_1([Q]) - p_{12}p_{13}p_{32}p_{42}^2p_{43}^2q^2(\partial_1([Q])[K] + p_{21}p_{31}[Q]\partial_1([K]))\\ =&\beta_2[U][Q] - \beta_2p_{12}p_{13}p_{21}p_{31}p_{32}p_{42}^2p_{43}^2q^2[Q][U] = \beta_2^2[U][Q] - \beta_2^2p_{42}p_{43}[T][S]\end{aligned}$$

and for $i = \{2,3,4\}$, we have $\partial_i([M]) = 0$.

Now, for $[N] = \big[[K], [R]\big] = [K][R] - p_{12}p_{13}^2p_{42}^2p_{43}^4q^2[R][K]$, we have

$$\begin{aligned}\partial_1([N]) =& \partial_1([K])[R] + p_{11}p_{21}p_{31}^2p_{41}^2[K]\partial_1([R]) - p_{12}p_{13}^2p_{42}^2p_{43}^4q^2(\partial_1([R])[K] + p_{21}p_{31}^2[R]\partial_1([K]))\\ =&\beta_2[U][R] - \beta_2p_{12}p_{13}^2p_{21}p_{31}^2p_{42}^2p_{43}^4q^2[R][U] = \beta_2^2[U][R] - \beta_2^2p_{42}p_{43}[T]^2\end{aligned}$$

and for $i = \{2,3,4\}$, we have $\partial_i([N]) = 0$.

Finally, for $[O] = \big[[N], x_2\big] = [N]x_2 - p_{12}p_{32}^4p_{42}^2q^4x_2[N]$, we have

$$\begin{aligned}\partial_1([O]) =& \partial_1([N])x_2 + p_{11}p_{21}^2p_{31}^4p_{41}^2[N]\partial_1(x_2) - p_{12}p_{32}^4p_{42}^2q^4(\partial_1(x_2)[N] + p_{21}x_2\partial_1([N]))\\ =&\beta_2^2[U][R]x_2 - \beta_2^2p_{42}p_{43}[T]^2x_2 - p_{12}p_{21}p_{32}^4p_{42}^2q^4(\beta_2^2x_2[U][R] - \beta_2^2p_{42}p_{43}x_2[T]^2)\\ =&\beta_2^3[U][R]x_2 - \beta_2^3p_{42}p_{43}[T]^2x_2 + \beta_2^3p_{32}p_{42}p_{43}q[T][S][Q] - \beta_1\beta_2^2p_{32}[U][Q]^2 -\\ &-\beta_1\beta_2^2p_{32}^3p_{42}q[S]^2[R]\end{aligned}$$

and for $i = \{2,3,4\}$, we have $\partial_i([O]) = 0$. □

Lemma 4.1.3. *Let* $[u]$ *be an element from list* (4.2)*. We have*

$$\big[\underbrace{[u], \big[[u], \cdots \big[[u]}_{l}, \partial_i([u])\big], \cdots\big]\big] = 0, \tag{4.3}$$

for $l = 1$ *if* $[u] \in \{[A], [B], [D], [E], [K], [L], [N], [O], [P], [R], [U], [V], [W], [X]\}$ *and* $l = 2$ *if* $[u] \in \{[C], [F], [G], [H], [I], [J], [M], [Q], [S], [T]\}$*, with* $i \in \{1,2,3,4\}$*.*

Proof. Here we use the list in the Appendix and formulas (2.2) and (2.3). From now on we consider $a, b, c, \cdots, x, y, z$ belonging to the field $\mathbf{k}$.

First if $[u]=[A]=x_1$ we have $\big[[A],\partial_1([A])\big]=[x_1,1]=0$ and if $[u]=[B]$, we have $\big[[B],\partial_1([B])\big]=\beta_2[B,x_2]=0$. In the case $[u]=[C]$, then

$$\big[[C],\partial_1([C])\big]=\beta_2\big[[C],[Q]\big]=\beta_2(ax_2[D]+b[E]),$$

$$\big[[C],\big[[C],\partial_1([C])\big]\big]=c\big[[C],x_2\big][D]+dx_2\big[[C],[D]\big]+e\big[[C],[E]\big]=0.$$

If $[u]=[D]$, we have $\big[[D],\partial_1([D])\big]=\beta_2\big[[D],[R]\big]=0$.

For $[u]=[E]$,

$$\big[[E],\partial_1([E])\big]=a\big[[E],[R]\big]x_2+b[R]\big[[E],x_2\big]+c\big[[E],[Q]\big][Q]+d[Q]\big[[E],[Q]\big]\big]=0.$$

If $[u]=[F]$, we have

$$\big[[F],\partial_1([F])\big]=\beta_2\big[[F],[S]\big]=ax_2[K]+b[L],$$

$$\big[[F],\big[[F],\partial_1([F])\big]\big]=c\big[[F],x_2\big][K]+dx_2\big[[F],[K]\big]+d\big[[F],[L]\big]=0.$$

For $[u]=[G]$,

$$\big[[G],\partial_1([G])\big]=\beta_2\big[[G],[T]\big]=a[N]+b[R][K],$$

$$\big[[G],\big[[G],\partial_1([G])\big]\big]=c\big[[G],[N]\big]+d\big[[G],[R]\big][K]+e[R]\big[[G],[K]\big]=0.$$

In the case $[u]=[H]$, we have

$$\begin{aligned}\big[[H],\partial_1([H])\big]=&a\big[[H],[T]\big][I]+b[T]\big[[H],[I]\big]+c\big[[H],[O]\big]+d\big[[H],[R]\big][L]+\\&+e[R]\big[[H],[L]\big]+f\big[[H],x_2\big][N]+gx_2\big[[H],[N]\big]+h\big[[H],[Q]\big][Q][K]+\\&+i[Q]\big[[H],[Q]\big][K]+j[Q]^2\big[[H],[K]\big]+k\big[[H],[S]\big][J]+l[J]\big[[H],[S]\big]+\\&+m\big[[H],[Q]\big][M]+n[Q]\big[[H],[M]\big]+o\big[[H],[R]\big]x_2[K]+p[R]\big[[H],x_2\big][K]+\\&+q[R]x_2\big[[H],[K]\big]\\=&r[N][I]^2+s[M][J][I]+t[Q][K][J][I]+u[R][K][I]^2+v[L][J]^2+wx_2[K][J]^2\end{aligned}$$

where $r=-\beta_1^2\beta_2p_{12}p_{13}^2p_{14}p_{23}^2p_{24}^2p_{34}q^3(1+q^2)$, $s=\beta_1\beta_2^2p_{12}p_{13}p_{14}p_{24}^2p_{34}^2q^2(1+q^2)$, $t=0$, $u=\beta_1^2\beta_2^2p_{12}^2p_{13}^4p_{14}p_{23}^2p_{43}^3q^6(1+q^2)$, $v=\beta_1\beta_2^2p_{12}p_{14}p_{24}^2p_{31}p_{32}^4p_{34}^4q^6(1+q^2)$,

$w=\beta_2^3p_{12}^2p_{14}p_{31}p_{32}^6p_{34}^4q^{10}(1+q^2)$, so they are all zero if $q^4=0$, and

$$\begin{aligned}\big[[H],\big[[H],\partial_1([H])\big]\big] =&r\big[[H],[N][I]^2\big]+s\big[[H],[M][J][I]\big]+v\big[[H],[L][J]^2\big]+\\ &+t\big[[H],[Q]\big][K][J][I]+x[Q]\big[[H],[K][J][I]\big]+u\big[[H],[R]\big][K][I]^2+\\ &+y[R]\big[[H],[K][I]^2\big]+w\big[[H],x_2\big][K][J]^2+zx_2\big[[H],[K][J]^2\big].\end{aligned}$$

As $=\big[[H],[I]\big]=\big[[H],[J]\big]=\big[[H],[K]\big]=\big[[H],[L]\big]=\big[[H],[M]\big]=\big[[H],[N]\big]=0$, we have

$$\begin{aligned}\big[[H],\big[[H],\partial_1([H])\big]\big] =&t\big[[H],[Q]\big][K][J][I]+u\big[[H],[R]\big][K][I]^2+w\big[[H],x_2\big][K][J]^2\\ =&-\beta_1\beta_2^3p_{12}^3p_{13}^3p_{14}p_{32}p_{42}p_{43}q^6(\beta_1+\beta_2q+\beta_1q+\beta_2q^2)[J][I][K][J][I]+\\ &+\beta_1\beta_2^3p_{12}^3p_{13}^5p_{14}p_{23}^2p_{42}p_{43}^4q^9(\beta_1+\beta_2q)[J]^2[K][I]^2+\\ &+\beta_1\beta_2^3p_{12}^3p_{14}p_{31}p_{32}^8p_{34}^4p_{42}q^{10}(\beta_1+\beta_2q)[I]^2[K][J]^2.\end{aligned}$$

Commuting the terms so that they are elements of the base, that is, in the form $[K][J]^2[I]^2$, we have $\big[[H],\big[[H],\partial_1([H])\big]\big]=0$.

If $[u]=[I]$, we have

$$\begin{aligned}\big[[I],\partial_1([I])\big] =&a\big[[I],[S]\big][Q]+b[S]\big[[I],[Q]\big]+c\big[[I],[T]\big]x_2+d[T]\big[[I],x_2\big]\\ =&ex_2[O]+f[Q]^2[L]+g[R]x_2[L],\end{aligned}$$

$$\begin{aligned}\big[[I],\big[[I],\partial_1([I])\big]\big] =&e\big[[I],x_2[O]\big]+f\big[[I],[Q]\big][Q][L]+h[Q]\big[[I],[Q]\big][L]+g\big[[I],[R]\big]x_2[L]+\\ &+i[R]\big[[I],x_2[L]\big]\\ =&-\beta_1^2\beta_2^2p_{12}^3p_{13}^2p_{32}^3p_{42}^3p_{43}^3q^8x_2[J][Q][L]-\beta_1^2\beta_2^2p_{12}^4p_{13}^3p_{32}^3p_{42}^4p_{43}^4q^{10}[Q]x_2[J][L]+\\ &+\beta_1\beta_2^3p_{12}^2p_{13}^3p_{42}^3p_{43}^4q^7[Q][J]x_2[L].\end{aligned}$$

Placing the elements in the form $[Q]x_2[L][J]$ we have $\big[[I],\big[[I],\partial_1([I])\big]\big]=0$.

In the case $[u]=[J]$, we have

$$\begin{aligned}\big[[J],\partial_1([J])\big] =&a\big[[J],[T]\big][Q]+b[T]\big[[J],[Q])\big)+c\big[[J],[S]\big][R]+d[S]\big[[J],[R]\big]\\ =&e[Q]^2[N]+f[R][Q][M]+g[R]x_2[N]+h[R][O],\end{aligned}$$

as $\big[[J],[M]\big]=\big[[J],[N]\big]=\big[[J],[O]\big]=\big[[J],x_2\big]=\big[[J],[Q]\big]=\big[[J],[R]\big]=0$, we have $\big[[J],\big[[J],\partial_1([J])\big]\big]=0$.

If $[u]=[K]$, then $\big[[K],\partial_1([K])\big]=\beta_2\big[[K],[U]\big]=0$.

For $[u]=[L]$, we have

$$\big[[L],\partial_1([L])\big]=a\big[[L],[U]\big]x_2+b[U]\big[[L],x_2\big]+c\big[[L],[S]\big][S]+d[S]\big[[L],[S]\big]=0,$$

and for $[u]=[M]$,

$$\begin{aligned}\big[[M],\partial_1([M])\big]=&a\big[[M],[U]\big][Q]+b[U]\big[[M],[Q]\big]+c\big[[M],[T]\big][S]+d[T]\big[[M],[S]\big]\\=&e[U][O]+f[U]x_2[N]+g[S]^2[N].\end{aligned}$$

Since $\big[[M],[N]\big]=\big[[M],[O]\big]=\big[[M],x_2\big]=\big[[M],[S]\big]=\big[[M],[U]\big]=0$, we obtain $\big[[M],\big[[M],\partial_1([M])\big]\big]=0$.

If $[u]=[N]$, we have

$$\big[[N],\partial_1([N])\big]=a\big[[N],[U]\big][R]+b[U]\big[[N],[R]\big]+c\big[[N],[T]\big][T]+d[T]\big[[N],[T]\big]=0.$$

In the case $[u]=[O]$, we have

$$\begin{aligned}\big[[O],\partial_1([O])\big]=&a\big[[O],[U]\big][R]x_2+b[U]\big[[O],[R]x_2\big]+c\big[[O],[T]^2\big]x_2+d[T]^2\big[[O],x_2\big]+\\&+e\big[[O],[T]\big][S][Q]+f[T]\big[[O],[S][Q]\big]+g\big[[O],[U]\big][Q]^2+h[U]\big[[O],[Q]^2\big]+\\&+i\big[[O],[S]^2\big][R]+j[S]^2\big[[O],[R]\big]=0.\end{aligned}$$

Since $\partial_i([u])=0$ for $[u]\in\{[A],[B],[C],[D],[E],[F],[G],[H],[I],[J],[K],[L],[M],[N],[O]\}$, $i\in\{2,3,4\}$, we have $\big[[u],\partial_i([u])\big]=0$.

If $[u]=[P]=x_2$, we have $\big[[P],\partial_2([P])\big]=[x_2,1]=0$.

For $[u]=[Q]$, we have $\big[[Q],\partial_2([Q])\big]=\beta_2\big[[Q],x_3\big]=\beta_2[R]$ and

$$\big[[Q],\big[[Q],\partial_2([Q])\big]\big]=\beta_2\big[[Q],[R]\big]=0.$$

For $[u]=[R]$, we have $\big[[R],\partial_2([R])\big]=\beta_1\beta_2\big[[R],x_3^2\big]=0$, since $\big[[R],x_3\big]=0$.

If $[u]=[S]$, we have

$$\big[[S],\big[[S],\partial_2([S])\big]\big]=\beta_2\big[[S],\big[[S],[W]\big]\big]=\beta_2\big[[S],[U]\big]=0.$$

In the case $[u]=[T]$, we have

$$\big[[T],\partial_2([T])\big]=a\big[[T],[W]\big]x_3+b[W]\big[[T],x_3\big]=cx_3^2[U].$$

Since $\big[[T],x_3\big]=\big[[T],[U]\big]=0$, we obtain $\big[[T],\big[[T],\partial_2([T])\big]\big]=0$.

If $[u]=[U]$, we have $\big[[U],\partial_2([U])\big]=\beta_1\beta_2\big[[U],[W]^2\big]=0$.

Since $\partial_i([u])=0$ for $[u]\in\{[P],[Q],[R],[S],[T],[U]\}$, $i\in\{1,3,4\}$, we have $\big[[u],\partial_i([u])\big]=0$.

For $[u]=[V]=x_3$, we have $\big[[V],\partial_3([V])\big]=[x_3,1]=0$ and if $[u]=[W]$, we have $\big[[W],\partial_3([W])\big]=\beta_1\big[[W],x_4\big]=0$.

Since $\partial_i([u])=0$ for $[u]\in\{[V],[W]\}$, $i\in\{1,2,4\}$, we have $\big[[u],\partial_i([u])\big]=0$.

Finally, if $[u]=[X]=x_4$, we have $\big[[X],\partial_4([X])\big]=[x_4,1]=0$ and $\big[[X],\partial_i([X])\big]=0$ for $i\in\{1,2,3\}$. □

Theorem 4.1.4. *If q has finite multiplicative order t, $t\geqslant 3$, then the values in $u_q^+(F_4)$ of the elements from list (4.2) form a set of PBW-generators for $u_q^+(F_4)$ over $\mathbf{k}[G]$. The height h of $[u]\in\{[C],[F],[G],[I],[J],[M],[Q],[S],[T],[V],[W],[X]\}$ equals t. For $[u]\in\{[A],[B],[D],[E],[H],[K],[L],[N],[O],[P],[R],[U]\}$ we have $h=t$ if t is odd and $h=\frac{t}{2}$ if t is even. In all cases $[u]^h=0$ in $u_q^+(F_4)$.*

Proof. This statement is true due to the fact that the hyper-letters of the list (4.2) are hard hyper-letters in $u_q^+(F_4)$. From Theorem 2.3.3 we have that the elements from list (4.2) form a set of PBW-generators for $u_q^+(F_4)$ over $\mathbf{k}[G]$.

Now we prove their heights.

We notice that, if $p(u,u)$ is a h_u-th primitive root of 1 and

$$\underbrace{\big[[u],\big[[u],\cdots\big[[u]}_{h_u-1},\partial_i([u])\big],\cdots\big]\big]=0$$

then from Lemma 2.6.2 we have $\partial_i([u]^{h_u})=0$ in $u_q^+(F_4)$.

For $[u]\in\{[C],[F],[G],[I],[J],[M],[Q],[S],[T],[V],[W]\}$, we have $p(u,u)=q$. As q is a primitive t-th root of 1 then $h_u=t$. From Lemmas 2.6.2 and 4.1.3, we have $\partial_i([u]^t)=0$ in $u_q^+(F_4)$ for $i=1,2,3,4$ and $t\geq 3$. We apply the Milinski-Schneider criterion (Lemma 2.6.3) and we obtain $[u]^t=0$. So $h([u])=t$ for $[u]\in\{[C],[F],[G],[I],[J],[M],[Q],[S],[T],[V],[W],[X]\}$.

In the case $[u]\in\{[A],[B],[D],[E],[H],[K],[L],[N],[O],[P],[R],[U]\}$ we have $p(u,u)=q^2$. Again q is a primitive t-th root of 1 then $h_u=t$ if t is odd and $h_u=\frac{t}{2}$ if t is even. From Lemmas 2.6.2 and 4.1.3, we have $\partial_i([u^{h_u}])=0$ in $u_q^+(F_4)$ for $i=1,2,3,4$ and $t\geq 3$. We notice that, as explained in the proof of the previous Lemma, altought $\big[[H],\partial_1([H])\big]$ is not zero in general, it annuls itself in the specific case $q^4=1$ as all the coefficients have the term $1+q^2$. By Milinski-Schneider criterion, we have $[u]^{h_u}=0$. Then the height of $[u]\in\{[A],[B],[D],[E],[H],[K],[L],[N],[O],[P],[R],[U],[X]\}$ is t or $\frac{t}{2}$. □

4.2 The coproduct formula

From now on we suppose that x_i is an element from the set $\{x_1, x_2, x_3, x_4\}$. Similarly g_i belongs to the set $\{g_1, g_2, g_3, g_4\}$ and for simplicity we denote the group-like element $g_{i_1} g_{i_2} \dots g_{i_n}$ by $g_{i_1 i_2 \dots i_n}$.

In the next lemmas we explicit some formulas that are useful to prove Theorem 4.2.5.

