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EXTERIOR DIRICHLET PROBLEMS FOR DEGENERATE p-LAPLACIAN TYPE EQUATIONS AND THE FRACTIONAL p-LAPLACIAN EQUATION

Porto Alegre, RS
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Tese apresentada ao Programa de Pós-Graduação em Matemática, Área de Equações Diferenciais Parciais, da Universidade Federal do Rio Grande do Sul (UFRGS, RS), como requisito parcial para obtenção do título de Doutor em Matemática.

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ABSTRACT

EXTERIOR DIRICHLET PROBLEMS FOR DEGENERATE p-LAPLACIAN TYPE EQUATIONS AND THE FRACTIONAL p-LAPLACIAN EQUATION

Author: Filipe Jung dos Santos

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We prove the existence of a unique bounded weak solution in $C(\overline{\mathbb{R}^n \setminus K}) \cap W^{1,p}_{loc}(\mathbb{R}^n \setminus K)$ of the exterior Dirichlet problem

$$\begin{cases} -\mathrm{div}\big(\,|\nabla u|^{p-2}A(\,|\nabla u|\,)\nabla u\,\big) = f & \text{in } \mathbb{R}^n \backslash K \\ \qquad\qquad u = \phi & \text{in } \partial K \end{cases}$$

for any nonempty compact $K \subset \mathbb{R}^n$ and boundary values $\phi \in C(\partial K)$, provided that $p > n$ and $f \in L^\infty(\mathbb{R}^n)$ satisfy for positive constants C_f, ϵ,

$$|f(x)| \;\leq\; C_f\,|x|^{-p-\epsilon}\,, \quad \text{for all} \;\; |x| \;\; \text{sufficiently large.} \tag{0.1}$$

We also show that, for any $p > 1$, any semibounded solution u of the equation on an exterior domain converge at infinity, with a possible infinite limit in case u is unbounded, and we prove the convergence rate has a positive order in case u is bounded and $p > n$.

On the fractional p-Laplacian operator

$$(-\Delta)^s_p\, u(x) \;=\; \text{p.v.} \int\limits_{\mathbb{R}^n} \frac{\,|\,u(x) - u(y)\,|^{\,p-2}\,\big(\,u(x) - u(y)\,\big)}{|\,x - y\,|^{n+sp}}\, dy$$

we prove that the radially symmetric functions $|x|^{\frac{sp-n}{p-1}}$, if $sp \neq n$, and $\log|x|$, if $sp = n$, are solutions of the fractional p-Laplacian equation $(-\Delta)^s_p u = 0$ in $\mathbb{R}^n \setminus \{0\}$; we then extend the existence result above, obtaining in case $sp > n$ the existence and uniqueness of continuous up to the boundary solutions to the exterior Dirichlet problem for the homogeneous p-Laplacian equation.

Keywords: Exterior Problem; p-Laplacian Equations; Fractional p-Laplacian.

RESUMO

PROBLEMAS DE DIRICHLET EXTERIORES PARA EQUAÇÕES DEGENERADAS E DO TIPO p-LAPLACIANO FRACIONÁRIO

Autor: Filipe Jung dos Santos

Orientador: Leonardo Prange Bonorino

Provamos a existência de uma única solução fraca limitada em $C(\overline{\mathbb{R}^n \setminus K}) \cap W^{1,p}_{loc}(\mathbb{R}^n \setminus K)$ para o problema de Dirichlet exterior

$$\begin{cases} -\mathrm{div}\big(\,|\nabla u|^{p-2} A(\,|\nabla u|\,)\nabla u\,\big) = f & \text{in } \mathbb{R}^n \backslash K \\ u = \phi & \text{in } \partial K \end{cases}$$

para quaisquer compacto não-vazio $K \subset \mathbb{R}^n$ e dado de fronteira $\phi \in C(\partial K)$, desde que $p > n$ e $f \in L^\infty(\mathbb{R}^n)$ satisfaça para constantes positivas C_f, ϵ,

$$|f(x)| \;\leq\; C_f\,|x|^{-p-\epsilon}\,, \quad \text{para todo } \;|x|\; \text{ suficientemente grande.} \tag{0.2}$$