Lemma 4.2.1. *Let u, v be homogeneous elements in $U_q^+(F_4)$. If $[u, w] = 0$ then $[u, [v, w]] = [[u, v], w]$.*

Proof. From $[u, w] = 0$ we have $uw = p_{uw} wu$ and

$$\begin{aligned}
[u, [v, w]] &= u[v, w] - p_{uv} p_{uw} [v, w] u \\
&= uvw - p_{vw} uwv - p_{uv} p_{uw} vwu + p_{uv} p_{uw} p_{vw} wvu \\
&= uvw - p_{uw} p_{vw} wuv - p_{uv} p_{uw} p_{uw}^{-1} vuw + p_{uv} p_{uw} p_{vw} wvu \\
&= (uv - p_{uv} vu) w - p_{uw} p_{vw} w (uv - p_{uv} vu) \\
&= [[u, v], w].
\end{aligned}$$

□

Lemma 4.2.2. *Let $x_{ij} = [x_i, x_j]$ and $g_{ij} = g_i g_j$, with $i, j \in \{1, 2, 3, 4\}$. We have that $\Delta(x_{ij}) = x_{ij} \otimes 1 + g_{ij} \otimes x_{ij} + (1 - p_{ij} p_{ji}) x_i g_j \otimes x_j$.*

Proof. As $\Delta(x_i) = x_i \otimes 1 + g_i \otimes x_i$ and Δ is linear and multiplicative we have

$$\begin{aligned}
\Delta(x_{ij}) =& \Delta(x_i x_j - p_{ij} x_j x_i) = \Delta(x_i)\Delta(x_j) - p_{ij} \Delta(x_j)\Delta(x_i) \\
=& (x_i \otimes 1 + g_i \otimes x_i)(x_j \otimes 1 + g_j \otimes x_j) - p_{ij}(x_j \otimes 1 + g_j \otimes x_j)(x_i \otimes 1 + g_i \otimes x_i) \\
=& x_i x_j \otimes 1 + x_i g_j \otimes x_j + g_i x_j \otimes x_i + g_{ij} \otimes x_i x_j - \\
& - p_{ij} x_j x_i \otimes 1 - p_{ij} x_j g_i \otimes x_i - p_{ij} g_j x_i \otimes x_j - p_{ij} g_{ij} \otimes x_j x_i \\
=& x_{ij} \otimes 1 + g_{ij} \otimes x_{ij} + x_i g_j \otimes x_j - p_{ij} p_{ji} x_i g_j \otimes x_j + p_{ij} x_j g_i \otimes x_i - p_{ij} x_j g_i \otimes x_i \\
=& x_{ij} \otimes 1 + g_{ij} \otimes x_{ij} + (1 - p_{ij} p_{ji}) x_i g_j \otimes x_j.
\end{aligned}$$

□

Lemma 4.2.3. *The coproduct of the element* $[[x_i, x_j], x_k]$ *is given by the formula*

$$\begin{aligned}\Delta([[x_i,x_j],x_k]) =&[[x_i,x_j],x_k]\otimes 1 + g_{ijk}\otimes [[x_i,x_j],x_k]\\ &+(1-p_{ik}p_{ki}p_{jk}p_{kj})x_{ij}g_k\otimes x_k + (1-p_{ij}p_{ji})p_{jk}x_{ik}g_j\otimes x_j+\\ &+(1-p_{ij}p_{ji})(1-p_{ik}p_{ki})x_i g_{jk}\otimes x_j x_k + (1-p_{ij}p_{ji})p_{ik}p_{ki}x_i g_{jk}\otimes x_{jk}.\end{aligned}$$

Proof. Using that Δ is linear and multiplicative and from the previous lemma we have

$$\begin{aligned}\Delta([[x_i,x_j],x_k]) =&\Delta(x_{ij}x_k - p_{ik}p_{jk}x_k x_{ij}) = \Delta(x_{ij})\Delta(x_k) - p_{ik}p_{jk}\Delta(x_k)\Delta(x_{ij})\\ =&(x_{ij}\otimes 1 + g_{ij}\otimes x_{ij} + (1-p_{ij}p_{ji})x_i g_j\otimes x_j)(x_k\otimes 1 + g_k\otimes x_k)-\\ &-p_{ik}p_{jk}(x_k\otimes 1 + g_k\otimes x_k)(x_{ij}\otimes 1 + g_{ij}\otimes x_{ij} + (1-p_{ij}p_{ji})x_i g_j\otimes x_j)\\ =&x_{ij}x_k\otimes 1 + g_{ij}x_k\otimes x_{ij} + (1-p_{ij}p_{ji})x_i g_j x_k\otimes x_j + x_{ij}g_k\otimes x_k+\\ &+g_{ij}g_k\otimes x_{ij}x_k + (1-p_{ij}p_{ji})x_i g_j g_k\otimes x_j x_k - p_{ik}p_{jk}x_k x_{ij}\otimes 1 - p_{ik}p_{jk}x_k g_{ij}\otimes x_{ij}-\\ &-p_{ik}p_{jk}(1-p_{ij}p_{ji})x_k x_i g_j\otimes x_j - p_{ik}p_{jk}g_k x_{ij}\otimes x_k - p_{ik}p_{jk}g_k g_{ij}\otimes x_k x_{ij}-\\ &-p_{ik}p_{jk}(1-p_{ij}p_{ji})g_k x_i g_j\otimes x_k x_j\\ =&(x_{ij}x_k - p_{ik}p_{jk}x_k x_{ij})\otimes 1 + g_{ijk}\otimes(x_{ij}x_k - p_{ik}p_{jk}x_k x_{ij}) + (1-p_{ik}p_{ki}p_{jk}p_{kj})x_{ij}g_k\otimes x_k+\\ &+(1-p_{ij}p_{ji})(p_{jk}x_i x_k - p_{ik}p_{jk}x_k x_i)g_j\otimes x_j + (1-p_{ij}p_{ji})x_i g_{jk}\otimes(x_j x_k - p_{ik}p_{ki}p_{jk}x_k x_j)\\ =&[[x_i,x_j],x_k]\otimes 1 + g_{ijk}\otimes[[x_i,x_j],x_k]+\\ &+(1-p_{ik}p_{ki}p_{jk}p_{kj})x_{ij}g_k\otimes x_k + (1-p_{ij}p_{ji})p_{jk}x_{ik}g_j\otimes x_j+\\ &+(1-p_{ij}p_{ji})(1-p_{ik}p_{ki})x_i g_{jk}\otimes x_j x_k + (1-p_{ij}p_{ji})p_{ik}p_{ki}x_i g_{jk}\otimes x_{jk}.\end{aligned}$$

□

Lemma 4.2.4. *The coproduct of the element* $[[x_i, [x_j, x_k]], x_l]$ *is given by the formula*

$$\begin{aligned}\Delta([[x_i,[x_j,x_k]],x_l]) =& [[x_i,[x_j,x_k]],x_l]\otimes 1 + g_{ijkl}\otimes[[x_i,[x_j,x_k]],x_l]+\\ &+ (1-p_{il}p_{li}p_{jl}p_{lj}p_{kl}p_{lk})[x_i,[x_j,x_k]]g_l\otimes x_l + p_{jl}p_{kl}(1-p_{ij}p_{ji}p_{ik}p_{ki})x_{il}g_{jk}\otimes x_{jk}+\\ &+ p_{ij}p_{il}p_{kl}(1-p_{jk}p_{kj})x_{jl}g_{ik}\otimes x_{ik} + (1-p_{ij}p_{ji}p_{ik}p_{ki})x_i g_{jkl}\otimes[[x_j,x_k],x_l]+\\ &+ p_{kl}(1-p_{jk}p_{kj})([[x_i,x_j],x_l] + p_{ij}(1-p_{ik}p_{ki})x_j x_{il} + p_{ij}p_{il}(1-p_{ik}p_{ki})x_{jl}x_i)g_k\otimes x_k+\\ &+ p_{jl}p_{kl}(1-p_{il}p_{li})(1-p_{ij}p_{ji}p_{ik}p_{ki})x_i g_{jkl}\otimes x_l x_{jk} + p_{ij}(1-p_{jk}p_{kj})x_j g_{ikl}\otimes[[x_i,x_k],x_l]\\ &+ p_{ij}p_{il}p_{kl}(1-p_{jk}p_{kj})(1-p_{jl}p_{lj})x_j g_{ikl}\otimes x_l x_{ik}+\\ &+ (1-p_{jk}p_{kj})(x_{ij} + p_{ij}(1-p_{ik}p_{ki})x_j x_i)g_{kl}\otimes x_{kl}+\\ &+ p_{kl}(1-p_{il}p_{li}p_{jl}p_{lj})(1-p_{jk}p_{kj})(x_{ij} + p_{ij}(1-p_{ik}p_{ki})x_j x_i)g_{kl}\otimes x_l x_k.\end{aligned}$$

Proof. First we notice that

$$\begin{aligned}
\Delta([x_i,[x_j,x_k]]) &= \Delta(x_i)\Delta(x_{jk}) - p_{ij}p_{ik}\Delta(x_{jk})\Delta(x_i)\\
&= (x_i\otimes 1 + g_i\otimes x_i)(x_{jk}\otimes 1 + g_{jk}\otimes x_{jk} + (1-p_{jk}p_{kj})x_jg_k\otimes x_k)\\
&- p_{ij}p_{ik}(x_{jk}\otimes 1 + g_{jk}\otimes x_{jk} + (1-p_{jk}p_{kj})x_jg_k\otimes x_k)(x_i\otimes 1 + g_i\otimes x_i)\\
&= x_ix_{jk}\otimes 1 + x_ig_{jk}\otimes x_{jk} + (1-p_{jk}p_{kj})x_ix_jg_k\otimes x_k + g_ix_{jk}\otimes x_i +\\
&+ g_{ijk}\otimes x_ix_{jk} + (1-p_{jk}p_{kj})g_ix_jg_k\otimes x_ix_k - p_{ij}p_{ik}x_{jk}x_i\otimes 1 - p_{ij}p_{ik}g_{jk}x_i\otimes x_{jk} -\\
&- p_{ij}p_{ik}(1-p_{jk}p_{kj})x_jg_kx_i\otimes x_k - p_{ij}p_{ik}x_{jk}g_i\otimes x_i - p_{ij}p_{ik}g_{ijk}\otimes x_{jk}x_i -\\
&- p_{ij}p_{ik}(1-p_{jk}p_{kj})x_jg_{ik}\otimes x_kx_i\\
&= [x_i,[x_j,x_k]]\otimes 1 + g_{ijk}\otimes [x_i,[x_j,x_k]] + (1-p_{ij}p_{ji}p_{ik}p_{ki})x_ig_{jk}\otimes x_{jk} +\\
&+ p_{ij}(1-p_{jk}p_{kj})x_jg_{ik}\otimes x_{ik} + (1-p_{jk}p_{kj})(x_{ij} + p_{ij}(1-p_{ik}p_{ki})x_jx_i)g_k\otimes x_k.
\end{aligned}$$

Now we have

$$
\begin{aligned}
\Delta([[x_i,[x_j,x_k]],x_l]) &= \Delta([x_i,[x_j,x_k]])\Delta(x_l) - p_{il}p_{jl}p_{kl}\Delta(x_l)\Delta([x_i,[x_j,x_k]]) \\
&= ([x_i,[x_j,x_k]]\otimes 1 + g_{ijk}\otimes [x_i,[x_j,x_k]] + (1-p_{ij}p_{ji}p_{ik}p_{ki})x_i g_{jk}\otimes x_{jk} + \\
&+ p_{ij}(1-p_{jk}p_{kj})x_j g_{ik}\otimes x_{ik} + (1-p_{jk}p_{kj})(x_{ij} + p_{ij}(1-p_{ik}p_{ki})x_j x_i)g_k\otimes x_k) \\
&(x_l\otimes 1 + g_l\otimes x_l) - p_{il}p_{jl}p_{kl}(x_l\otimes 1 + g_l\otimes x_l) \\
&([x_i,[x_j,x_k]]\otimes 1 + g_{ijk}\otimes [x_i,[x_j,x_k]] + (1-p_{ij}p_{ji}p_{ik}p_{ki})x_i g_{jk}\otimes x_{jk} + \\
&+ p_{ij}(1-p_{jk}p_{kj})x_j g_{ik}\otimes x_{ik} + (1-p_{jk}p_{kj})(x_{ij} + p_{ij}(1-p_{ik}p_{ki})x_j x_i)g_k\otimes x_k) \\
&= [x_i,[x_j,x_k]]x_l\otimes 1 + g_{ijk}x_l\otimes [x_i,[x_j,x_k]] + (1-p_{ij}p_{ji}p_{ik}p_{ki})x_i x_l g_{jk}\otimes x_{jk} + \\
&+ p_{ij}(1-p_{jk}p_{kj})x_j x_l g_{ik}\otimes x_{ik} + (1-p_{jk}p_{kj})(x_{ij}x_l + p_{ij}(1-p_{ik}p_{ki})x_j x_i x_l)g_k\otimes x_k + \\
&+ [x_i,[x_j,x_k]]g_l\otimes x_l + g_{ijkl}\otimes [x_i,[x_j,x_k]]x_l + (1-p_{ij}p_{ji}p_{ik}p_{ki})x_i g_{jkl}\otimes x_{jk}x_l + \\
&+ p_{ij}(1-p_{jk}p_{kj})x_j g_{ikl}\otimes x_{ik}x_l + (1-p_{jk}p_{kj})(x_{ij} + p_{ij}(1-p_{ik}p_{ki})x_j x_i)g_{kl}\otimes x_k x_l - \\
&- p_{il}p_{jl}p_{kl}(x_l[x_i,[x_j,x_k]]\otimes 1 + x_l g_{ijk}\otimes [x_i,[x_j,x_k]] + (1-p_{ij}p_{ji}p_{ik}p_{ki})x_l x_i g_{jk}\otimes x_{jk} + \\
&+ p_{ij}(1-p_{jk}p_{kj})x_l x_j g_{ik}\otimes x_{ik} + (1-p_{jk}p_{kj})(x_l x_{ij} + p_{ij}(1-p_{ik}p_{ki})x_l x_j x_i)g_k\otimes x_k + \\
&+ g_l[x_i,[x_j,x_k]]\otimes x_l + g_{ijkl}\otimes x_l[x_i,[x_j,x_k]] + (1-p_{ij}p_{ji}p_{ik}p_{ki})g_l x_i g_{jk}\otimes x_l x_{jk} + \\
&+ p_{ij}(1-p_{jk}p_{kj})g_l x_j g_{ik}\otimes x_l x_{ik} + (1-p_{jk}p_{kj})g_l(x_{ij} + p_{ij}(1-p_{ik}p_{ki})x_j x_i)g_k\otimes x_l x_k) \\
&= [[x_i,[x_j,x_k]],x_l]\otimes 1 + g_{ijkl}\otimes [[x_i,[x_j,x_k]],x_l] + (1-p_{il}p_{li}p_{jl}p_{lj}p_{kl}p_{lk})[x_i,[x_j,x_k]]g_l \\
&+ (1-p_{ij}p_{ji}p_{ik}p_{ki})(p_{jl}p_{kl}x_i x_l - p_{il}p_{jl}p_{kl}x_l x_i)g_{jk}\otimes x_{jk} + \\
&+ p_{ij}(1-p_{jk}p_{kj})(p_{il}p_{kl}x_j x_l - p_{il}p_{jl}p_{kl}x_l x_j)g_{ik}\otimes x_{ik} + \\
&+ (1-p_{jk}p_{kj})(p_{kl}x_{ij}x_l - p_{il}p_{jl}p_{kl}x_l x_{ij})g_k\otimes x_k + \\
&+ p_{ij}(1-p_{jk}p_{kj})(1-p_{ik}p_{ki})(p_{kl}x_j x_i x_l - p_{il}p_{jl}p_{kl}x_l x_j x_i)g_k\otimes x_k + \\
&+ (1-p_{ij}p_{ji}p_{ik}p_{ki})x_i g_{jkl}\otimes (x_{jk}x_l - p_{il}p_{li}p_{jl}p_{kl}x_l x_{jk}) + \\
&+ p_{ij}(1-p_{jk}p_{kj})x_j g_{ikl}\otimes (x_{ik}x_l - p_{il}p_{jl}p_{kl}p_{lj}x_l x_{ik}) + \\
&+ (1-p_{jk}p_{kj})x_{ij}g_{kl}\otimes (x_k x_l - p_{il}p_{jl}p_{kl}p_{li}p_{lj}x_l x_k) + \\
&+ p_{ij}(1-p_{jk}p_{kj})(1-p_{ik}p_{ki})x_j x_i g_{kl}\otimes (x_k x_l - p_{il}p_{jl}p_{kl}p_{li}p_{lj}x_l x_k).
\end{aligned}
$$

As $x_{uv}=[x_u,x_v]=x_ux_v-p_{uv}x_vx_u$, for all u,v, we have

$$\begin{aligned}\Delta([[x_i,[x_j,x_k]],x_l]) = {} & [[x_i,[x_j,x_k]],x_l]\otimes 1+g_{ijkl}\otimes[[x_i,[x_j,x_k]],x_l]+\\
&+(1-p_{il}p_{li}p_{jl}p_{lj}p_{kl}p_{lk})[x_i,[x_j,x_k]]g_l\otimes x_l+p_{jl}p_{kl}(1-p_{ij}p_{ji}p_{ik}p_{ki})x_{il}g_{jk}\otimes x_{jk}+\\
&+p_{ij}p_{il}p_{kl}(1-p_{jk}p_{kj})x_{jl}g_{ik}\otimes x_{ik}+(1-p_{ij}p_{ji}p_{ik}p_{ki})x_ig_{jkl}\otimes[[x_j,x_k],x_l]+\\
&+p_{kl}(1-p_{jk}p_{kj})([[x_i,x_j],x_l]+p_{ij}(1-p_{ik}p_{ki})x_jx_{il}+p_{ij}p_{il}(1-p_{ik}p_{ki})x_{jl}x_i)g_k\otimes x_k+\\
&+p_{jl}p_{kl}(1-p_{il}p_{li})(1-p_{ij}p_{ji}p_{ik}p_{ki})x_ig_{jkl}\otimes x_lx_{jk}+p_{ij}(1-p_{jk}p_{kj})x_jg_{ikl}\otimes[[x_i,x_k],x_l]\\
&+p_{ij}p_{il}p_{kl}(1-p_{jk}p_{kj})(1-p_{jl}p_{lj})x_jg_{ikl}\otimes x_lx_{ik}+\\
&+(1-p_{jk}p_{kj})(x_{ij}+p_{ij}(1-p_{ik}p_{ki})x_jx_i)g_{kl}\otimes x_{kl}+\\
&+p_{kl}(1-p_{il}p_{li}p_{jl}p_{lj})(1-p_{jk}p_{kj})(x_{ij}+p_{ij}(1-p_{ik}p_{ki})x_jx_i)g_{kl}\otimes x_lx_k.\end{aligned}$$

□

Now, using the previous lemmas, we are able to present the coproducts of the PBW-generators, which we are going to use to obtain the combinatorial rank of the considered quantum algebra.

Theorem 4.2.5. *The explicit coproduct formulas for the PBW-generators of list* (4.2) *are:*

- $\Delta([A])=\Delta(x_1)=x_1\otimes 1+g_1\otimes x_1$
- $\Delta([B])=[B]\otimes 1+g_{12}\otimes[B]+\beta_2x_1g_2\otimes x_2$
- $\Delta([C])=[C]\otimes 1+g_{123}\otimes[C]+\beta_2[B]g_3\otimes x_3+\beta_2x_1g_{23}\otimes[Q]$
- $\Delta([D])=[D]\otimes 1+g_{1233}\otimes[D]+\beta_2x_1g_{233}\otimes[R]+\beta_1\beta_2[B]g_{33}\otimes x_3^2+\beta_2p_{33}[C]g_3\otimes x_3$
- $\Delta([E])=[E]\otimes 1+g_{12233}\otimes[E]+\beta_2[D]g_2\otimes x_2+\beta_2p_{32}^2q[B]g_{233}\otimes[R]+\beta_2^2x_1g_{2233}\otimes [R]x_2-\beta_1\beta_2p_{32}x_1g_{2233}\otimes[Q]^2+\beta_1\beta_2^2[B]g_{233}\otimes x_3^2x_2-\beta_1\beta_2p_{32}(1+q)[B]g_{233}\otimes x_3[Q]+\beta_2^2q[C]g_{23}\otimes x_3x_2-\beta_2p_{32}q[C]g_{23}\otimes[Q]$
- $\Delta([F])=[F]\otimes 1+g_{1234}\otimes[F]+\beta_2x_1g_{234}\otimes[S]+\beta_2[B]g_{34}\otimes[W]+\beta_1[C]g_4\otimes x_4$
- $\Delta([G])=[G]\otimes 1+g_{12334}\otimes[G]+\beta_2x_1g_{2334}\otimes[T]+\beta_1[F]g_3\otimes x_3+\beta_1^2[C]g_{34}\otimes x_4x_3+\beta_1p_{43}q[C]g_{34}\otimes[W]+\beta_1p_{43}[D]g_4\otimes x_4+\beta_1\beta_2[B]g_{334}\otimes[W]x_3$
- $\Delta([H])=[H]\otimes 1+g_{11222333344}\otimes[H]+\beta_1\beta_2[G]^2g_2\otimes x_2+\beta_1\beta_2p_{42}p_{43}q[G][E]g_4\otimes x_4-\beta_1\beta_2p_{32}q[G][F]g_{23}\otimes[Q]+p_{32}^2p_{42}q^2(\beta_2^2[G][B]+\beta_1^2p_{31}p_{32}(2q+\beta_1)[F][C])g_{2334}\otimes[T]+\beta_1\beta_2p_{31}p_{32}^2p_{42}p_{43}q^3[F][E]g_{34}\otimes[W]-\beta_1p_{32}p_{42}p_{43}q^2(\beta_2[G][C]+\beta_1^2p_{31}p_{32}[F][D])g_{234}\otimes$

$[S]+\beta_1^3p_{12}p_{32}^2p_{41}p_{42}^3p_{43}^4q^4[E][D]g_{44}\otimes x_4^2+\beta_1\beta_2^2q[G][F]g_{23}\otimes x_3x_2+\beta_1\beta_2^2p_{43}q^2([G][C]+\beta_1p_{31}p_{32}q[F][D])g_{234}\otimes[W]x_2+\beta_1\beta_2^2p_{43}q[G][D]g_{24}\otimes x_4x_2-$

$-\beta_1^2\beta_2^2p_{31}^2p_{32}^4p_{41}p_{42}^2q^3[B]x_1g_{22333344}\otimes[T]^2+\beta_2^3q[G]x_1g_{22334}\otimes[T]x_2-\beta_1^2\beta_2p_{32}q([G][C]+p_{31}p_{32}q[F][D])g_{234}\otimes x_4[Q]-\beta_1\beta_2^2p_{32}q[G]x_1g_{22334}\otimes[S][Q]-\beta_1\beta_2p_{32}p_{42}p_{43}q^2(\beta_2[G][B]+\beta_1^2p_{31}p_{32}q[F][C])g_{2334}\otimes x_3[S]+\beta_2q[G]g_{122334}\otimes[I]-\beta_1\beta_2p_{32}q(\beta_2[G][B]+\beta_1p_{31}p_{32}q(1+q+\beta_1)[F][C])g_{2334}\otimes[W][Q]-\beta_2p_{31}p_{32}^2p_{34}q^3[F]g_{1223334}\otimes[J]+$

$+\beta_1p_{12}p_{32}^2p_{41}p_{42}^3p_{43}^3q^4[E]g_{123344}\otimes[K]+\beta_1p_{32}^2p_{41}p_{42}^3p_{43}^3q^2(\beta_1p_{31}p_{32}[C]^2+\beta_2^2q^2[D][B]+\beta_2p_{12}q^2[E]x_1)g_{23344}\otimes[U]+\beta_1\beta_2p_{32}^2p_{41}p_{42}^2p_{43}q^2(\beta_2[D][B]-\beta_1^2p_{31}p_{32}q[C]^2)g_{23344}\otimes x_4[T]+\beta_1\beta_2^2p_{31}p_{32}^3p_{34}q^3[F]x_1g_{223334}\otimes[S][R]-\beta_2^3p_{31}p_{32}^2p_{34}q^4[F]x_1g_{223334}\otimes[T][Q]-\beta_1\beta_2^2p_{31}p_{32}^2p_{41}p_{42}^2p_{43}^2q^2[C]x_1g_{2233344}\otimes[T][S]+\beta_1^2p_{31}p_{32}^3p_{34}q^2[F]^2g_{233}\otimes[R]-$

$-\beta_1^2\beta_2p_{31}p_{32}^2p_{34}q^2[F]^2g_{233}\otimes x_3[Q]+\beta_1^3\beta_2p_{31}p_{32}p_{34}q[F]^2g_{233}\otimes x_3^2x_2+$

$+\beta_1^2\beta_2p_{12}p_{32}^2p_{41}p_{42}^3p_{43}^3q^5[E][C]g_{344}\otimes x_4[W]+\beta_1p_{41}p_{42}p_{43}^3q(\beta_2q-1)[D]g_{1223344}\otimes[L]+\beta_1^2\beta_2^2q([G][C]+p_{31}p_{32}q[F][D])g_{234}\otimes x_4x_3x_2+\beta_1^3\beta_2p_{41}p_{42}^3p_{43}^2[D]^2g_{244}\otimes x_4^2x_2+\beta_1\beta_2q(\beta_2^2[G][B]+2\beta_1^2p_{31}p_{32}q[F][C])g_{2334}\otimes[W]x_3x_2+\beta_1\beta_2q[F]g_{1223334}\otimes x_3[I]+\beta_1\beta_2p_{41}p_{42}p_{43}^3q^2(\beta_2q-1)[C]g_{12233344}\otimes x_3[L]+\beta_1\beta_2^2p_{12}p_{32}^2p_{41}p_{42}^3p_{43}^3q^5[C]g_{12233344}\otimes x_3x_2[K]+\beta_1^2p_{43}q[C]g_{12233344}\otimes[W][I]+\beta_1\beta_2p_{12}p_{32}^2p_{41}p_{42}^3p_{43}^3q^4[D]g_{1223344}\otimes x_2[K]+\beta_1\beta_2p_{43}q[D]g_{1223344}\otimes x_4[I]+\beta_1p_{31}p_{32}^2p_{41}p_{42}p_{43}q^2(1-\beta_2q)[C]g_{12233344}\otimes[M]-\beta_1\beta_2p_{12}p_{32}^3p_{41}p_{42}^3p_{43}^3q^5[C]g_{12233344}\otimes[Q][K]-\beta_1^2p_{31}p_{32}^2p_{34}q^2[C]g_{12233344}\otimes x_4[J]+\beta_1^2\beta_2q[C]g_{12233344}\otimes x_4x_3[I]+\beta_1^2\beta_2p_{41}p_{42}p_{43}^3q(\beta_2q-1)[B]g_{122333344}\otimes x_3^2[L]+\beta_1^2\beta_2^2p_{12}p_{32}^2p_{41}p_{42}^3p_{43}^3q^4[B]g_{122333344}\otimes x_3^2x_2[K]+\beta_1\beta_2^2q[B]g_{122333344}\otimes[W]x_3[I]+\beta_1\beta_2p_{31}p_{32}^2p_{41}p_{42}p_{43}q^2(1-\beta_2q)[B]g_{122333344}\otimes x_3[M]-$

$-\beta_1\beta_2^2p_{12}p_{32}^3p_{41}p_{42}^3p_{43}^3q^5[B]g_{122333344}\otimes x_3[Q][K]-\beta_2^2p_{31}p_{32}^2p_{34}q^3[B]g_{122333344}\otimes[W][J]+\beta_1\beta_2^3p_{31}q[F]x_1g_{223334}\otimes x_3[T]x_2-\beta_1^2\beta_2^2p_{31}p_{32}q[F]x_1g_{223334}\otimes x_3[S][Q]+\beta_1\beta_2^3p_{31}p_{41}p_{43}q^2[C]x_1g_{2233344}\otimes[W][T]x_2+\beta_1^2\beta_2^3p_{31}p_{41}p_{42}p_{43}^3q^3[C]x_1g_{2233344}\otimes x_3[U]x_2-\beta_1^2\beta_2^2p_{31}p_{32}p_{41}p_{43}q^2[C]x_1g_{2233344}\otimes[W][S][Q]+\beta_1\beta_2^3p_{41}p_{43}q[D]x_1g_{223344}\otimes x_4[T]x_2+\beta_1^2\beta_2^2p_{41}p_{42}p_{43}^3q^2[D]x_1g_{223344}\otimes[U]x_2-\beta_1^2\beta_2^2p_{32}p_{41}p_{43}q[D]x_1g_{223344}\otimes x_4[S][Q]-\beta_1^2\beta_2^2p_{31}p_{32}^2p_{34}p_{41}q^3[C]x_1g_{2233344}\otimes x_4[T][Q]-\beta_1^2\beta_2^2p_{31}p_{32}^2p_{41}p_{42}p_{43}q^3[C]x_1g_{2233344}\otimes[U][Q]+\beta_1^2\beta_2^3p_{31}p_{41}q[C]x_1g_{2233344}\otimes x_4x_3[T]x_2-\beta_1^3\beta_2^2p_{31}p_{32}p_{41}q[C]x_1g_{2233344}\otimes x_4x_3[S][Q]+\beta_1\beta_2^4p_{31}^2p_{41}q[B]x_1g_{22333344}\otimes[W]x_3[T]x_2+\beta_1^3\beta_2^3p_{31}^2p_{41}p_{42}p_{43}^3q^2[B]x_1g_{22333344}\otimes x_3^2[U]x_2-\beta_1^2\beta_2^3p_{31}^2p_{41}q[B]x_1g_{22333344}\otimes[W]x_3[S][Q]-\beta_1\beta_2^3p_{31}^2p_{32}^2p_{41}p_{42}^2p_{43}^2q^2[B]x_1g_{22333344}\otimes x_3[T][S]-\beta_1^2\beta_2^3p_{31}^2p_{32}^2p_{41}p_{42}p_{43}q^3[B]x_1g_{22333344}\otimes x_3[U][Q]-\beta_1\beta_2^3p_{31}^2p_{32}^2p_{34}p_{41}q^3[B]x_1g_{22333344}\otimes[W][T][Q]+\beta_1\beta_2^3p_{31}^2p_{32}^3p_{34}p_{41}q^3[B]x_1g_{22333344}\otimes[W][S][R]+\beta_1\beta_2^3p_{21}p_{31}^2p_{41}x_1^2g_{222333344}\otimes[T]^2x_2-\beta_1\beta_2^3p_{21}p_{31}^2p_{32}p_{41}qx_1^2g_{222333344}\otimes[T][S][Q]+\beta_1^2\beta_2^2p_{21}p_{31}^2p_{32}^3p_{34}p_{41}q^2x_1^2g_{222333344}\otimes[S]^2[R]+\beta_1^2\beta_2p_{31}p_{32}^3p_{34}q^3[F][C]g_{2334}\otimes x_4[R]+\beta_1^3\beta_2^2p_{31}p_{32}p_{34}q^2[F][C]g_{2334}\otimes x_4x_3^2x_2-\beta_1^2\beta_2^2p_{31}p_{32}^2p_{34}q^3[F][C]g_{2334}\otimes x_4x_3[Q]+\beta_1^2\beta_2^3p_{31}p_{32}p_{34}q^2[F][B]g_{23334}\otimes[W]x_3^2x_2-\beta_1\beta_2^3p_{31}p_{32}^2p_{34}q^3[F][B]g_{23334}\otimes[W]x_3[Q]+\beta_1\beta_2^2p_{31}p_{32}^3p_{34}q^3[F][B]g_{23334}\otimes[W][R]+$

$\beta_1^2\beta_2^2p_{41}p_{42}p_{43}^3q^3[D][C]g_{2344}\otimes x_4[W]x_2+\beta_1\beta_2p_{32}p_{41}p_{42}^2p_{43}^3q^2(\beta_1p_{31}p_{32}[C]^2+\beta_2[D][B])g_{23344}\otimes$
$[W][S]+\beta_1^3\beta_2p_{41}p_{42}p_{43}^3q^2(p_{31}p_{32}q(1+\beta_1\beta_2q)[C]^2+\beta_2[D][B])g_{23344}\otimes[W]^2x_2+$
$\beta_1p_{32}p_{41}p_{42}p_{43}(\beta_1^2p_{31}p_{32}(\beta_1+\beta_2+\beta_3)[C]^2-\beta_2^3q[D][B])g_{23344}\otimes x_4[W][Q]+$
$+\beta_1^2\beta_2p_{32}p_{41}p_{42}^2p_{43}^2q^2(\beta_1p_{31}p_{32}[C]^2-\beta_2[D][B])g_{23344}\otimes x_4x_3[S]+\beta_1^2\beta_2p_{41}p_{42}p_{43}q(\beta_2^2[D][B]+$
$\beta_1^2p_{31}p_{32}(2q+\beta_1+\beta_2q^2)[C]^2)g_{23344}\otimes x_4[W]x_3x_2+\beta_1^2\beta_2^3p_{31}p_{32}p_{41}p_{42}p_{43}q^3[C][B]g_{233344}\otimes$
$[W]^2x_3x_2-\beta_1^2\beta_2^2p_{31}p_{32}^2p_{41}p_{42}p_{43}q^3[C][B]g_{233344}\otimes[W]^2[Q]+\beta_1^2\beta_2^2p_{31}p_{32}p_{41}p_{42}^2p_{43}^3q^2[C]x_1g_{2233344}\otimes$
$x_3[S]^2+\beta_1^3\beta_2^2p_{41}p_{42}p_{43}^2q[D][C]g_{2344}\otimes x_4^2x_3x_2+\beta_1^2\beta_2p_{32}p_{41}p_{42}^2p_{43}^3q[D]x_1g_{223344}\otimes$
$[S]^2-\beta_1^3\beta_2p_{32}p_{41}p_{42}p_{43}^2q[D][C]g_{2344}\otimes x_4^2[Q]+\beta_1^2\beta_2p_{12}p_{32}^2p_{41}p_{42}^3p_{43}^3q^4[B][E]g_{3344}\otimes$
$[W]^2-\beta_1^2\beta_2^3p_{31}p_{32}^2p_{34}p_{41}p_{42}q^3[C][B]g_{233344}\otimes x_4[W]x_3[Q]+\beta_1^2\beta_2^2p_{31}p_{32}^3p_{34}p_{41}p_{42}q^3[C][B]g_{233344}\otimes$
$x_4[W][R]+\beta_1^3\beta_2^3p_{31}p_{32}p_{34}p_{41}p_{42}q^2[C][B]g_{233344}\otimes x_4[W]x_3^2x_2++\beta_1^3\beta_2^3p_{31}^2p_{32}^2p_{34}p_{41}p_{42}q[B]^2g_{2333344}\otimes$
$[W]^2x_3^2x_2+\beta_1^2\beta_2^2p_{31}^2p_{32}^4p_{34}p_{41}p_{42}q^2[B]^2g_{2333344}\otimes[W]^2[R]-\beta_1^2p_{31}p_{32}^2p_{34}q^3[C]g_{12233344}\otimes$
$x_3[J]+\beta_1^4p_{31}p_{32}^3p_{41}p_{42}q[C]^2g_{23344}\otimes x_4^2[R]+\beta_1^5\beta_2p_{31}p_{32}p_{41}p_{42}[C]^2g_{13344}\otimes x_4^2x_3^2x_2-$
$\beta_1^4\beta_2p_{31}p_{32}^2p_{41}p_{42}q[C]^2g_{23344}\otimes x_4^2x_3[Q]+\beta_1^3\beta_2p_{31}p_{32}^3p_{34}p_{41}q^3[C]x_1g_{2233344}\otimes x_4[S][R]+$
$\beta_1^3\beta_2^2p_{31}^2p_{32}p_{41}p_{42}^2p_{43}^3q[B]x_1g_{22333344}\otimes x_3^2[S]^2+\beta_1^2\beta_2^2p_{31}^2p_{32}^4p_{34}p_{41}p_{42}q^4[B]x_1g_{22333344}\otimes$
$[U][R]+\beta_2p_{31}^2p_{32}^4p_{34}p_{41}p_{42}q^3(\beta_1q^2-1)(1+q)^{-1}[B]g_{122333344}\otimes[N]+$
$+\beta_1\beta_2p_{12}p_{32}^4p_{41}p_{42}^3p_{43}^3q^5[B]g_{122333344}\otimes[R][K]+\beta_2p_{21}p_{24}p_{31}^2p_{34}p_{41}(1+q)^{-1}(1-$
$q^2+q^{-1})x_1g_{1222333344}\otimes[O]+\beta_1\beta_2p_{31}^2p_{32}^4p_{34}p_{41}p_{42}q^4(1-q^{-1}-q^{-2})x_1g_{1222333344}\otimes$
$x_2[N]+\beta_1\beta_2p_{41}p_{42}p_{43}^3(-1-q+q^2)x_1g_{1222333344}\otimes[R][L]+\beta_1\beta_2^2p_{12}p_{32}^2p_{41}p_{42}^3p_{43}^3q^4x_1g_{1222333344}\otimes$
$[R]x_2[K]+\beta_2^2qx_1g_{1222333344}\otimes[T][I]+\beta_1\beta_2p_{31}p_{32}^2p_{41}p_{42}p_{43}q^2(1-q+q^{-1})x_1g_{1222333344}\otimes$
$[Q][M]-\beta_1^2\beta_2p_{12}p_{32}^3p_{41}p_{42}^3p_{43}^3q^4x_1g_{1222333344}\otimes[Q]^2[K]-$
$-\beta_2^2p_{31}p_{32}^2p_{34}q^3x_1g_{1222333344}\otimes[S][J]$

- $\Delta([I])=[I]\otimes 1+g_{122334}\otimes[I]+\beta_2[G]g_2\otimes x_2+\beta_2p_{32}^2p_{42}q[B]g_{2334}\otimes[T]+\beta_1p_{42}p_{43}[E]g_4\otimes x_4-\beta_1p_{32}[F]g_{23}\otimes[Q]+\beta_1\beta_2[F]g_{23}\otimes x_3x_2-\beta_1p_{32}p_{42}p_{43}q[C]g_{234}\otimes[S]+\beta_1\beta_2p_{43}q[C]g_{234}\otimes[W]x_2+\beta_1\beta_2p_{43}[D]g_{24}\otimes x_4x_2-\beta_1^2p_{32}[C]g_{234}\otimes x_4[Q]+\beta_1^2\beta_2[C]g_{234}\otimes x_4x_3x_2+\beta_1\beta_2^2[B]g_{2334}\otimes[W]x_3x_2-\beta_1\beta_2p_{32}p_{42}p_{43}q[B]g_{2334}\otimes x_3[S]-\beta_1\beta_2p_{32}[B]g_{2334}\otimes[W][Q]+\beta_2^2x_1g_{22334}\otimes[T]x_2-\beta_1\beta_2p_{32}x_1g_{22334}\otimes[S][Q]$

- $\Delta([J])=[J]\otimes 1+g_{1223334}\otimes[J]+\beta_1[I]g_3\otimes x_3+\beta_1p_{32}p_{42}p_{43}[C]g_{2334}\otimes[T]+\beta_1\beta_2x_1g_{223334}\otimes[T][Q]-\beta_1\beta_2p_{32}x_1g_{223334}\otimes[S][R]+\beta_1^2[F]g_{233}\otimes x_3[Q]-\beta_1p_{32}[F]g_{233}\otimes[R]+\beta_1^2p_{43}q[C]g_{2334}\otimes[W][Q]-\beta_1^2p_{42}p_{43}^2(1+q)[C]g_{2334}\otimes x_3[S]+\beta_1^2p_{43}[D]g_{234}\otimes x_4[Q]-\beta_1p_{42}p_{43}^2[D]g_{234}\otimes[S]+\beta_1^3[C]g_{2334}\otimes x_4x_3[Q]-\beta_1^2p_{32}[C]g_{2334}\otimes x_4[R]+\beta_1^2p_{42}p_{43}[E]g_{34}\otimes x_4x_3+\beta_1^2\beta_2[B]g_{23334}\otimes[W]x_3[Q]+\beta_1\beta_2p_{32}p_{42}p_{43}[B]g_{23334}\otimes x_3[T]-\beta_1^2\beta_2p_{42}p_{43}^2[B]g_{23334}\otimes x_3^2[S]-\beta_1\beta_2p_{32}[B]g_{23334}\otimes[W][R]$

- $\Delta([K])=[K]\otimes 1+g_{123434}\otimes[K]+\beta_2x_1g_{23344}\otimes[U]+\beta_2q[F]g_{34}\otimes[W]+\beta_1\beta_2[B]g_{3344}\otimes[W]^2+\beta_1^2p_{43}[D]g_{44}\otimes x_4^2+\beta_1(1+q)[G]g_4\otimes x_4+\beta_1\beta_2q[C]g_{344}\otimes x_4[W]$

- $\Delta([L]) = [L]\otimes 1 + g_{1223344}\otimes [L] + \beta_2[K]g_2\otimes x_2 + \beta_2 p_{32}^2 p_{42}^2 q[B]g_{23344}\otimes [U] + \beta_1^2 p_{42}^2 p_{43}[E]g_{44}\otimes x_4^2 + \beta_2^2 q[G]g_{24}\otimes x_4x_2 + \beta_2^2 q[F]g_{234}\otimes [W]x_2 - \beta_2 p_{32}p_{42}q[F]g_{234}\otimes [S] + \beta_1^2\beta_2 p_{43}[D]g_{244}\otimes x_4^2x_2 + \beta_1\beta_2^2 q[C]g_{2344}\otimes x_4[W]x_2 - \beta_1\beta_2 p_{32}p_{42}q[C]g_{2344}\otimes x_4[S] + \beta_2 p_{42}q[I]g_4\otimes x_4 - \beta_2^2 p_{32}p_{42}q[B]g_{223344}\otimes [W][S] + \beta_1\beta_2^2[B]g_{23344}\otimes [W]^2x_2 + \beta_2^2 x_1 g_{223344}\otimes [U]x_2 - \beta_1\beta_2 p_{32}p_{42}x_1 g_{223344}\otimes [S]^2$