Mostramos também que, para $p > 1$, as soluções limitadas acima ou abaixo u da equação em um domínio exterior convergem no infinito, possivelmente para um limite infinito caso u seja ilimitada, e provamos no caso $p > n$ que a solução tem uma ordem de convergência positiva no infinito. Para o operador p-Laplaciano fracionário

$$(-\Delta)^s_p\, u(x) \;=\; \text{p.v.} \int\limits_{\mathbb{R}^n} \frac{\big|\, u(x) - u(y)\,\big|^{\,p-2}\,\big(\,u(x) - u(y)\,\big)}{|\; x - y\;|^{n+sp}}\, dy$$

provamos que as funções $|x|^{\frac{sp-n}{p-1}}$, se $sp \neq n$, e $\log|x|$, se $sp = n$, são soluções da equação homogênea $(-\Delta)^s_p\, u = 0$ em $\mathbb{R}^n \backslash \{0\}$; estendemos o resultado de existência acima, obtendo para the $sp > n$ existêcia e unicidade de uma solução contínua até a fronteira do problema de Dirichlet exterior para a equação homogênea $(-\Delta)_p\, u = 0$.

Palavras-Chave: Problema Exterior; Equações do Tipo p-Laplaciano; p-Laplaciano Fracioário.

Contents

INTRODUCTION

In the first part of this work, we consider p-laplacian type equations driven by the degenerate divergence form operator

$$-\mathrm{div}\big(\,|\nabla u|^{p-2}A(|\nabla u|)\nabla u\,\big) \tag{0.3}$$

defined in the weak sense for functions $u \in W^{1,p}$; the function A is assumed to satisfy

$$\begin{aligned}
&i)\;\; A \in C^1\big([0,+\infty]\big)\,,\;\; A(0) > 0;\\
&ii)\;\; \delta \le A \le L\,,\;\text{ for positive constants }\; \delta, L\,;\\
&iii)\;\; \delta'\,t^{p-2} \;\le\; \frac{d}{dt}\Big\{\,t^{p-1}A(t)\,\Big\} \le L'\,t^{p-2}\,,\;\text{ for positive constants }\; \delta', L'\,,\;\text{ for all }\; t \ge 0\,.
\end{aligned} \tag{0.4}$$

This generalizes, for example, the p-laplacian operator

$$\Delta_p\, u \;=\; \mathrm{div}\big(\,|\nabla u|^{p-2}\,\nabla u\,\big)$$

In case $p = 2$, $A(t) = \frac{1}{\sqrt{1+t^2}}$ and it is known a priori that $|\nabla u| \le C$, our operator also cover the mean-curvature operator

$$\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)$$

Our first result states the existence and uniqueness of continuous bounded weak solutions for Dirichlet problems on exterior domains in case $p > n$.

Theorem 1. Let $K \subset \mathbb{R}^n$ be a nonempty compact set and $\phi \in C(\partial K)$. Assume that the function A satisfies (0.4) and $f \in L^\infty(\mathbb{R}^n)$ be such that, for positive constants C_f and ϵ,

$$|f(x)| \;\le\; C_f\,|x|^{-(p+\epsilon)} \tag{0.5}$$

for all $|x|$ sufficiently large. Then, if $p > n$, there exists a unique bounded solution $u \in C(\overline{\mathbb{R}^n \setminus K}) \cap W^{1,p}_{loc}(\mathbb{R}^n \setminus K)$ of

$$\begin{cases} -\mathrm{div}\big(\,|\nabla u|^{p-2}A(\,|\nabla u|\,)\nabla u\,\big) = f & \text{in } \mathbb{R}^n \setminus K \\ u = \phi & \text{in } \partial K \end{cases} \tag{0.6}$$

In addition, if ϕ is α-Hölder continuous in K, with $\alpha = \frac{p-n}{p-1}$, then $u \in C^\alpha(\mathbb{R}^n)$.