- $\Delta([M]) = [M]\otimes 1 + g_{12233344}\otimes [M] + \beta_2[K]g_{23}\otimes [Q] + \beta_2[L]g_3\otimes x_3 + \beta_2 p_{32}p_{42}^2 p_{43}^2 q[C]g_{23344}\otimes [U] - \beta_2 p_{42}p_{43}[I]g_{34}\otimes [W] + \beta_2 p_{42}p_{43}q[J]g_4\otimes x_4 + \beta_2^2 x_1 g_{2233344}\otimes [U][Q] - \beta_2^2 p_{42}p_{43}x_1 g_{2233344}\otimes [T][S] + \beta_2^2 p_{32}p_{42}^2 p_{43}^2 q[B]g_{233344}\otimes x_3[U] + \beta_2^2 q[F]g_{2334}\otimes [W][Q] - \beta_2 p_{32}p_{42}q[F]g_{2334}\otimes [T] - \beta_1\beta_2 p_{42}p_{43}[F]g_{2334}\otimes x_3[S] + \beta_1\beta_2^2[B]g_{233344}\otimes [W]^2[Q] - \beta_2^2 p_{32}p_{42}q[B]g_{233344}\otimes [W][T] - \beta_1\beta_2^2 p_{42}p_{43}[B]g_{233344}\otimes [W]x_3[S] + \beta_1^2\beta_2 p_{43}[D]g_{2344}\otimes x_4^2[Q] - \beta_1\beta_2 p_{42}p_{43}^2[D]g_{2344}\otimes x_4[S] + \beta_2^2 q[G]g_{234}\otimes x_4[Q] - \beta_2 p_{42}p_{43}[G]g_{234}\otimes [S] + \beta_1^2\beta_2 p_{42}^2 p_{43}[E]g_{344}\otimes x_4^2x_3 - \beta_1\beta_2 p_{42}^2 p_{43}^2[E]g_{344}\otimes x_4[W] + \beta_2^2 p_{42}q[I]g_{34}\otimes x_4x_3 + \beta_1\beta_2^2 q[C]g_{23344}\otimes x_4[W][Q] - \beta_1\beta_2 p_{42}p_{43}^2 q[C]g_{23344}\otimes [W][S] - \beta_1\beta_2 p_{32}p_{42}q[C]g_{23344}\otimes x_4[T] - \beta_1^2\beta_2 p_{42}p_{43}[C]g_{23344}\otimes x_4x_3[S]$

- $\Delta([N]) = [N]\otimes 1 + g_{122333344}\otimes [N] + \beta_2[K]g_{233}\otimes [R] + \beta_1\beta_2[L]g_{33}\otimes x_3^2 + \beta_2 q[M]g_3\otimes x_3 + \beta_2 p_{42}^2 p_{43}^4 q[D]g_{23344}\otimes [U] - \beta_2 p_{42}p_{43}^2 q(1+q)[J]g_{34}\otimes [W] + \beta_1\beta_2 p_{42}^2 p_{43}^4 q[E]g_{3344}\otimes [W]^2 + \beta_2^2 x_1 g_{22333344}\otimes [U][R] - \beta_2^2 p_{42}p_{43}x_1 g_{22333344}\otimes [T]^2 + \beta_2^2 q[F]g_{23334}\otimes [W][R] - \beta_2^2 p_{42}p_{43}q[F]g_{23334}\otimes x_3[T] + \beta_1\beta_2^2[B]g_{2333344}\otimes [W]^2[R] + \beta_1\beta_2^2 p_{42}^2 p_{43}^4 q[B]g_{2333344}\otimes x_3^2[U] - \beta_2^3 p_{42}p_{43}q[B]g_{2333344}\otimes Wx_3T + \beta_1^2\beta_2 p_{43}[D]g_{23344}\otimes x_4^2[R] - \beta_2^2 p_{42}p_{43}^2 q[D]g_{23344}\otimes x_4[T] + \beta_2^2 q[G]g_{2334}\otimes x_4[R] - \beta_2 p_{42}p_{43}(1+q)[G]g_{2334}\otimes [T] + \beta_1\beta_2^2 q[C]g_{233344}\otimes x_4[W][R] + \beta_2^2 p_{42}^2 p_{43}^4 q^2[C]g_{233344}\otimes x_3[U] - \beta_2^2 p_{42}p_{43}^2 q^2[C]g_{233344}\otimes [W][T] - \beta_1\beta_2^2 p_{42}p_{43}q[C]g_{233344}\otimes x_4x_3[T] - \beta_2^2 qp_{42}p_{43}[I]g_{334}\otimes [W]x_3 + \beta_2^2 p_{42}p_{43}q^2[J]g_{34}\otimes x_4x_3 + \beta_1\beta_2^2 p_{42}q[I]g_{334}\otimes x_4x_3^2 + \beta_1^3\beta_2 p_{42}^2 p_{43}[E]g_{3344}\otimes x_4^2x_3^2 - \beta_1\beta_2^2 p_{42}^2 p_{43}^2 q[E]g_{3344}\otimes x_4[W]x_3$

- $\Delta([O]) = [O]\otimes 1 + g_{1222333344}\otimes [O] + \beta_2[N]g_2\otimes x_2 + \beta_2 p_{32}^2 q[L]g_{233}\otimes [R] + \beta_2 p_{32}^2 p_{42}^4 p_{43}^4 q^2[E]g_{23344}\otimes [U] + \beta_2^2 p_{32}^4 p_{42}^2 q^2[B]g_{22333344}\otimes [U][R] - \beta_2^2 p_{32}^4 p_{42}^3 p_{43}q^2[B]g_{22333344}\otimes [T]^2 - \beta_2 p_{32}^2 p_{42}^2 p_{43}q(1+q)[I]g_{2334}\otimes [T] + \beta_1^2\beta_2 p_{32}^2 p_{42}^2 p_{43}q[E]g_{23344}\otimes x_4^2[R] - \beta_2^2 p_{32}^2 p_{42}^3 p_{43}^2 q^2[E]g_{23344}\otimes x_4[T] + \beta_2^2 p_{32}^2 p_{42}q^2[I]g_{2334}\otimes x_4[R] - \beta_2 p_{32}q[M]g_{23}\otimes [Q] - \beta_1\beta_2 p_{32}[K]g_{2233}\otimes [Q]^2 + \beta_2 p_{32}p_{42}^2 p_{43}^2 q(1+q)[J]g_{234}\otimes [S] - \beta_1\beta_2 p_{32}p_{42}^3 p_{43}^4 q[D]g_{223344}\otimes [S]^2 + \beta_2^2[K]g_{2233}\otimes [R]x_2 + \beta_2^2 q[M]g_{23}\otimes x_3x_2 + \beta_1\beta_2^2[L]g_{233}\otimes x_3^2x_2 - \beta_2^2 p_{32}q[L]g_{233}\otimes x_3[Q] + \beta_2^2 p_{42}^2 p_{43}^4 q[D]g_{223344}\otimes [U]x_2 - \beta_2^2 p_{42}p_{43}^2 q(1+q)[J]g_{234}\otimes [W]x_2 + \beta_1\beta_2^2 p_{42}^2 p_{43}^4 q[E]g_{23344}\otimes [W]^2x_2 - \beta_2^2 p_{32}p_{42}^3 p_{43}^4 q^2[E]g_{23344}\otimes [W][S] - \beta_2^2 p_{32}^2 p_{42}^2 p_{43}^2 q^2[C]g_{2233344}\otimes [U][Q] - \beta_1\beta_2^2 p_{32}x_1 g_{222333344}\otimes [U][Q]^2 - \beta_2^2 p_{32}^2 p_{42}q^2[F]g_{223334}\otimes [S][R] - \beta_1\beta_2^2 p_{32}^3 p_{42}qx_1 g_{222333344}\otimes [S]^2[R] - \beta_2^3 p_{42}p_{43}x_1 g_{222333344}\otimes [T]^2x_2 + \beta_2^2 p_{32}p_{42}p_{43}q[I]g_{2334}\otimes [W][Q] - \beta_1\beta_2^2 p_{32}q[F]g_{223334}\otimes [W][Q]^2 - \beta_1^2\beta_2^2 p_{32}[B]g_{22333344}\otimes [W]^2[Q]^2 + \beta_2^2 p_{32}^2 p_{42}q^2[F]g_{223334}\otimes [T][Q] + \beta_2^2 p_{32}p_{42}^2 p_{43}^2 q^2[I]g_{2334}\otimes x_3[S] - \beta_1\beta_2^2 p_{32}p_{42}^3 p_{43}^4 q^2[C]g_{2233344}\otimes x_3[S]^2 - \beta_1^2\beta_2^2 p_{32}p_{42}^3 p_{43}^4 q[B]g_{22333344}\otimes x_3^2[S]^2 -$

$\beta_2^2 p_{32}p_{42}p_{43}q^2[J]g_{234}\otimes x_4[Q]-\beta_1\beta_2^2p_{32}q[G]g_{22334}\otimes x_4[Q]^2-\beta_1^3\beta_2p_{32}p_{43}[D]g_{223344}\otimes x_4^2[Q]^2 - \beta_2^2p_{42}p_{43}(1+q)[G]g_{22334}\otimes[T]x_2+\beta_2^2p_{32}p_{42}p_{43}q[G]g_{22334}\otimes[S][Q]+\beta_2^2p_{32}^2p_{42}^3p_{43}^3q^2[C]g_{2233344}\otimes[T][S]+\beta_2^3x_1g_{222333344}\otimes[U][R]x_2+\beta_2^3p_{32}p_{42}p_{43}qx_1g_{222333344}\otimes[T][S][Q]+\beta_2^3q[F]g_{223334}\otimes[W][R]x_2-\beta_2^3p_{42}p_{43}q[F]g_{223334}\otimes x_3[T]x_2+\beta_1\beta_2^2p_{32}p_{42}p_{43}q[F]g_{223334}\otimes x_3[S][Q]+\beta_1\beta_2^3[B]g_{2233344}\otimes[W]^2[R]x_2-\beta_2^3p_{32}^3p_{42}q^2[B]g_{22333344}\otimes[W][S][R]+\beta_2^3p_{42}^2p_{43}^4q^2[C]g_{2233344}\otimes x_3[U]x_2+\beta_1\beta_2^3p_{42}^2p_{43}^4q[B]g_{22333344}\otimes x_3^2[U]x_2+\beta_2^3p_{32}^2p_{42}^3p_{43}^3q^2[B]g_{22333344}\otimes x_3[T][S]-\beta_2^3p_{32}^2p_{42}^2p_{43}^2q^2[B]g_{22333344}\otimes x_3[U][Q]-\beta_2^3p_{42}p_{43}q[I]g_{2334}\otimes[W]x_3x_2-\beta_2^4p_{42}p_{43}q[B]g_{22333344}\otimes[W]x_3[T]x_2+\beta_2^3p_{32}^2p_{42}q^2[B]g_{22333344}\otimes[W][T][Q]+\beta_1\beta_2^3p_{32}p_{42}p_{43}q[B]g_{22333344}\otimes[W]x_3[S][Q]-\beta_2^3p_{42}p_{43}^2q^2[C]g_{2233344}\otimes[W][T]x_2+\beta_1\beta_2^2p_{32}p_{42}p_{43}^2q^2[C]g_{2233344}\otimes[W][S][Q]+\beta_1^2\beta_2^2p_{43}[D]g_{223344}\otimes x_4^2[R]x_2-\beta_2^3p_{42}p_{43}^2q[D]g_{223344}\otimes x_4[T]x_2+\beta_1\beta_2^2p_{32}p_{42}p_{43}^2q[D]g_{223344}\otimes x_4[S][Q]+\beta_2^3q[G]g_{22334}\otimes x_4[R]x_2+\beta_1\beta_2^3q[C]g_{2233344}\otimes x_4[W][R]x_2-\beta_1\beta_2^2p_{32}^3p_{42}q^2[C]g_{2233344}\otimes x_4[S][R]-\beta_1^2\beta_2^2p_{32}q[C]g_{2233344}\otimes x_4[W][Q]^2-\beta_1\beta_2^3p_{42}p_{43}q[C]g_{2233344}\otimes x_4x_3[T]x_2+\beta_1\beta_2^2p_{32}^2p_{42}q^2[C]g_{2233344}\otimes x_4[T][Q]+\beta_1^2\beta_2^2p_{32}p_{42}p_{43}q[C]g_{2233344}\otimes x_4x_3[S][Q]+\beta_2^3p_{42}p_{43}q^2[J]g_{234}\otimes x_4x_3x_2+\beta_1\beta_2^3p_{42}q[I]g_{2334}\otimes x_4x_3^2x_2-\beta_2^3p_{32}p_{42}q^2[I]g_{2334}\otimes x_4x_3[Q]+\beta_1^3\beta_2^2p_{42}^2p_{43}[E]g_{23344}\otimes x_4^2x_3^2x_2-\beta_1^2\beta_2^2p_{32}p_{42}^2p_{43}q[E]g_{23344}\otimes x_4^2x_3[Q]-\beta_1\beta_2^3p_{42}^2p_{43}^2q[E]g_{23344}\otimes x_4[W]x_3x_2+\beta_1\beta_2^2p_{32}p_{42}^3p_{43}^3q^2[E]g_{23344}\otimes x_4x_3[S]+\beta_1\beta_2^2p_{32}p_{42}^2p_{43}^2q[E]g_{23344}\otimes x_4[W][Q]$

- $\Delta([P]) = \Delta(x_2) = x_2\otimes 1 + g_2\otimes x_2$
- $\Delta([Q]) = [Q]\otimes 1 + g_{23}\otimes[Q] + \beta_2x_2g_3\otimes x_3$
- $\Delta([R]) = [R]\otimes 1 + g_{233}\otimes R + \beta_1\beta_2x_2g_{33}\otimes x_3^2 + \beta_2p_{33}[Q]g_3\otimes x_3$
- $\Delta([S]) = [S]\otimes 1 + g_{234}\otimes[S] + \beta_2x_2g_{34}\otimes[W] + \beta_1[Q]g_4\otimes x_4$
- $\Delta([T]) = [T]\otimes 1 + g_{2334}\otimes[T] + \beta_1\beta_2x_2g_{334}\otimes[W]x_3 + \beta_1p_{43}q[Q]g_{34}\otimes[W] + \beta_1^2[Q]g_{34}\otimes x_4x_3 + \beta_1p_{43}[R]g_4\otimes x_4 + \beta_1[S]g_3\otimes x_3$
- $\Delta([U]) = [U]\otimes 1 + g_{23344}\otimes[U] + \beta_1\beta_2x_2g_{3344}\otimes[W]^2 + \beta_2q[S]g_{34}\otimes[W] + \beta_1^2p_{43}[R]g_{44}\otimes x_4^2 + \beta_1\beta_2q[Q]g_{344}\otimes x_4[W] + \beta_1(1+q)[T]g_4\otimes x_4$
- $\Delta([V]) = \Delta(x_3) = x_3\otimes 1 + g_3\otimes x_3$
- $\Delta([W]) = [W]\otimes 1 + g_{34}\otimes[W] + \beta_1x_3g_4\otimes x_4$
- $\Delta([X]) = \Delta(x_4) = x_4\otimes 1 + g_4\otimes x_4$.

Proof. The coproduct of the generators x_1, x_2, x_3 and x_4 are given by definition of the algebra $U_q^+(F_4)$. Using Lemma 4.2.2 with $i=1$ and $j=2$ we obtain $\Delta([B])$. Analogously we have the coproduct of $[Q]$ and $[W]$. By Lemmas 4.2.1 and 4.2.3 we have the coproduct formula for $[C]$ with $i=1$, $j=2$ and $k=3$. In the same way we obtain the coproduct formula for the PBW-generators $[R]$ and $[S]$. Applying the Lemma 4.2.4 for $i=1$, $j=2$ and $k=l=3$ we have the coproduct formula of $[D]$. Similarly we obtain the coproduct formula for the PBW-generators of degree 4, which are $[D], [F]$ and $[T]$. Using Lemmas 4.2.2, 4.2.3 and 4.2.4 and the fact that the coproduct is multiplicative we obtain the coproduct formula of the PBW-generators of degree $5, 6, 7, 8, 9, 10$ and 11. □

Corollary 4.2.6. *The only skew-primitive PBW-generators of $U_q^+(F_4)$ are x_1, x_2, x_3 and x_4.*

4.3 Skew-primitive elements

In this section we list all the skew-primitive homogeneous elements of $U_q^+(F_4)$.

Lemma 4.3.1. *The coproduct of the element x_i^n, where n is a natural number and $x_i \in \{x_1, x_2, x_3, x_4\}$, is given by the formula*

$$\Delta(x_i^n) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p_{ii}} x_i^{n-k} g_i^k \otimes x_i^k,$$

where $\begin{bmatrix} n \\ k \end{bmatrix}_{p_{ii}} = \frac{[n]!_{p_{ii}}}{[k]!_{p_{ii}}[n-k]!_{p_{ii}}}$, $[n]!_{p_{ii}} = [n]_{p_{ii}}[n-1]_{p_{ii}}...[2]_{p_{ii}}[1]_{p_{ii}}$ and $[n]_{p_{ii}} = 1 + p_{ii} + p_{ii}^2 ... + p_{ii}^{n-1}$.

Proof. We prove by induction on n. If $n=1$ then the equality reduces to $\Delta(x_i) = x_i \otimes 1 + g_i \otimes x_i$ since $\begin{bmatrix} 1 \\ 0 \end{bmatrix}_{p_{ii}} = 1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{p_{ii}}$.

We note that $\begin{bmatrix} n \\ k \end{bmatrix}_{p_{ii}}$ satisfy two p_{ii}-Pascal identities $\begin{bmatrix} n+1 \\ k \end{bmatrix}_{p_{ii}} = \begin{bmatrix} n \\ k-1 \end{bmatrix}_{p_{ii}} + p_{ii}^k \begin{bmatrix} n \\ k \end{bmatrix}_{p_{ii}}$

and $\begin{bmatrix} n+1 \\ k \end{bmatrix}_{p_{ii}} = \begin{bmatrix} n \\ k-1 \end{bmatrix}_{p_{ii}} p_{ii}^{n-k+1} + \begin{bmatrix} n \\ k \end{bmatrix}_{p_{ii}}$, so we have the following equalities

$$\begin{aligned}
\Delta(x_i^{n+1}) &= \Delta(x_i)\Delta(x_i^n) = (x_i \otimes 1 + g_i \otimes x_i)(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p_{ii}} x_i^{n-k} g_i^k \otimes x_i^k) \\
&= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p_{ii}} x_i^{n-k+1} g_i^k \otimes x_i^k + \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p_{ii}} p_{ii}^{n-k} x_i^{n-k} g_i^{k+1} \otimes x_i^{k+1} \\
&= x_i^{n+1} \otimes 1 + \begin{bmatrix} n \\ 1 \end{bmatrix}_{p_{ii}} x_i^n g_i \otimes x_i + \ldots + \begin{bmatrix} n \\ n-1 \end{bmatrix}_{p_{ii}} x_i^2 g_i^{n-1} \otimes x_i^{n-1} + x_i g_i^n \otimes x_i^n + \\
&+ p_{ii}^n x_i^n g_i \otimes x_i + \begin{bmatrix} n \\ 1 \end{bmatrix}_{p_{ii}} p_{ii}^{n-1} x_i^{n-1} g_i^2 \otimes x_i^2 + \ldots + \begin{bmatrix} n \\ n-1 \end{bmatrix}_{p_{ii}} p_{ii} x_i g_i^n \otimes x_i^n + g_i^{n+1} \otimes x_i^{n+1} \\
&= x_i^{n+1} \otimes 1 + (\begin{bmatrix} n \\ 1 \end{bmatrix}_{p_{ii}} + p_{ii}^n) x_i^n g_i \otimes x_i + (\begin{bmatrix} n \\ 2 \end{bmatrix}_{p_{ii}} + \begin{bmatrix} n \\ 1 \end{bmatrix}_{p_{ii}} p_{ii}^{n-1}) x_i^{n-1} g_i^2 \otimes x_i^2 + \ldots + \\
&+ (1 + \begin{bmatrix} n \\ n-1 \end{bmatrix}_{p_{ii}} p_{ii}) x_i g_i^n \otimes x_i^n + g_i^{n+1} \otimes x_i^{n+1} \\
&= \begin{bmatrix} n+1 \\ 0 \end{bmatrix}_{p_{ii}} x_i^{n+1} \otimes 1 + \begin{bmatrix} n+1 \\ 1 \end{bmatrix}_{p_{ii}} x_i^n g_i \otimes x_i + \begin{bmatrix} n+1 \\ 2 \end{bmatrix}_{p_{ii}} x_i^{n-1} g_i^2 \otimes x_i^2 + \ldots + \\
&+ \begin{bmatrix} n+1 \\ n \end{bmatrix}_{p_{ii}} p_{ii} x_i g_i^n \otimes x_i^n + \begin{bmatrix} n+1 \\ n+1 \end{bmatrix}_{p_{ii}} g_i^{n+1} \otimes x_i^{n+1} \\
&= \sum_{k=0}^{n+1} \begin{bmatrix} n+1 \\ k \end{bmatrix}_{p_{ii}} x_i^{n+1-k} g_i^k \otimes x_i^k.
\end{aligned}$$

□

Theorem 4.3.2. *If q is not a root of the unit, the only homogeneous skew-primitive elements of $U_q^+(F_4)$ are x_i for every i in $\{1,2,3,4\}$. If $q^t = 1$, the skew-primitive elements are in the form x_i and $x_i^{h_i}$ where h_i is the order of p_{ii}.*

Proof. From Lemma 2.2.11, if $v \in U_q^+(F_4)$ is an homogeneous skew-primitive element, then $v = \alpha[u]^h + \sum \alpha_i W_i$ where $[u]$ is an element from list (4.2) and W_i are basis words in super-letters smaller than $[u]$ with the same degree as $[u]^h$. If p_{uu} is not a root of the unit we have $h = 1$. If p_{uu} is a primitive t-th root of unit, then $h = 1$ or $h = t$.

If $[u] = x_1$, then clearly there are no other basis words W_i of degree $(h, 0, 0, 0)$ so $v = \alpha[u]^h$. The same holds for $[u] = x_i$ with $i \in \{2,3,4\}$. If $[u] = [x_1, x_2]$ then $W_1 = x_2^{s_1}[x_1, x_2]^{s_2} x_1^{s_3}$ is a basis word with the same degree as $[x_1, x_2]^h$, where $s_1 + s_2 = h$ and $s_2 + s_3 = h$. However, x_1 is greater than $[x_1, x_2]$, so again $v = \alpha[u]^h$. If $[u] = [x_1, [x_2, x_3]]$ we have

$$W_1 = x_3^{s_1}[[x_2, x_3], x_3]^{s_2}[x_2, x_3]^{s_3} x_2^{s_4}[[x_1, [[x_2, x_3], x_3]], x_2]^{s_5}[x_1, [[x_2, x_3], x_3]]^{s_6}[x_1, [x_2, x_3]]^{s_7}[x_1, x_2]^{s_8} x_1^{s_9}$$

is a basis word with the same degree as $[x_1, [x_2, x_3]]^h$. As x_1 and $[x_1, x_2]$ are greater than $[x_1, [x_2, x_3]]$, we have $s_8 = s_9 = 0$, $s_1 + 2s_2 + s_3 + 2s_5 + 2s_6 + s_7 = h$, $s_2 + s_3 +$

$s_4+2s_5+s_6+s_7=h$ and $s_5+s_6+s_7=h$. Since each degree s_i is a non negative integer, we obtain $v=\alpha[u]^h$. Analysing the degree of the hard super-letters, it is easy to see that the same occurs for every $[u]$ in the list (4.2). This provides that the possible skew-primitive elements are multiples of elements in the form $[u]^h$. If $h=1$, then Corollary 4.2.6 shows that the only skew-primitive PBW-generators are x_1, x_2, x_3 and x_4.

Now we suppose that $q^t=1$ and $h=h_u$ is the multiplicative order of p_{uu}. First we consider the case $[u]=x_i$ for every $i\in\{1,2,3,4\}$ and see that from Lemma 4.3.1 we obtain that $x_i^{h_i}$ are skew-primitive. If $p_{ii}^{h_i}=1$ we have $[h_i]_{p_{ii}}=0$, so $\begin{bmatrix} h_i \\ 0 \end{bmatrix}_{p_{ii}}=1=\begin{bmatrix} h_i \\ h_i \end{bmatrix}_{p_{ii}}$ and $\begin{bmatrix} h_i \\ k \end{bmatrix}_{p_{ii}}=0$, for all $k\in\{1,2,3,...,h_i-1\}$. Therefore $\Delta(x_i^{h_i})=x_i^{h_i}\otimes 1+g_i^{h_i}\otimes x_i^{h_i}$ and $x_i^{h_i}$ is skew-primitive.

If $[u]=[x_1,x_2]=[B]$, then Theorem 4.2.5 provides $\Delta([B])=[B]\otimes 1+g_{12}\otimes [B]+\beta_2x_1g_2\otimes x_2$. Using the fact that the subalgebra generated by the elements $[u]^h$ is a normal Hopf subalgebra of $U_q^+(F_4)$ (see [5, Lemma 4.10]), where $[u]$ belongs to the list (4.2) and h is the height of $[u]$, we have

$$\Delta([B]^n)=\sum_{u_i}u_1u_2\cdots u_n,$$

where $u_i\in\{[B]\otimes 1, g_{12}\otimes[B], \beta_2x_1g_2\otimes x_2\}$, for any $n\in\mathbb{N}$. Then we obtain

$$\Delta([B]^n)=[B]^n\otimes 1+g_{12}^n\otimes[B]^n+ax_1^ng_2^n\otimes x_2^n+\sum\gamma yg_z\otimes z,$$

where the degree of y plus the degree of z equals the degree of $[B]^n$. We note that the only way to have x_2^n in the second tensorand is by taking $u_1=\cdots=u_n=\beta_2x_1g_2\otimes x_2$. So we obtain

$$(\beta_2x_1g_2\otimes x_2)^n=\beta_2^n(x_1g_2)^n\otimes x_2^n=\beta_2^np_{21}^{\frac{n(n-1)}{2}}x_1^ng_2^n\otimes x_2^n.$$

Therefore $a=\beta_2^np_{21}^{\frac{n(n-1)}{2}}\neq 0$. In particular $[B]^h$ is not skew-primitive.

Using the same idea as above, for case $[u]=[C]$ we obtain

$$\Delta([C]^n)=[C]^n\otimes 1+g_{123}^n\otimes[C]^n+\beta_2^n(p_{31}p_{32})^{\frac{n(n-1)}{2}}[B]^ng_3^n\otimes x_3^n+\sum\gamma yg_z\otimes z,$$

where the degree of y plus the degree of z equals the degree of $[C]^n$. Therefore $[C]^h$ is not skew-primitive.

In the same way we have that

$$\Delta([D]^n)=[D]^n\otimes 1+g_{1233}^n\otimes[D]^n+\beta_2^np_{21}^{\frac{n(n-1)}{2}}p_{31}^{n(n-1)}x_1^ng_{233}^n\otimes[R]^n+\sum\gamma yg_z\otimes z;$$

$\Delta([E]^n) = [E]^n\otimes 1+g^n_{12233}\otimes[E]^n+\beta_2^n p_{21}^{\frac{n(n-1)}{2}}(p_{23}q)^{n(n-1)}[D]^n g_2^n\otimes x_2^n+\sum\gamma y g_z\otimes z;$

$\Delta([F]^n) = [F]^n\otimes 1+g^n_{1234}\otimes[F]^n+\beta_1^n(p_{41}p_{42}p_{43})^{\frac{n(n-1)}{2}}[C]^n g_4^n\otimes x_4^n+\sum\gamma y g_z\otimes z;$

$\Delta([G]^n) = [G]^n\otimes 1+g^n_{12334}\otimes[G]^n+\beta_1^n(p_{31}p_{32}p_{34}q)^{\frac{n(n-1)}{2}}[F]^n g_3^n\otimes x_3^n+\sum\gamma y g_z\otimes z;$

$\Delta([H]^n) = [H]^n\otimes 1+g^n_{11222333344}\otimes[H]^n+\beta_1^n\beta_2^n(p_{21}p_{24})^{\frac{n(n-1)}{2}}(p_{23}q)^{n(n-1)}[G]^{2n}g^n_{123344}\otimes x_2^n+\sum\gamma y g_z\otimes z;$

$\Delta([I]^n) = [I]^n\otimes 1+g^n_{122334}\otimes[I]^n+\beta_2^n(p_{21}p_{24})^{\frac{n(n-1)}{2}}(p_{23}q)^{n(n-1)}[G]^n g_2^n\otimes x_2^n+\sum\gamma y g_z\otimes z;$

$\Delta([J]^n) = [J]^n\otimes 1+g^n_{1223334}\otimes[J]^n+\beta_1^n(p_{31}p_{34})^{\frac{n(n-1)}{2}}(p_{32}q)^{n(n-1)}[I]^n g_3^n\otimes x_3^n+\sum\gamma y g_z\otimes z;$

$\Delta([K]^n) = [K]^n\otimes 1+g^n_{123344}\otimes[K]^n+\beta_2^n p_{21}^{\frac{n(n-1)}{2}}(p_{31}p_{41})^{n(n-1)}x_1^n g^n_{23344}\otimes[U]^n+\sum\gamma y g_z\otimes z;$

$\Delta([L]^n) = [L]^n\otimes 1+g^n_{1223344}\otimes[L]^n+\beta_2^n p_{21}^{\frac{n(n-1)}{2}}(p_{23}p_{24}q)^{n(n-1)}[K]^n g_2^n\otimes x_2^n+\sum\gamma y g_z\otimes z;$

$\Delta([M]^n) = [M]^n\otimes 1+g^n_{12233344}\otimes[M]^n+\beta_2^n p_{31}^{\frac{n(n-1)}{2}}(p_{32}p_{34}q)^{n(n-1)}[L]^n g_3^n\otimes x_3^n+\sum\gamma y g_z\otimes z;$

$\Delta([N]^n) = [N]^n\otimes 1+g^n_{122333344}\otimes[N]^n+\beta_1^n\beta_2^n p_{31}^{n(n-1)}(p_{32}p_{34}q)^{2n(n-1)}[L]^n g_3^{2n}\otimes x_3^{2n}+\sum\gamma y g_z\otimes z;$

$\Delta([O]^n) = [O]^n\otimes 1+g^n_{1222333344}\otimes[O]^n+\beta_2^n p_{21}^{\frac{n(n-1)}{2}}(p_{23}q)^{2n(n-1)}p_{24}^{n(n-1)}[N]^n g_2^n\otimes x_2^n+\sum\gamma y g_z\otimes z;$

$\Delta([Q]^n) = [Q]^n\otimes 1+g^n_{23}\otimes[Q]^n+\beta_2^n p_{32}^{\frac{n(n-1)}{2}}x_2^n g_3^n\otimes x_3^n+\sum\gamma y g_z\otimes z;$

$\Delta([R]^n) = [R]^n\otimes 1+g^n_{233}\otimes[R]^n+\beta_1^n\beta_2^n p_{32}^{\frac{n(n-1)}{2}}x_2^n g_3^{2n}\otimes x_3^{2n}+\sum\gamma y g_z\otimes z;$

$\Delta([S]^n) = [S]^n\otimes 1+g^n_{234}\otimes[S]^n+\beta_2^n(p_{32}p_{42})^{\frac{n(n-1)}{2}}x_2^n g_{34}^n\otimes[W]^n+\sum\gamma y g_z\otimes z;$

$\Delta([T]^n) = [T]^n\otimes 1+g^n_{2334}\otimes[T]^n+\beta_1^n(p_{32}p_{34}q)^{\frac{n(n-1)}{2}}[S]^n g_3^n\otimes x_3^n+\sum\gamma y g_z\otimes z;$

$\Delta([U]^n) = [U]^n\otimes 1+g^n_{23344}\otimes[U]^n+\beta_1^{2n}p_{42}^{n(n-1)}p_{43}^{n(2n-1)}[R]^n g_4^{2n}\otimes x_4^{2n}+\sum\gamma y g_z\otimes z;$

$\Delta([W]^n) = [W]^n\otimes 1+g^n_{34}\otimes[W]^n+\beta_1^n p_{43}^{\frac{n(n-1)}{2}}x_3^n g_4^n\otimes x_4^n+\sum\gamma y g_z\otimes z,$

proving that $[u]^h$ is not skew-primitive, where $[u]$ belongs to list (4.2) except x_1, x_2, x_3 and x_4, and where $\gamma\in\mathbf{k}$. □

4.4 The combinatorial rank of the quantum groups of type F_4

In this section we obtain $\kappa(u_q^+(F_4))$.

Proposition 4.4.1. *The elements $[u]^h$ are skew central in $U_q^+(F_4)$, where $[u]$ is an element from list* (4.2) *and h is the height of $[u]$.*

Proof. It is enough to prove that $[u]^h x_i = \alpha x_i [u]^h$, for $i = \{1,2,3,4\}$, $\alpha \in \mathbf{k}$. We notice that for every element $[u]$ in the PBW-basis $p_{uu} = q$ or $p_{uu} = q^2$. If t is odd we have that the height of $[u]$ is $h = t$ and $p_{uu}^h = q^t = 1$ or $p_{uu}^h = (q^2)^t = 1$. For the case where t is even, we have that the height h is t for the elements $[u]$ such that $p_{uu} = q$ and when $p_{uu} = q^2$ the height of $[u]$ is $\frac{t}{2}$. So we also have $p_{uu}^h = q^t = 1$ or $p_{uu}^h = (q^2)^{\frac{t}{2}} = q^t = 1$. Thus in both cases we may use relations (2.6) and (2.7).

If $[u] = [A] = x_1$ clearly $x_1^h x_1 = x_1 x_1^h$. We have $[x_1,[x_1,x_2]] = \big[x_1,[B]\big] = 0$, then by (2.7) we obtain $[x_1^h, x_2] = [x_1,[\cdots[x_1,x_2]] = 0$. Thus $x_1^h x_2 = p_{12}^h x_2 x_1^h$, for $h > 1$. For $i = \{3,4\}$, we have $[x_1,x_i] = 0$ then $[x_1^h, x_i] = 0$, so $x_1^h x_i = p_{1i}^h x_i x_1^h$ for $h \geq 1$. Therefore x_1^h is skew central.

In the case $[u] = [B]$, we have $\big[x_1,[B]\big] = 0$ then $\big[x_1,[B]^h\big] = 0$ and $x_1[B]^h = p_{11}^h p_{12}^h [B]^h x_1$. We notice that $\big[[B],x_2\big] = 0$, $\big[[B],\big[[B],x_3\big]\big] = \big[[B],[C]\big] = 0$ and $\big[[B],x_4\big] = 0$ so if $i = \{2,3,4\}$ we obtain $\big[[B]^h, x_i\big] = 0$ and $[B]^h x_i = p_{1i}^h p_{2i}^h x_i [B]^h$ for $h \geq 2$. Thus $[B]^h$ is skew central.

For the case $[u] = [C]$ we notice that $\big[x_1,[C]\big] = 0$, then $\big[x_1,[C]^h\big] = 0$ and $x_1[C]^h = p_{11}^h p_{12}^h p_{13}^h [C]^h x_1$. We also have that $\big[[C],x_2\big] = 0$, $\big[[C],\big[[C],x_3\big]\big] = \big[[C],[D]\big] = 0$ and $\big[[C],\big[[C],x_4\big]\big] = \big[[C],[F]\big] = 0$, by (2.7) $\big[[C]^h, x_i\big] = 0$ for $i = \{2,3,4\}$. So $[C]^h x_i = p_{1i}^h p_{2i}^h p_{3i}^h x_i [C]^h$ for $h \geq 2$.

If $[u] = [D]$ we have $\big[x_1,[D]\big] = 0$ then $\big[x_1,[D]^h\big] = 0$. In other words $x_1[D]^h = p_{11}^h p_{12}^h p_{13}^{2h} [D]^h x_1$. For $i = \{2,3,4\}$ we obtain $\big[[D]^h, x_i\big] = 0$ because $\big[[D],\big[[D],x_2\big]\big] = \big[[D],[E]\big] = 0$, $\big[[D],x_3\big] = 0$ and $\big[[D],\big[[D],x_4\big]\big] = p_{34}(1+q)\big[[D],[G]\big] = 0$. Thus $[D]^h x_i = p_{1i}^h p_{2i}^h p_{3i}^{2h} x_i [D]^h$ for $h \geq 2$.

In the case $[u] = [E]$ we have $\big[\big[x_1,[E]\big],[E]\big] = \alpha\big[[D][B],[E]\big] + \beta\big[[C]^2,[E]\big] = 0$, $\big[[E],x_2\big] = 0$, $\big[[E],x_3\big] = 0$ and $\big[[E],\big[[E],x_4\big]\big] = p_{24}p_{34}(1+q)\big[[E],[I]\big] = 0$, where $\alpha, \beta \in \mathbf{k}$. By (2.6) and (2.7) we obtain $\big[x_1,[E]^h\big] = 0$ and $\big[[E]^h, x_i\big] = 0$ if $i = \{2,3,4\}$. Therefore $x_1[E]^h = p_{12}^h p_{12}^{2h} p_{13}^{2h} [E]^h$ and $[E]^h x_i = p_{1i}^h p_{2i}^{2h} p_{3i}^{2h} x_i [E]^h$, for $i = \{2,3,4\}$ and $h \geq 2$.

Now we suppose $[u] = [F]$. In this case we have $\big[x_1,[F]\big] = 0$, $\big[[F],x_2\big] = 0$, $\big[[F],\big[[F],x_3\big]\big] = \big[[F],[G]\big] = 0$ and $\big[[F],x_4\big] = 0$, so from formulas (2.6) and (2.7) we obtain $\big[x_1, [F]^h\big] = 0$ and $\big[[F]^h, x_i\big] = 0$ for $i = \{2,3,4\}$ and $h \geq 2$. Thus $x_1[F]^h = p_{11}^h p_{12}^h p_{13}^h p_{14}^h [F]^h x_1$ and $[F]^h x_i = p_{1i}^h p_{2i}^h p_{3i}^h p_{4i}^h x_i [F]^h$.

In the case $[u] = [G]$ we notice that $\big[x_1, [G]\big] = 0$, $\big[[G], \big[[G], \big[[G], x_2\big]\big]\big] = \big[[G], \big[[G], [I]\big]\big] = \big[[G], [H]\big] = 0$, $\big[[G], x_3\big] = 0$ and $\big[[G], \big[[G], x_4\big]\big] = \big[[G], [K]\big] = 0$, then by (2.6) and (2.7) we have $\big[x_1, [G]^h\big] = 0$ and $\big[[G]^h, x_i\big] = 0$ for $i = \{2,3,4\}$ and $h \geq 3$. Therefore $x_1[G]^h = p_{11}^h p_{12}^h p_{13}^{2h} p_{14}^h [G]^h x_1$ and $[G]^h x_i = p_{1i}^h p_{2i}^h p_{3i}^{2h} p_{4i}^h x_i [G]^h$.

Now if $[u] = [H]$ we observe that $\big[x_1, [H]\big] = \alpha[G]^2[B] + \beta[F]^2[D] + \gamma[G][F][C]$, where $\alpha, \beta, \gamma \in \mathbf{k}$ are in the appendix list. We obtain

$$\big[\big[x_1, [H]\big], [H]\big] = \alpha\big[[G]^2[B], [H]\big] + \beta\big[[F]^2[D], [H]\big] + \gamma\big[[G][F][C], [H]\big] = 0.$$

We also have $\big[[H], \big[[H], x_2\big]\big] = \lambda\big[[H], [I]^2\big] = 0$, $\big[[H], x_3\big] = 0$ and $\big[[H], \big[[H], x_4\big]\big] = \theta\big[[H], [K][I]\big] = 0$, where $\lambda, \theta \in \mathbf{k}$. By formulas (2.6) and (2.7) we obtain $\big[x_1, [H]^h\big] = 0$ and $\big[[H]^h, x_i\big] = 0$ for $i = \{2,3,4\}$ and $h \geq 3$. Thus $x_1[H]^h = p_{11}^{2h} p_{12}^{3h} p_{13}^{4h} p_{14}^{2h} [H]^h x_1$ and $[H]^h x_i = p_{1i}^{2h} p_{2i}^{3h} p_{3i}^{4h} p_{4i}^{2h} x_i [H]^h$.

For $[u] = [I]$ we notice that $\big[x_1, [I]\big] = \alpha[G][B] + \beta[F][C]$, where $\alpha, \beta \in \mathbf{k}$ are described in the appendix. In this way we have

$$\big[\big[x_1, [I]\big], [I]\big] = \alpha\big[[G][B], [I]\big] + \beta\big[[F][C], [I]\big] = \gamma[H][B] + \theta[F]^2[E],$$

where $\gamma, \theta \in \mathbf{k}$. So

$$\big[\big[\big[x_1, [I]\big], [I]\big], [I]\big] = \gamma\big[[H][B], [I]\big] + \theta\big[[F]^2[E], [I]\big] = 0,$$

since $\big[[H], [I]\big] = \big[[B], [I]\big] = \big[[F], [I]\big] = \big[[E], [I]\big] = 0$. We also have $\big[[I], x_2\big] = 0$, $\big[[I], \big[[I], x_3\big]\big] = \big[[I], [J]\big] = 0$ and $\big[[I], \big[[I], x_4\big]\big] = \lambda\big[[I], [L]\big] = 0$. The formulas (2.6) and (2.7) result that $\big[x_1, [I]^h\big] = 0$ and $\big[[I]^h, x_i\big] = 0$ for $i = \{2,3,4\}$ and $h \geq 3$. Therefore $x_1[I]^h = p_{11}^h p_{12}^{2h} p_{13}^{2h} p_{14}^h [H]^h x_1$ and $[I]^h x_i = p_{1i}^h p_{2i}^{2h} p_{3i}^{2h} p_{4i}^h x_i [H]^h$.

Now we suppose $[u] = [J]$. In this case we have $\big[x_1, [J]\big] = \alpha[G][C] + \beta[F][D]$, where $\alpha, \beta \in \mathbf{k}$ are in the appendix. So

$$\big[\big[x_1, [J]\big], [J]\big] = \alpha\big[[G][C], [J]\big] + \beta\big[[F][D], [J]\big] = \gamma[H][D] + \theta[G]^2[E] + \lambda[I][G][D],$$

where $\gamma, \theta, \lambda \in \mathbf{k}$. Since $\big[[H], [J]\big] = \big[[D], [J]\big] = \big[[E], [J]\big] = \big[[G], [J]\big] = \big[[I], [J]\big] = 0$ we obtain $\big[x_1, [J]^h\big] = 0$ for $h \geq 3$. We also have $\big[[J], x_2\big] = 0$, $\big[[J], x_3\big] = 0$ and $\big[[J], \big[[J], x_4\big]\big] = p_{24}p_{34}q(1+q)^{-1}\big[[J], [M]\big] = 0$. Then formula (2.7) implies $\big[[J]^h, x_i\big] = 0$ for $i = \{2,3,4\}$ and $h \geq 2$.

If $[u] = [K]$ we notice that $\big[x_1, [K]\big] = 0$, $\big[[K], \big[[K], x_2\big]\big] = \big[[K], [L]\big] = 0$,

$\big[[K],x_3\big]=0$ and $\big[[K],x_4\big]=0$. So by formulas (2.6) and (2.7) we have $\big[x_1,[K]^h\big]=0$ and $\big[[K]^h,x_i\big]=0$ for $i=\{2,3,4\}$ and $h\geq 2$.

For $[u]=[L]$ we observe that $\big[x_1,[L]\big]=\alpha[F]^2+\beta[K][B]$, where $\alpha,\beta\in\mathbf{k}$ are listed in the appendix. Then $\big[\big[x_1,[L]\big],[L]\big]=\alpha\big[[F]^2,[L]\big]+\beta\big[[K][B],[L]\big]=0$ since $\big[[B],[L]\big]=\big[[F],[L]\big]=\big[[K],[L]\big]=0$. We also have $\big[[L],x_2\big]=0$, $\big[[L],\big[[L],x_3\big]\big]=\big[[L],[M]\big]=0$ and $\big[[L],x_4\big]=0$, then (2.6) and (2.7) provide $\big[x_1,[L]^h\big]=0$ and $\big[[L]^h,x_i\big]=0$ if $i=\{2,3,4\}$ and $h\geq 2$.

Now we suppose $[u]=[M]$. In this case we have $\big[x_1,[M]\big]=\alpha[K][C]+\beta[G][F]$, where $\alpha,\beta\in\mathbf{k}$ are described in the list of formulas in the appendix. Thus

$\big[\big[x_1,[M]\big],[M]\big]=\alpha\big[[K][C],[M]\big]+\beta\big[[G][F],[M]\big]=\gamma[K]^2[E]+\theta[K][H]+\lambda[L][K][D]+\delta[L][G]^2$,

$\big[\big[\big[x_1,[M]\big],[M]\big],[M]\big]=\gamma\big[[K]^2[E],[M]\big]+\theta\big[[K][H],[M]\big]+\lambda\big[[L][K][D],[M]\big]+\delta\big[[L][G]^2,[M]\big]=0$,

and we obtain $\big[x_1,[M]^h\big]=0$ for $h\geq 3$, and then $x_1[M]^h=p_{11}^hp_{12}^{2h}p_{13}^{3h}p_{14}^{2h}[M]^hx_1$. We also have $\big[[M],x_2\big]=0$, $\big[[M],\big[[M],x_3\big]\big]=\big[[M],[N]\big]=0$ and $\big[[M],x_4\big]=0$ so formula (2.7) provides $\big[[M]^h,x_i\big]=0$ for $i=\{2,3,4\}$. Therefore $[M]^hx_i=p_{1i}^hp_{2i}^{2h}p_{3i}^{3h}p_{4i}^{2h}x_i[M]^h$.

If $[u]=[N]$ we have $\big[\big[x_1,[N]\big],[N]\big]=\alpha\big[[K][D],[N]\big]+\beta\big[[G]^2,[N]\big]=0$, where $\alpha,\beta\in\mathbf{k}$ are present in the appendix, so $x_1[N]^h=p_{11}^hp_{12}^{2h}p_{13}^{4h}p_{14}^{2h}[N]^hx_1$. Since $\big[[N],\big[[N],x_2\big]\big]=\big[[N],[O]\big]=0$, $\big[[N],x_3\big]=0$ and $\big[[N],x_4\big]=0$ we obtain $\big[[N]^h,x_i]\big]=0$ for $i=\{2,3,4\}$. Thus $[N]^hx_i=p_{1i}^hp_{2i}^{2h}p_{3i}^{4h}p_{4i}^{2h}x_i[N]^h$ for $i=\{2,3,4\}$ and $h\geq 2$.

Now if $[u]=[O]$ we notice that

$$\big[x_1,[O]\big]=\alpha[K][E]+\beta[L][D]+\gamma[M][C]+\theta[H]+\lambda[I][G]+\delta[N][B]+\rho[J][F],$$

where $\alpha,\beta,\gamma,\theta,\lambda,\delta,\rho\in\mathbf{k}$ are in the appendix. So

$$\big[\big[x_1,[O]\big],[O]\big]=\varepsilon[N][L][E]+\zeta[N][I]^2+\eta[L][J]^2+\vartheta[M][J][I]+\iota[M]^2[E],$$

where $\varepsilon,\zeta,\eta,\vartheta,\iota\in\mathbf{k}$. Since $\big[[E],[O]\big]=\big[[I],[O]\big]=\big[[J],[O]\big]=\big[[L],[O]\big]=\big[[M],[O]\big]=\big[[N],[O]\big]=0$ we obtain $\big[\big[\big[x_1,[O]\big],[O]\big],[O]\big]=0$. By formula (2.6) we have $\big[x_1,[O]^h\big]=0$ so $x_1[O]^h=p_{11}^hp_{12}^{3h}p_{13}^{4h}p_{14}^{2h}[O]^hx_1$ for $h\geq 3$. Although $\big[\big[x_1,[O]\big],[O]\big]$ is not zero in general, in the specific case where $h=2$ we have $q^4=1$ and the coefficients $\varepsilon,\zeta,\eta,\vartheta,\iota$ equal zero as we have $\varepsilon=\beta_2^2p_{12}^3p_{13}^6p_{14}^4p_{23}^2p_{24}^2q^6(1+q^2)$, $\zeta=\beta_2p_{12}^3p_{13}^6p_{14}^3p_{23}^2p_{43}q^4(1-q^4)$, $\eta=-\beta_2^2p_{12}^3p_{13}^3p_{14}^3p_{32}^4p_{34}^2q^7(1+q)(1+q^2)$, $\vartheta=$

$\beta_2^2 p_{12}^3 p_{13}^5 p_{14}^3 q^3(1+q)(1+q^2)$, $\iota = -\beta_1\beta_2 p_{12}^3 p_{13}^5 p_{14}^4 p_{24}^2 p_{34}^2 q^4(1+q^2)$. We also have $\big[[O], x_2\big] = 0$, $\big[[O], x_3\big] = 0$ and $\big[[O], x_4\big] = 0$ then for $i = \{2,3,4\}$ we obtain $\big[[O]^h, x_i\big] = 0$. Thus $[O]^h x_i = p_{1i}^h p_{2i}^{3h} p_{3i}^{4h} p_{4i}^{2h} x_i [O]^h$.

For $[u] = [P] = x_2$ we have $[[x_1, x_2], x_2] = \big[[B], x_2\big] = 0$ then by (2.6) $[x_1, x_2^h] = 0$, for $h > 1$. So $x_1 x_2^h = p_{12}^h x_2^h x_1$. Clearly $x_2^h x_2 = x_2 x_2^h$. We also have $[x_2, [x_2, x_3]] = \big[x_2, [Q]\big] = 0$. By (2.7) we obtain $[x_2^h, x_3] = 0$ and $x_2^h x_3 = p_{23}^h x_3 x_2^h$ for $h \geq 2$. Now $[x_2, x_4] = 0$, then $[x_2^h, x_4] = 0$ and $x_2^h x_4 = p_{24}^h x_4 x_2^h$.

We suppose that $[u] = [Q]$. In this case we have

$$\Big[\big[[x_1, [Q]], [Q]\big], [Q]\Big] = \Big[\big[[C], [Q]\big], [Q]\Big] = \alpha\big[x_2[D], [Q]\big] + \beta\big[[E], [Q]\big] = 0,$$

with $\alpha, \beta \in \mathbf{k}$, and $\big[x_2, [Q]\big] = 0$. We obtain that $\big[x_i, [Q]\big] = 0$ then $x_i[Q]^h = p_{i2}^h p_{i3}^h [Q]^h x_i$ for $i = \{1,2\}$. If $i = \{3,4\}$, we have $\big[[Q]^h, x_i\big] = 0$ since $\big[[Q], \big[[Q], x_3\big]\big] = \big[[Q], [R]\big] = 0$ and $\big[[Q], \big[[Q], x_4\big]\big] = \big[[Q], [S]\big] = 0$. Therefore $[Q]^h x_i = p_{2i}^h p_{3i}^h x_i [Q]^h$ for $i = \{3,4\}$ and $h \geq 2$.

In the case $[u] = [R]$ we have $\big[\big[x_1, [R]\big], [R]\big] = \big[[D], [R]\big] = 0$, $\big[\big[x_2, [R]\big], [R]\big] = \beta_1 p_{23} q^2 \big[[Q]^2, [R]\big] = 0$, $\big[[R], x_3\big] = 0$ and $\big[[R], \big[[R], x_4\big]\big] = p_{34}(1+q)\big[[R], [T]\big] = 0$. So by formulas (2.6) and (2.7) we obtain $x_i[R]^h = p_{i2}^h p_{i3}^{2h} [R]^h x_i$ and $[R]^h x_j = p_{2j}^h p_{3j}^{2h} x_j [R]^h$ for $i = \{1,2\}$, $j = \{3,4\}$ and $h \geq 2$.

If $[u] = [S]$ we notice that

$$\Big[\big[[x_1, [S]], [S]\big], [S]\Big] = \Big[\big[[F], [S]\big], [S]\Big] = \alpha\big[x_2[K], [S]\big] + \beta\big[[L], [S]\big] = 0,$$

where $\alpha, \beta \in \mathbf{k}$ are described in the list in appendix. We also have $\big[x_2, [S]\big] = 0$, $\big[[S], \big[[S], x_3\big]\big] = \big[[S], [T]\big] = 0$ and $\big[[S], x_4\big] = 0$. Then by formulas (2.6) and (2.7) we obtain $x_i[S]^h = p_{i2}^h p_{i3}^h p_{i4}^h [S]^h x_i$ and $[S]^h x_j = p_{2j}^h p_{3j}^h p_{4j}^h x_j [S]^h$ for $i = \{1,2\}$, $j = \{3,4\}$ and $h \geq 3$.

Now we suppose $[u] = [T]$. In this case we have

$$\Big[\big[[x_1, [T]], [T]\big], [T]\Big] = \Big[\big[[G], [T]\big], [T]\Big] = \alpha\big[[N], [T]\big] + \beta\big[[R][K], [T]\big] = 0,$$

where $\alpha, \beta \in \mathbf{k}$ are in the appendix. We also have

$$\Big[\big[[x_2, [T]], [T]\big], [T]\Big] = \gamma\Big[\big[[S][Q], [T]\big], [T]\Big] = \theta\big[[S]^2[R], [T]\big] = 0,$$

where $\gamma, \theta \in \mathbf{k}$, $\big[[T], x_3\big] = 0$ and $\big[[T], \big[[T], x_4\big]\big] = \big[[T], [U]\big] = 0$. So by formulas (2.6) and (2.7) we obtain $x_i[T]^h = p_{i2}^h p_{i3}^{2h} p_{i4}^h [T]^h x_i$ and $[T]^h x_j = p_{2j}^h p_{3j}^{2h} p_{4j}^h x_j [T]^h$ for $i = \{1,2\}$, $j = \{3,4\}$ and $h \geq 3$.

For $[u]=[U]$ we notice that $[\big[[x_1,[U]],[U]\big]=\big[[K],[U]\big]=0$, $[\big[[x_2,[U]],[U]\big]=\alpha\big[[S]^2,[U]\big]=0$, $\big[[U],x_3\big]=0$ and $\big[[U],x_4\big]=0$. Then by formulas (2.6) and (2.7) we have $x_i[T]^h=p_{i2}^h p_{i3}^{2h} p_{i4}^h [T]^h x_i$ and $[T]^h x_j=p_{2j}^h p_{3j}^{2h} p_{4j}^h x_j [T]^h$ for $i=\{1,2\}$, $j=\{3,4\}$ and $h\geq 2$.

In the case $[u]=[V]=x_3$ clearly $[x_1,x_3^h]=0$ since $[x_1,x_3]=0$. Then $x_1x_3^h=p_{13}^h x_3^h x_1$. Now $[[[x_2,x_3],x_3],x_3]=\big[[[Q],x_3],x_3\big]=\big[[R],x_3\big]=0$, so $[x_2,x_3^h]=0$ and $x_2x_3^h=p_{23}^h x_3^h x_2$ for $h\geq 3$. Evidently $x_3^h x_3=x_3x_3^h$. Lastly we have $[x_3,[x_3,x_4]]=\big[x_3,[W]\big]=0$ then $[x_3^h,x_4]=0$ and $x_3^h x_4=p_{34}^h x_4 x_3^h$.

If $[u]=[W]$ we have $\big[x_1,[W]\big]=0$, $\big[\big[[x_2,[W]],[W]\big],[W]\big]=\big[\big[[S],[W]\big],[W]\big]=\big[[U],[W]\big]=0$ and $\big[x_3,[W]\big]=0$, then by (2.6) we obtain $\big[x_i,[W]^h\big]=0$, for $i=\{1,2,3\}$, so $x_i[W]^h=p_{i3}^h p_{i4}^h [W]^h x_i$ for $h\geq 3$. Since $\big[[W],x_4\big]=0$ (2.7) provide $\big[[W]^h,x_4\big]=0$, so $[W]^h x_4=p_{34}^h p_{44}^h x_4 [W]^h$.

Finally if $[u]=[X]=x_4$, we have that $[x_i,x_4]=0$, for $i=\{1,2\}$, then $[x_i,x_4^h]=0$ and $x_ix_4^h=p_{i4}^h x_4^h x_i$ in these cases. We notice that $[[x_3,x_4],x_4]=\big[[W],x_4\big]=0$ so $[x_3,x_4^h]=0$ and $x_3x_4^h=p_{34}^h x_4^h x_3$ for $h\geq 2$. Obviously $x_4^h x_4=x_4x_4^h$. □

We remember that $\varphi: U_q^+(F_4)\to u_q^+(F_4)$. We have the following proposition.

Proposition 4.4.2. *The set $J=\ker\varphi$ is generated by the elements $[u]^h$, where $[u]$ is an element from list* (4.2) *and h is the height of* $[u]$.

Proof. Theorem 4.1.4 proves that $[u]^h=0$ in $u_q^+(F_4)$ for $[u]$ in the list (4.2), then the elements $[u]^h$ are contained in J. Now let $v=[X]^{n_1}[W]^{n_2}\cdots[B]^{n_{23}}[A]^{n_{24}}\in J=\ker\varphi\subseteq U_q^+(F_4)$. If $n_i<h_i$ for every $i=\{1,\cdots,24\}$, where h_i is the height of the corresponding element, then v is a basis element of $u_q^+(F_4)$ and thus $\varphi(v)\neq 0$, which is a contradiction. So we assume that $n_i\geq h_i$ for some fixed i and then v is a multiple of the respective element $[u]^{h_i}$ and belongs to the ideal generated by this element. Now we consider $v=\alpha_1v_1+\alpha_2v_2\in J=\ker\varphi$, where v_1,v_2 are such as v. If $\varphi(v_1)=0$ so $\varphi(v_2)=0$, then v_1 and v_2 are multiples of elements of the form $[u]^{h_i}$. Therefore v belongs to the ideal generated by these elements. If $\varphi(v_1)\neq 0$ and $\varphi(v_2)\neq 0$ with $v_1\neq\alpha v_2$ then $\varphi(v)$ is a sum of linearly independent basis elements of $u_q^+(F_4)$, so $\varphi(v)\neq 0$, which is a contradiction. Inductively we have the same result for $v=\alpha_1v_1+\cdots\alpha_kv_k\in\ker\varphi=J$. Thus we obtain that J is generated by the elements $[u]^h$. □

As a conclusion of the previous results, the Hopf ideal J is generated by linearly independent skew central elements $[u]^h$, with $[u]\in\{[A],[B],[C],\cdots[W],[X]\}$ and h the height of $[u]$. In fact, J is not just a Hopf ideal, but a Hopf subalgebra of $U_q^+(F_4)$ [5, Lemma 4.10]. Now we calculate the combinatorial rank of $u_q^+(F_4)$.

Proposition 4.4.3. *The combinatorial rank* $\kappa(u_q^+(F_4)) \leq 4$.

Proof. Let $J = \ker\varphi$ be the Hopf ideal of $U_q^+(F_4)$. We consider $q^t = 1$ and we have that for t odd the height of PBW-generators from list (4.2) is $h = t$ and for t even the height is $h = t$ or $h = \frac{t}{2}$. From Proposition 4.3.2 we have that the only skew-primitive elements in J are $[A]^{h_1} = x_1^{h_1}$, $[P]^{h_2} = x_2^{h_2}$, $[V]^{h_3} = x_3^{h_3}$ and $[X]^{h_4} = x_4^{h_4}$. We conclude that $\{x_1^{h_1}, x_2^{h_2}, x_3^{h^3}, x_4^{h_4}\} \subseteq J_1$.

Now we consider $[u]$ belonging to the list (4.2) that has a degree smaller than $2^2 = 4$. We note that the coproduct of these elements are given as follows

$$\Delta([u]) = [u] \otimes 1 + g_{[u]} \otimes [u] + \sum_j \alpha v_j g_w \otimes w_j,$$

where the degree of v_j plus the degree of w_j equals 2 or 3 for every index j. We notice that Δ is multiplicative. Thus

$$\Delta([u]^h) = [u]^h \otimes 1 + g_{[u]}^h \otimes [u]^h + \sum_j \gamma y_j g_z \otimes z_j.$$

Suppose that t is odd. Then all PBW-generators $[u]$ have the same height t. The fact that the elements $[u]^t$ generate a Hopf subalgebra of $U_q^+(\mathfrak{g})$ implies that necessarily y_j or z_j belongs to $\{x_1^t, x_2^t, x_3^t, x_4^t\}$. So all terms from the sum depending on j are zero in $\frac{J}{J_1}$. We obtain that the the PBW-generators of degree 2 or 3 belong to J_2, as they are skew-primitive elements in $\frac{J}{J_1}$. We notice that we are not proving that the elements with total degree greater than 3 are not in J_2, as we can not guarantee that.

Let us suppose by induction that every $[u]$ with degree smaller than 2^n satisfies that $[u]^t$ belongs to J_n. If $[v]$ has degree smaller than 2^{n+1} by analysing its coproduct we have $\sum_j \alpha y_j g_y \otimes z_j$. Let us call A the degree of y_j and B the degree of z_j so $A + B < 2^{n+1} = 2.2^n$ then the degree of A is smaller than 2^n or the degree of B is smaller of 2^n. If we write y_j and z_j in the PBW-basis, using that J is a Hopf subalgebra, for every j we obtain at least one factor $[w]^{\alpha_w t}$ of y_j or z_j where the degree of $[w]$ is smaller than 2^n. By induction, $[w]^t \in J_n$ and therefore $[v]^t$ belongs to J_{n+1}.

Now we suppose that t is even. Then the PBW-generators may have height t or $\frac{t}{2}$ and we can not prove the result in general as in the previous case. However, analysing case by case, it is not difficult to see that we still have that if the total degree of $[u]$ is smaller than 2^n, than $[u]^h \in J_n$. Again we use the notation

$$\Delta([u]^h) = [u]^h \otimes 1 + g_{[u]}^h \otimes [u]^h + \sum_j \gamma y_j g_z \otimes z_j.$$

For the elements of degree one we have already proven that they are in J_1. If $[u]$ is a generator of degree 2 and $h = t$ we may have the following possibilities:

$$y_j = [v_1]^t,\ z_j = [v_2]^t, \text{ where } v_1, v_2 \text{ have degree } 1,$$

$$y_j = [v_1]^{\frac{t}{2}},\ z_j = [v_2]^t, \text{ where } v_1, v_2 \text{ have degree 2 and 1, respectively,}$$

$$y_j = [v_1]^{\frac{t}{2}},\ z_j = [v_2]^{\frac{t}{2}}, \text{ where } v_1, v_2 \text{ have degree } 2.$$

In the third case, if we had both v_1 and v_2 with degree 2, we could have a summand that would not be zero in $\frac{J}{J_1}$. However, it is easy to see that the only elements with degree 2 that have $h = t$ are $[Q] = [x_2, x_3]$ and $[W] = [x_3, x_4]$. However, it is impossible to obtain the degrees $(0, t, t, 0)$ and $(0, 0, t, t)$ as a sum of the degree $(\frac{t}{2}, \frac{t}{2}, 0, 0)$ of $[B]$. So all the elements with degree two belong to J_2.

Similarly, if $[u]$ is a generator of degree 3 we could have the following cases that would not be zero in $\frac{J}{J_1}$:

$$y_j = [v_1]^{\frac{t}{2}},\ z_j = [v_2]^{\frac{t}{2}}, \text{ where } v_1, v_2 \text{ have degree } 3,$$

$$y_j = [v_1]^{\frac{t}{2}},\ z_j = [v_2]^{\frac{t}{2}}[v_3]^{\frac{t}{2}}, \text{ or vice versa, where } v_1, v_2, v_3 \text{ have degree } 2.$$

Once again, we can not obtain the degrees of the elements of degree 3 with height t as a sum of elements of degree 2 or 3 with height $\frac{t}{2}$. Proceeding in this same way for the degrees 4,5,6,7,8,9 and 10 we obtain the wanted result. We notice that the cases with degree over 8 are even more trivial as we have only one PBW-generator in each degree.

Finally, as all the elements of the PBW-basis have degree smaller than $12 < 16 = 2^4$, the combinatorial rank of the algebra is not greater than 4.

□

Theorem 4.4.4. *The combinatorial rank* $\kappa(u_q^+(F_4)) = 4$.

Proof. The Proposition 4.4.3 shows that the combinatorial rank of $u_q^+(F_4)$ is less than or equal to 4. To prove that it is equal to 4, we show that there is a non zero element in $J_4 - J_3$. First we consider t odd.

From Theorem 4.3.2 we have that the only skew-primitive elements in J are $[A]^{h_1} = x_1^{h_1}$, $[P]^{h_2} = x_2^{h_2}$, $[V]^{h_3} = x_3^{h_3}$ and $[X]^{h_4} = x_4^{h_4}$. We define J_1 as the Hopf ideal of J generated by $x_1^{h_1}$, $x_2^{h_2}$, $x_3^{h_3}$ and $x_4^{h_4}$. Now we prove that $[u]^h \notin J_1$ for $[u]$ in the list (4.2) except x_1, x_2, x_3 and x_4. Since the generators of J are skew central, we may consider J_1 as a right (or left) ideal. Suppose that

$$[u]^h = \alpha_1 y_1 x_1^{h_1} + \alpha_2 y_2 x_2^{h_2} + \alpha_3 y_3 x_3^{h_3} + \alpha_4 y_4 x_4^{h_4}.$$

We may write $y_1, y_2, y_3, y_4 \in U_q^+(F_4)$ in the PBW-basis and then skew-commute $x_1^{h_1}$, $x_2^{h_2}$, $x_3^{h_3}$ and $x_4^{h_4}$, writing $[u]^h$ as a linear combination of basis elements of $U_q^+(F_4)$. Then, on both sides of the equality we have linear combinations of basis elements, however, on the right side we have necessarily $x_i^{h_i}$ on every term. This provides that $[u]^h$ is not one of the elements on the right side, so we have a contradiction. Therefore $[u]^h \notin J_1$, unless $[u] = x_i$ for $i = \{1, 2, 3, 4\}$.

Using the proof of the Theorem 4.3.2 we have $[B]^t, [R]^t$ and $[W]^t$ belonging to $J_2 - J_1$ due to the fact that the coproduct of these elements has a nonzero coefficient for a term $\alpha y g_z \otimes z$, where y and z belong to $\{x_1^t, x_2^t, x_3^t, x_4^t\}$.

Now we consider $n \in \mathbb{N}$. Using the formula of $\Delta([E])$, we have that the coproduct of element $[E]^n$ has a term $\alpha [B]^n g_{233}^n \otimes [R]^n$. Let us calculate the coefficient α. Analyzing the degree of the elements on the right side of the tensor of each term of the coproduct of $[E]$, we have that the only possibility to obtain the element $[R]^n$ is to multiply n times the term $\beta_2 p_{32}^2 q [B] g_{233} \otimes [R]$. Indeed, when we multiply the element $[R]$ by $x_3^2 x_2$ and $x_3[Q]$ we will always have an element starting with x_3. Thus $\alpha = \beta_2^n p_{21}^{\frac{n(n-1)}{2}} p_{31}^{n(n-1)} p_{32}^{n(n+1)} q^{n^2} \neq 0$ and

$$\Delta([E]^n) = [E]^n \otimes 1 + g_{12233}^n \otimes [E]^n + \beta_2^n p_{21}^{\frac{n(n-1)}{2}} p_{31}^{n(n-1)} p_{32}^{n(n+1)} q^{n^2} [B]^n g_{233}^n \otimes [R]^n + \sum_j \gamma w_j g_z \otimes z_j,$$

where the degree of w_j plus the degree of z_j is the degree of $[E]^n$. This is true for $n = t$ odd, so $[E]^t \notin J_1, J_2$, and then $[E]^t$ belongs to $J_3 - J_2$.

Analogously, we have that

$$\Delta([K]^n) = [K]^n \otimes 1 + g_{123344}^n \otimes [K]^n + \beta_1^n \beta_2^n (p_{31} p_{32} p_{41} p_{42})^{n(n-1)} [B]^n g_{34}^{2n} \otimes [W]^{2n} + \sum_j \gamma y_j g_z \otimes z_j,$$

where degree of y_j plus degree of z_j is the degree of $[K]^n$. In particular, $[K]^t$ belongs to $J_3 - J_2$ for t odd.

Finally we have that the coproduct of $[H]^n$ has a term of the form $\lambda[E]^n g_{123344}^n \otimes [K]^n$. Analysing the elements on the left side of each term of coproduct of $[H]$, we have that the possibility of having the degree of $[E]^n$ would be a combination of degree of the terms x_1, $[B]$, $[C]$, $[D]$ and $[E]$ in this way

$$(n, 2n, 2n, 0) = a_1(1, 0, 0, 0) + a_2(1, 1, 0, 0) + a_3(1, 1, 1, 0) + a_4(1, 1, 2, 0) + a_5(1, 2, 2, 0),$$

where $(n, 2n, 2n, 0)$, $(1, 0, 0, 0)$, $(1, 1, 0, 0)$, $(1, 1, 1, 0)$, $(1, 1, 2, 0)$ and $(1, 2, 2, 0)$ are the degree of $[E]^n$, x_1, $[B]$, $[C]$, $[D]$ and $[E]$, respectively. As a_i is a positive integer, the only way to have this equality is $a_5 = n$, that is, multiplying n times the term $\beta_1 p_{12} p_{32}^2 p_{41} p_{42}^3 p_{43}^3 q^4 [E] g_{123344} \otimes [K]$. Then we have

$$\Delta([H]^n) = [H]^n \otimes 1 + g_{11222333344}^n \otimes [H]^n + \lambda[E]^n g_{123344}^n \otimes [K]^n + \sum_j \gamma y_j g_z \otimes z_j,$$

where $\lambda = \beta_1^n p_{12}^{\frac{n(n+1)}{2}} p_{32}^{n(n+1)} p_{41}^{n^2} (p_{42}p_{43})^{n(2n+1)} q^{2n(n+1)} \neq 0$. If $n = t$ is odd we have that $[H]^t$ belongs to $J_4 - J_3$.

Analogously, when t is even, we consider $s = \frac{t}{2}$, then J_1 is generated by x_1^s, x_2^s, x_3^t and x_4^t. By the proof of Proposition 4.3.2 we have that $[B]^s, [R]^s$ and $[W]^t$ belong to $J_2 - J_1$, $[E]^s$ and $[K]^s$ belong to $J_3 - J_2$ and $[H]^t$ belongs to $J_4 - J_3$. Therefore $\kappa(u_q^+(F_4)) = 4$. □

We notice that, similarly to [15, Theorem 6.1], the result $\kappa(u_q^+(F_4)) = 4$ provides immediately the same combinatorial rank for the negative quantum Borel subalgebra $u_q^-(F_4)$. As a consequence, using the triangular decomposition we also obtain $\kappa(u_q(F_4)) = 4$.

Chapter 5

Appendix

In this appendix we list the skew commutators between the basis elements in the case F_4.

1. $\big[x_1,[B]\big]=0$
2. $\big[x_1,[C]\big]=0$
3. $\big[x_1,[D]\big]=0$
4. $\big[x_1,[E]\big]=\beta_2p_{12}p_{13}^2q^2[D][B]-\beta_1p_{12}p_{13}p_{32}q^2[C]^2$
5. $\big[x_1,[F]\big]=0$
6. $\big[x_1,[G]\big]=0$
7. $\big[x_1,[H]\big]=\beta_1\beta_2p_{12}^2p_{13}^4p_{14}^2q^4[G]^2[B]+\beta_1^2p_{12}^2p_{13}p_{14}^2p_{32}^3p_{34}q^6[F]^2[D]-\beta_1\beta_2p_{12}^2p_{13}^3p_{14}^2p_{32}q^5[G][F][C]$
8. $\big[x_1,[I]\big]=\beta_2p_{12}p_{13}^2p_{14}q^2[G][B]-\beta_1p_{12}p_{13}p_{14}p_{32}q^2[F][C]$
9. $\big[x_1,[J]\big]=\beta_1p_{12}p_{13}^2p_{14}q^2[G][C]-\beta_1p_{12}p_{13}p_{14}p_{32}q^2[F][D]$
10. $\big[x_1,[K]\big]=0$
11. $\big[x_1,[L]\big]=-\beta_1p_{12}p_{13}p_{14}p_{32}p_{42}q^2[F]^2+\beta_2p_{12}p_{13}^2p_{14}^2q^2[K][B]$
12. $\big[x_1,[M]\big]=\beta_2p_{12}p_{13}^2p_{14}^2q^2[K][C]-\beta_2p_{12}p_{13}^2p_{14}p_{42}p_{43}q^2[G][F]$
13. $\big[x_1,[N]\big]=\beta_2p_{12}p_{13}^2p_{14}^2q^2[K][D]-\beta_2p_{12}p_{13}^2p_{14}p_{42}p_{43}q^2[G]^2$
14. $\big[x_1,[O]\big]=\beta_2\beta_3p_{12}p_{13}^2p_{14}^2q^2[K][E]+\beta_2p_{12}^2p_{13}^2p_{14}^2p_{32}^2q^3[L][D]-\beta_2p_{12}^2p_{13}^3p_{14}^2p_{32}q^3[M][C]+$ $p_{12}p_{13}^2p_{14}p_{42}p_{43}(1+q)(q^{-2}+q^{-1}-q)[H]-\beta_2p_{12}^2p_{13}^2p_{14}p_{32}^2p_{42}^2p_{43}q^3(1+q)[I][G]+$ $\beta_2p_{12}^2p_{13}^4p_{14}^2q^2[N][B]+\beta_2p_{12}^2p_{13}^3p_{14}p_{32}p_{42}^2p_{43}^2q^3(1+q)[J][F]$

15. $[x_1, x_2] = [B]$

16. $\big[x_1, [Q]\big] = [C]$

17. $\big[x_1, [R]\big] = [D]$

18. $\big[x_1, [S]\big] = [F]$

19. $\big[x_1, [T]\big] = [G]$

20. $\big[x_1, [U]\big] = [K]$

21. $[x_1, x_3] = 0$

22. $\big[x_1, [W]\big] = 0$

23. $[x_1, x_4] = 0$

24. $\big[[B], [C]\big] = 0$

25. $\big[[B], [D]\big] = \beta_1 p_{13} p_{23} q^2 [C]^2$

26. $\big[[B], [E]\big] = 0$

27. $\big[[B], [F]\big] = 0$

28. $\big[[B], [G]\big] = \beta_1 p_{13} p_{14} p_{23} p_{24} q^2 [F][C]$

29. $\big[[B], [H]\big] = \beta_1^2 p_{13} p_{14}^2 p_{23} p_{24}^2 p_{34} q^3 [F]^2 [E]$

30. $\big[[B], [I]\big] = 0$

31. $\big[[B], [J]\big] = -\beta_1 p_{13} p_{14} p_{23}^2 p_{24} q^2 [F][E] + \beta_1 p_{12} p_{13}^2 p_{14} p_{23}^2 p_{24} q^4 [I][C]$

32. $\big[[B], [K]\big] = \beta_1 p_{13} p_{14} p_{23} p_{24} q^2 [F]^2$

33. $\big[[B], [L]\big] = 0$

34. $\big[[B], [M]\big] = -\beta_2 p_{12} p_{13}^2 p_{14} p_{23}^2 p_{43} q^4 [I][F] + \beta_2 p_{12} p_{13}^2 p_{14}^2 p_{23}^2 p_{24}^2 q^4 [L][C]$

35. $\big[[B], [N]\big] = p_{13}^2 p_{14} p_{23}^4 p_{24} p_{43} q^2 (q^3 - 2q - 1)[H] + \beta_2 p_{13}^2 p_{14}^2 p_{23}^4 p_{24}^2 q^3 [K][E] - \beta_2^2 p_{12} p_{13}^2 p_{14} p_{23}^2 p_{43} q^5 [I][G] - \beta_2 p_{12} p_{13}^3 p_{14} p_{23}^3 p_{43}^2 q^5 (1+q)[J][F] + \beta_1 \beta_2 p_{12} p_{13}^2 p_{14}^2 p_{23}^2 p_{24}^2 q^4 [L][D] + \beta_2 p_{12} p_{13}^3 p_{14}^2 p_{23}^3 p_{24}^2 q^5 [M][C]$

36. $\big[[B], [O]\big] = \beta_2 p_{12} p_{13}^2 p_{14}^2 p_{23}^2 p_{24}^2 q^4 [L][E] - \beta_2 p_{12} p_{13}^2 p_{14} p_{23}^2 p_{43} q^4 [I]^2$

37. $\big[[B], x_2\big] = 0$

38. $\big[[B],[Q]\big] = \beta_2 p_{12} q^2 x_2 [C]$

39. $\big[[B],[R]\big] = \beta_1\beta_2 p_{12} q^2 x_2 [D] + \beta_2 p_{12} p_{13} p_{23} q^3 [Q][C] + p_{23}^2 q [E]$

40. $\big[[B],[S]\big] = \beta_2 p_{12} q^2 x_2 [F]$

41. $\big[[B],[T]\big] = \beta_1\beta_2 p_{12} q^2 x_2 [G] + p_{23}^2 p_{24} q [I] + \beta_1 p_{12} p_{13} p_{14} p_{23} p_{24} q^2 [S][G] + \beta_1 p_{12} p_{13} p_{23} p_{43} q^3 [Q][F]$

42. $\big[[B],[U]\big] = \beta_1\beta_2 p_{12} q^2 x_2 [K] + p_{23}^2 p_{24}^2 q [L] + \beta_2 p_{12} p_{13} p_{14} p_{23} p_{24} q^3 [S][F]$

43. $\big[[B],x_3\big] = [C]$

44. $\big[[B],[W]\big] = [F]$

45. $\big[[B],x_4\big] = 0$

46. $\big[[C],[D]\big] = 0$

47. $\big[[C],[E]\big] = 0$

48. $\big[[C],[F]\big] = 0$

49. $\big[[C],[G]\big] = \beta_1 p_{14} p_{24} p_{34} q [F][D]$

50. $\big[[C],[H]\big] = \beta_1\beta_2 p_{13} p_{14}^2 p_{23} p_{24}^2 p_{34}^2 q^4 [G][F][E]$

51. $\big[[C],[I]\big] = \beta_1 p_{14} p_{24} p_{34} q [F][E]$

52. $\big[[C],[J]\big] = \beta_1 p_{12} p_{13} p_{14} p_{24} p_{34} q^2 [I][D] - \beta_1 p_{13} p_{14} p_{23}^2 p_{24} p_{34} q^2 [G][E]$

53. $\big[[C],[K]\big] = \beta_2 p_{13} p_{14} p_{23} p_{24} p_{34} q^3 [G][F]$

54. $\big[[C],[L]\big] = \beta_2 p_{12} p_{13} p_{14} p_{34} q^3 [I][F]$

55. $\big[[C],[M]\big] = -\beta_2 p_{13} p_{14}^2 p_{23}^2 p_{24}^2 p_{34}^2 q^2 [K][E] + p_{13} p_{14} p_{23}^2 p_{24} p_{34} (1 + 2q - q^3)[H] + \beta_2 p_{12} p_{13} p_{14} p_{34} q (q^2 - q - 1)[I][G] + \beta_2 p_{12} p_{13}^2 p_{14} p_{23} q^3 [J][F] + \beta_2 p_{12} p_{13} p_{14}^2 p_{24}^2 p_{34}^2 q^2 [L][D]$

56. $\big[[C],[N]\big] = -\beta_2 p_{12} p_{13}^2 p_{14} p_{23} q^2 (1+q)[J][G] + \beta_2 p_{12} p_{13}^2 p_{14}^2 p_{23} p_{24}^2 p_{34}^2 q^4 [M][D]$

57. $\big[[C],[O]\big] = -\beta_2 p_{12} p_{13}^2 p_{14} p_{23} q^2 (1+q)[J][I] + \beta_2 p_{12} p_{13}^2 p_{14}^2 p_{23} p_{24}^2 p_{34}^2 q^4 [M][E]$

58. $\big[[C],x_2\big] = 0$

59. $\big[[C],[Q]\big] = \beta_2 p_{12} p_{32} q^2 x_2 [D] - p_{23} [E]$

60. $\big[[C],[R]\big] = \beta_2 p_{12} p_{13} q^2 [Q][D]$

61. $\big[[C],[S]\big] = -p_{23}p_{24}p_{34}q[I] + \beta_2 p_{12}p_{32}p_{34}q^3x_2[G] + \beta_1 p_{12}p_{13}q[Q][F]$

62. $\big[[C],[T]\big] = p_{23}p_{24}p_{34}[J] + \beta_1 p_{12}p_{13}q(1+\beta_2 q)[Q][G] + \beta_1 p_{12}p_{13}p_{14}p_{24}p_{34}q[S][D] + \beta_1 p_{12}p_{13}^2p_{23}p_{43}q^2[R][F]$

63. $\big[[C],[U]\big] = p_{23}p_{24}^2p_{34}^2q[M] + \beta_1\beta_2 p_{12}p_{13}q^2[Q][K] + \beta_2 p_{12}p_{13}^2p_{14}p_{23}p_{24}p_{34}q^3[T][F] + \beta_2 p_{12}p_{13}p_{14}p_{24}p_{34}^2q^3[S][G]$

64. $\big[[C],x_3\big] = [D]$

65. $\big[[C],[W]\big] = p_{34}q[G] + \beta_1 p_{13}p_{23}qx_3[F]$

66. $\big[[C],x_4\big] = [F]$

67. $\big[[D],[E]\big] = 0$

68. $\big[[D],[F]\big] = 0$

69. $\big[[D],[G]\big] = 0$

70. $\big[[D],[H]\big] = \beta_1\beta_2 p_{14}^2p_{24}^2p_{34}^4q^4[G]^2[E]$

71. $\big[[D],[I]\big] = \beta_2 p_{14}p_{24}p_{34}^2q^2[G][E]$

72. $\big[[D],[J]\big] = 0$

73. $\big[[D],[K]\big] = \beta_2 p_{14}p_{24}p_{34}^3q^3[G]^2$

74. $\big[[D],[L]\big] = p_{14}p_{24}p_{34}^3(q^3-2q-1)[H] + \beta_2 p_{14}^2p_{24}^2p_{34}^4q^2[K][E] + \beta_2 p_{12}p_{14}p_{32}^2p_{34}^3q^5(1+q)[I][G]$

75. $\big[[D],[M]\big] = \beta_2 p_{12}p_{13}p_{14}p_{32}p_{34}^2q^4(1+q)[J][G]$

76. $\big[[D],[N]\big] = 0$

77. $\big[[D],[O]\big] = \beta_2 p_{12}p_{13}^2p_{14}^2p_{24}^2p_{34}^4q^4[N][E] - \beta_2 p_{12}p_{13}p_{14}p_{32}^2p_{34}^2q^4(1+q)[J]^2$

78. $\big[[D],x_2\big] = [E]$

79. $\big[[D],[Q]\big] = 0$

80. $\big[[D],[R]\big] = 0$

81. $\big[[D],[S]\big] = -p_{24}p_{34}^2(1+q)[J] + \beta_2 p_{12}p_{13}p_{32}p_{34}q^3[Q][G]$

82. $\big[[D],[T]\big] = \beta_2 p_{12}p_{13}^2q^2[R][G]$

83. $\big[[D],[U]\big] = p_{24}^2p_{34}^4q[N] + \beta_2p_{12}p_{13}^2p_{14}p_{24}p_{34}^3q^3(1+q)[T][G] + \beta_1\beta_2p_{12}p_{13}^2q^2[R][K]$

84. $\big[[D],x_3\big] = 0$

85. $\big[[D],[W]\big] = \beta_2p_{13}p_{23}p_{34}q^3x_3[G]$

86. $\big[[D],x_4\big] = p_{34}(1+q)[G]$

87. $\big[[E],[F]\big] = 0$

88. $\big[[E],[G]\big] = 0$

89. $\big[[E],[H]\big] = 0$

90. $\big[[E],[I]\big] = 0$

91. $\big[[E],[J]\big] = 0$

92. $\big[[E],[K]\big] = p_{14}p_{21}p_{23}^2p_{24}^3p_{34}^3q^4(1+q)[H]$

93. $\big[[E],[L]\big] = \beta_2p_{14}p_{24}^2p_{34}^3q^3[I]^2$

94. $\big[[E],[M]\big] = \beta_2p_{13}p_{14}p_{23}^2p_{24}^2p_{34}^2q^4(1+q)[J][I]$

95. $\big[[E],[N]\big] = \beta_2p_{13}p_{14}p_{23}^2p_{24}^2p_{34}^2q^4(1+q)[J]^2$

96. $\big[[E],[O]\big] = 0$

97. $\big[[E],x_2\big] = 0$

98. $\big[[E],[Q]\big] = 0$

99. $\big[[E],[R]\big] = 0$

100. $\big[[E],[S]\big] = \beta_2p_{12}p_{13}p_{24}p_{34}q^3[Q][I] - \beta_2p_{12}p_{24}p_{32}^2p_{34}^2q^4(1+q)x_2[J]$

101. $\big[[E],[T]\big] = -\beta_2p_{12}p_{13}p_{24}p_{34}q^2[Q][J] + \beta_2p_{12}p_{13}^2p_{23}^2p_{24}q^4[R][I]$

102. $\big[[E],[U]\big] = -\beta_1\beta_2p_{12}p_{13}p_{24}^2p_{34}^2q^3[Q][M] + \beta_1\beta_2p_{12}p_{13}^2p_{14}p_{23}^2p_{24}^3p_{34}^3q^6[T][I] + \beta_1\beta_2p_{12}p_{24}^2p_{32}^2p_{34}^4q^5x_2[N] + p_{23}^2p_{24}^4p_{34}^4q^2[O] - \beta_2p_{12}p_{13}p_{14}p_{24}^3p_{34}^4q^3(1+q)[S][J] + \beta_1\beta_2p_{12}p_{13}^2p_{23}^2p_{24}^2q^4[R][L]$

103. $\big[[E],x_3\big] = 0$

104. $\big[[E],[W]\big] = -p_{24}p_{34}^2(1+q)J + \beta_2p_{13}p_{23}^2p_{24}p_{34}q^3x_3[I]$

105. $\big[[E],x_4\big] = p_{24}p_{34}(1+q)[I]$

106. $\big[[F],[G]\big] = 0$

107. $\big[[F],[H]\big] = 0$

108. $\big[[F],[I]\big] = 0$

109. $\big[[F],[J]\big] = -p_{13}p_{23}^2p_{43}q^2[H] + \beta_1 p_{12}p_{13}p_{42}p_{43}q^2[I][G]$

110. $\big[[F],[K]\big] = 0$

111. $\big[[F],[L]\big] = 0$

112. $\big[[F],[M]\big] = \beta_2 p_{12}p_{13}p_{14}q^2[L][G] - \beta_2 p_{13}p_{14}p_{23}^2p_{24}q^2[K][I]$

113. $\big[[F],[N]\big] = -\beta_2 p_{13}p_{14}p_{23}^2p_{24}q^2(1+q)[K][J] + \beta_2 p_{12}p_{13}^2p_{14}p_{23}p_{43}q^4[M][G]$

114. $\big[[F],[O]\big] = \beta_2 p_{12}p_{13}^2p_{14}p_{23}p_{43}q^4[M][I] - \beta_2 p_{12}p_{13}p_{14}p_{32}q^2(1+q)[L][J]$

115. $\big[[F],x_2\big] = 0$

116. $\big[[F],[Q]\big] = \beta_2 p_{12}p_{32}p_{42}q^2x_2[G] - p_{23}[I]$

117. $\big[[F],[R]\big] = \beta_2 p_{12}p_{13}p_{42}p_{43}q^2[Q][G] - p_{23}(1+q)[J]$

118. $\big[[F],[S]\big] = \beta_2 p_{12}p_{32}p_{42}q^2x_2[K] - p_{23}p_{24}[L]$

119. $\big[[F],[T]\big] = -p_{23}p_{24}q(1+q)^{-1}[M] + \beta_1 p_{12}p_{13}p_{14}q[S][G] + \beta_1 p_{12}p_{13}p_{42}p_{43}^2q^2[Q][K]$

120. $\big[[F],[U]\big] = \beta_2 p_{12}p_{13}p_{14}q^2[S][K]$

121. $\big[[F],x_3\big] = [G]$

122. $\big[[F],[W]\big] = [K]$

123. $\big[[F],x_4\big] = 0$

124. $\big[[G],[H]\big] = 0$

125. $\big[[G],[I]\big] = [H]$

126. $\big[[G],[J]\big] = 0$

127. $\big[[G],[K]\big] = 0$

128. $\big[[G],[L]\big] = \beta_2 p_{14}p_{24}p_{34}^2q^2[K][I]$

129. $\big[[G],[M]\big] = \beta_2 p_{14}p_{24}p_{34}^2q^2[K][J]$

130. $\big[[G],[N]\big] = 0$

131. $\big[[G],[O]\big] = \beta_2 p_{12} p_{13}^2 p_{14} q^2 [N][I] - \beta_2 p_{12} p_{13} p_{14} p_{32}^2 p_{34} q^4 [M][J]$

132. $\big[[G],x_2\big] = [I]$

133. $\big[[G],[Q]\big] = [J]$

134. $\big[[G],[R]\big] = 0$

135. $\big[[G],[S]\big] = -p_{24} p_{34} (1+q)^{-1} [M] + \beta_1 p_{12} p_{13} p_{32} p_{42} p_{43} q^2 [Q][K]$

136. $\big[[G],[T]\big] = -p_{24} p_{34} (1+q)^{-1} [N] + \beta_1 p_{12} p_{13}^2 p_{42} p_{43}^3 q^2 [R][K]$

137. $\big[[G],[U]\big] = \beta_2 p_{12} p_{13}^2 p_{14} q^2 [T][K]$

138. $\big[[G],x_3\big] = 0$

139. $\big[[G],[W]\big] = \beta_1 p_{13} p_{23} p_{43} q^2 x_3 [K]$

140. $\big[[G],x_4\big] = [K]$

141. $\big[[H],[I]\big] = 0$

142. $\big[[H],[J]\big] = 0$

143. $\big[[H],[K]\big] = 0$

144. $\big[[H],[L]\big] = \beta_1 \beta_2 p_{14}^2 p_{24}^3 p_{34}^4 q^4 [K][I]^2$

145. $\big[[H],[M]\big] = \beta_2^2 p_{13} p_{14}^2 p_{23}^2 p_{24}^3 p_{34}^3 q^6 [K][J][I]$

146. $\big[[H],[N]\big] = \beta_2^2 p_{13} p_{14}^2 p_{23}^2 p_{24}^3 p_{34}^3 q^6 [K][J]^2$

147. $\big[[H],[O]\big] = \beta_2^2 p_{12}^2 p_{13} p_{14}^2 p_{24} p_{32}^4 p_{34}^3 q^9 [L][J]^2 - \beta_2^2 p_{12}^2 p_{13}^3 p_{14}^2 p_{24} p_{34} q^5 [M][J][I] + \beta_1 \beta_2 p_{12}^2 p_{13}^4 p_{14}^2 p_{23}^2 p_{24} q^6 [N][$

148. $\big[[H],x_2\big] = \beta_1 p_{12} p_{32}^2 p_{42} q^4 [I]^2$

149. $\big[[H],[Q]\big] = \beta_2 p_{12} p_{13} p_{42} p_{43} q^3 [J][I]$

150. $\big[[H],[R]\big] = \beta_2 p_{12} p_{13} p_{42} p_{43} q^3 [J]^2$

151. $\big[[H],[S]\big] = \beta_2 p_{12} p_{14} p_{24}^2 p_{32}^2 p_{34}^3 q^4 [L][J] - \beta_1 p_{12} p_{13} p_{14} p_{24}^2 p_{34}^2 q^2 [M][I] - \beta_2^2 p_{12}^2 p_{14} p_{32}^4 p_{34}^3 q^8 x_2 [K][J] +$
$\beta_1 \beta_2 p_{12}^2 p_{13}^2 p_{14} p_{32} q^4 [Q][K][I]$

152. $\big[[H],[T]\big] = -\beta_1 p_{12} p_{13}^2 p_{14} p_{23}^2 p_{24}^2 p_{34} q^3 [N][I] + \beta_1 p_{12} p_{13} p_{14} p_{24}^2 p_{34}^2 q [M][J] - \beta_1 \beta_2 p_{12}^2 p_{13}^2 p_{14} p_{32} q^3 [Q][K][J]$
$\beta_1 \beta_2 p_{12}^2 p_{13}^4 p_{14} p_{23}^2 p_{43}^3 q^6 [R][K][I]$

153. $\big[[H],[U]\big] = \beta_1 p_{12}p_{13}p_{14}p_{24}^3p_{34}^3q^2(1+q)^{-1}[M]^2+\beta_2^2p_{12}^2p_{13}^4p_{14}^3p_{23}^2p_{24}^3p_{34}^2q^7[T][K][I]- \beta_1p_{12}p_{13}^2p_{14}p_{23}^2p_{24}^3p_{34}q^3[N][L]+\beta_1p_{12}^2p_{13}^2p_{14}p_{24}p_{34}q[O][K]+\beta_1^2\beta_2p_{12}^3p_{13}^2p_{14}p_{32}^4p_{34}p_{42}q^8x_2[N][K]+ \beta_1^2\beta_2p_{12}^3p_{13}^4p_{14}p_{42}p_{43}^3q^6[R][L][K]+\beta_1\beta_2^2p_{12}^3p_{13}^3p_{14}p_{32}^2p_{42}p_{43}q^8[Q][M][K]-\beta_2^2p_{12}^2p_{13}^2p_{14}^3p_{24}^3p_{32}p_{34}^5q^7[S][K][$

154. $\big[[H],x_3\big] = 0$

155. $\big[[H],[W]\big] = \beta_1\beta_2p_{13}^2p_{14}p_{23}^3p_{24}^2q^4x_3[K][I] - \beta_2p_{14}p_{24}^2p_{34}^3q^2[K][J]$

156. $\big[[H],x_4\big] = \beta_2p_{14}p_{24}^2p_{34}^2q^2[K][I]$

157. $\big[[I],[J]\big] = 0$

158. $\big[[I],[K]\big] = 0$

159. $\big[[I],[L]\big] = 0$

160. $\big[[I],[M]\big] = \beta_2p_{14}p_{24}^2p_{34}^2q^2[L][J]$

161. $\big[[I],[N]\big] = \beta_2p_{13}p_{14}p_{23}^2p_{24}^2p_{34}q^4[M][J]$

162. $\big[[I],[O]\big] = 0$

163. $\big[[I],x_2\big] = 0$

164. $\big[[I],[Q]\big] = \beta_2p_{12}p_{32}^2p_{42}q^4x_2[J]$

165. $\big[[I],[R]\big] = \beta_2p_{12}p_{13}p_{42}p_{43}q^3[Q][J]$

166. $\big[[I],[S]\big] = -\beta_1p_{12}p_{32}^2p_{34}q^3x_2[M] + \beta_1p_{12}p_{13}p_{43}q^2[Q][L]$

167. $\big[[I],[T]\big] = -p_{23}^2p_{24}^2p_{34}q(1+q)^{-1}[O] - \beta_1^2p_{12}p_{32}^2p_{34}q^3x_2[N] - \beta_1p_{12}p_{13}p_{43}q^2(\beta_1 - (1+q)^{-1})[Q][M] + \beta_1p_{12}p_{13}^2p_{23}^2p_{43}^3q^4[R][L] + \beta_1p_{12}p_{13}p_{14}p_{24}p_{34}q^2[S][J]$

168. $\big[[I],[U]\big] = \beta_2p_{12}p_{13}^2p_{14}p_{23}^2p_{24}^2q^4[T][L] - \beta_1p_{12}p_{13}p_{14}p_{24}^2p_{34}^2q^2[S][M]$

169. $\big[[I],x_3\big] = [J]$

170. $\big[[I],[W]\big] = -p_{24}p_{34}(1+q)^{-1}[M] + \beta_1p_{13}p_{23}^2p_{24}p_{43}q^2x_3[L]$

171. $\big[[I],x_4\big] = p_{24}[L]$

172. $\big[[J],[K]\big] = 0$

173. $\big[[J],[L]\big] = 0$

174. $\big[[J],[M]\big] = 0$

175. $\big[[J],[N]\big]=0$

176. $\big[[J],[O]\big]=0$

177. $\big[[J],x_2\big]=0$

178. $\big[[J],[Q]\big]=0$

179. $\big[[J],[R]\big]=0$

180. $\big[[J],[S]\big]=-\beta_1p_{12}p_{32}^3p_{34}^2q^4x_2[N]+\beta_1p_{12}p_{13}p_{32}q^3(1+q)^{-1}[Q][M]+p_{23}p_{24}^2p_{34}^2q(1+q)^{-1}[O]$

181. $\big[[J],[T]\big]=-\beta_1p_{12}p_{13}p_{32}q^2(1+q)^{-1}[Q][N]+\beta_1p_{12}p_{13}^2p_{23}p_{43}^2q^4(1+q)^{-1}[R][M]$

182. $\big[[J],[U]\big]=\beta_1p_{12}p_{13}^2p_{14}p_{23}p_{24}^2p_{34}^2q^4[T][M]-\beta_1p_{12}p_{13}p_{14}p_{24}^2p_{32}p_{34}^4q^4[S][N]$

183. $\big[[J],x_3\big]=0$

184. $\big[[J],[W]\big]=-p_{24}p_{34}^2q(1+q)^{-1}[N]+\beta_1p_{13}p_{23}^2p_{24}q^3(1+q)^{-1}x_3[M]$

185. $\big[[J],x_4\big]=p_{24}p_{34}q(1+q)^{-1}[M]$

186. $\big[[K],[L]\big]=0$

187. $\big[[K],[M]\big]=0$

188. $\big[[K],[N]\big]=0$

189. $\big[[K],[O]\big]=\beta_2p_{12}p_{13}^2p_{42}^2p_{43}^4q^4[N][L]-\beta_1p_{12}p_{13}p_{32}^2p_{42}^2p_{43}^2q^4[M]^2$

190. $\big[[K],x_2\big]=[L]$

191. $\big[[K],[Q]\big]=[M]$

192. $\big[[K],[R]\big]=[N]$

193. $\big[[K],[S]\big]=0$

194. $\big[[K],[T]\big]=0$

195. $\big[[K],[U]\big]=0$

196. $\big[[K],x_3\big]=0$

197. $\big[[K],[W]\big]=0$

198. $\big[[K], x_4\big] = 0$

199. $\big[[L], [M]\big] = 0$

200. $\big[[L], [N]\big] = \beta_1 p_{13} p_{23}^2 p_{43}^2 q^4 [M]^2$

201. $\big[[L], [O]\big] = 0$

202. $\big[[L], x_2\big] = 0$

203. $\big[[L], [Q]\big] = \beta_2 p_{12} p_{32}^2 p_{42}^2 q^4 x_2 [M]$

204. $\big[[L], [R]\big] = p_{23}^2 q[O] + \beta_1 \beta_2 p_{12} p_{32}^2 p_{42}^2 q^4 x_2 [N] + \beta_2 p_{12} p_{13} p_{42}^2 p_{43}^2 q^3 [Q][M]$

205. $\big[[L], [S]\big] = 0$

206. $\big[[L], [T]\big] = \beta_1 p_{12} p_{13} p_{14} q^2 [S][M]$

207. $\big[[L], [U]\big] = 0$

208. $\big[[L], x_3\big] = [M]$

209. $\big[[L], [W]\big] = 0$

210. $\big[[L], x_4\big] = 0$

211. $\big[[M], [N]\big] = 0$

212. $\big[[M], [O]\big] = 0$

213. $\big[[M], x_2\big] = 0$

214. $\big[[M], [Q]\big] = -p_{23}[O] + \beta_2 p_{12} p_{32}^3 p_{42}^2 q^4 x_2 [N]$

215. $\big[[M], [R]\big] = \beta_2 p_{12} p_{13} p_{32} p_{42}^2 p_{43}^2 q^4 [Q][N]$

216. $\big[[M], [S]\big] = 0$

217. $\big[[M], [T]\big] = \beta_1 p_{12} p_{13} p_{14} p_{32} p_{34} q^3 [S][N]$

218. $\big[[M], [U]\big] = 0$

219. $\big[[M], x_3\big] = [N]$

220. $\big[[M], [W]\big] = 0$

221. $\big[[M], x_4\big] = 0$

222. $\big[[N],[O]\big]=0$

223. $\big[[N],x_2\big]=[O]$

224. $\big[[N],[Q]\big]=0$

225. $\big[[N],[R]\big]=0$

226. $\big[[N],[S]\big]=0$

227. $\big[[N],[T]\big]=0$

228. $\big[[N],[U]\big]=0$

229. $\big[[N],x_3\big]=0$

230. $\big[[N],[W]\big]=0$

231. $\big[[N],x_4\big]=0$

232. $\big[[O],x_2\big]=0$

233. $\big[[O],[Q]\big]=0$

234. $\big[[O],[R]\big]=0$

235. $\big[[O],[S]\big]=0$

236. $\big[[O],[T]\big]=0$

237. $\big[[O],[U]\big]=0$

238. $\big[[O],x_3\big]=0$

239. $\big[[O],[W]\big]=0$

240. $\big[[O],x_4\big]=0$

241. $\big[x_2,[Q]\big]=0$

242. $\big[x_2,[R]\big]=\beta_1 p_{23} q^2 [Q]^2$

243. $\big[x_2,[S]\big]=0$

244. $\big[x_2,[T]\big]=\beta_1 p_{23} p_{24} q^2 [S][Q]$

245. $\big[x_2,[U]\big]=\beta_1 p_{23} p_{24} q^2 [S]^2$

246. $[x_2, x_3] = [Q]$

247. $\big[x_2, [W]\big] = [S]$

248. $[x_2, x_4] = 0$

249. $\big[[Q], [R]\big] = 0$

250. $\big[[Q], [S]\big] = 0$

251. $\big[[Q], [T]\big] = \beta_1 p_{24} p_{34} q [S][R]$

252. $\big[[Q], [U]\big] = \beta_2 p_{23} p_{24} p_{34} q^3 [T][S]$

253. $\big[[Q], x_3\big] = [R]$

254. $\big[[Q], [W]\big] = p_{34} q [T] + \beta_1 p_{23} q x_3 [S]$

255. $\big[[Q], x_4\big] = [S]$

256. $\big[[R], [S]\big] = 0$

257. $\big[[R], [T]\big] = 0$

258. $\big[[R], [U]\big] = \beta_2 p_{24} p_{34}^3 q^3 [T]^2$

259. $\big[[R], x_3\big] = 0$

260. $\big[[R], [W]\big] = \beta_2 p_{23} p_{34} q^3 x_3 [T]$

261. $\big[[R], x_4\big] = p_{34}(1 + q)[T]$

262. $\big[[S], [T]\big] = 0$

263. $\big[[S], [U]\big] = 0$

264. $\big[[S], x_3\big] = [T]$

265. $\big[[S], [W]\big] = [U]$

266. $\big[[S], x_4\big] = 0$

267. $\big[[T], [U]\big] = 0$

268. $\big[[T], x_3\big] = 0$

269. $\big[[T], [W]\big] = \beta_1 p_{23} p_{43} q^2 x_3 [U]$

270. $\big[[T], x_4\big] = [U]$

271. $\big[[U], x_3\big] = 0$

272. $\big[[U], [W]\big] = 0$

273. $\big[[U], x_4\big] = 0$

274. $\big[x_3, [W]] = 0$

275. $[x_3, x_4] = W$

276. $\big[[W], x_4\big] = 0$

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