We point out in the full generality allowed for the boundary ∂K, for which no regularity has to be assumed. Moreover, as it is straightforward from the proof, the result also holds on bounded domains, being necessary only to assume $f \in L^\infty(\mathbb{R}^n)$.

Many efforts were directed to elliptic problems on unbounded domains. For instance, Meyers and Serrin [39] have made important clarifications on the existence and uniqueness of bounded solutions of linear exterior problems, and some of their results were extended to several classes of semilinear equations by Kusano [28], Ogata [44], Noussair [42, 43], Furusho et. al. [13, 14], Phuong Các [50], among others.

In existence results for boundary value problems, some smoothness of the domain is in general required to ensure that solutions continuously attain the prescribed boundary data. In the classical potential theory, the domains for which there is a solution of the Laplace equation continuously attaining any prescribed continuous boundary data were called regular domains and its boundary points characterized by a criterion introduced by Wiener. Later a Wiener-type condition involving Serrin's concept of p-capacity was introduced by Maz'ya and its sufficiency for regularity of boundary points was proven for a large class of quasilinear equations in divergence form. The necessity of Maz'ya's condition was then established by Kilpeläinen and Malý. In our result on regularity up to the boundary, we make crucial use of barrier arguments following the ideas of Serrin [61], where the main concern was the Liouville property for entire solutions. An extension of the Liouville property for exterior solutions was obtained by Bonorino et. al. [3], which motivated our first theorem, as it generalizes a result in [3] which states that, in case $p > n$, for a finite set $P \subset \mathbb{R}^n$, there exists a bounded p-harmonic function in $\mathbb{R}^n \setminus P$ attaining any prescribed data in P.

Another question arising on exterior problems is the behavior of solutions at infinity. This relates to the theory of singularities of solutions and a variety of results on removable singularities and the asymptotic behavior for several equations were obtained by Serrin [57, 58, 59, 60], Serrin and Weinberger [62], and others. [58] presents a detailed description of the asymptotic behavior at the origin and at infinity of positive solutions of the homogeneous quasilinear equation $\mathrm{div}\mathcal{A}(x, Du) = 0$. It was shown that positive solutions u always converge at infinity to a possibly infinite limit ℓ and, moreover, either u satisfies a *maximum principle at infinity* or else ℓ is infinite if $p \geq n$ and finite if $p < n$, and it holds

$$\begin{aligned} u &\approx r^{(p-n)/(p-1)}, & p > n \\ u &\approx \log r, & p = n \\ u - \ell &\approx \pm r^{(p-n)/(p-1)}, & p < n \end{aligned}$$

where $u \approx v$ means that there exists positive constants c, C such that $c\,v \leq u \leq C\,v$. The function u is said to satisfy the maximum principle property at infinity if in any neighbourhood of infinity either u is constant or else takes on values both greater and less than $\ell < +\infty$. More recently, it was proved in [15] the existence of limit near singularities for nonnegative solutions of

$$-\Delta_p u + V\,|u|^{p-2}\,u = 0$$

assuming that near the singularity the potential V belong to a *Kato class* and

$$V \in L^\infty_{loc}\,,\quad |x|^p\,|V(x)| \le C\,,\ \text{ for some constant } C\,.$$

In our second result, for $p > 1$, we obtain the existence of the limit at infinity for nonnegative solutions $u \in C^1(\mathbb{R}^n \setminus K)$ of

$$\operatorname{div}\big(\,|\,\nabla u\,|^{p-2}A(\,|\,\nabla u\,|\,)\nabla u\,\big) = f \ \text{ in } \ \mathbb{R}^n \setminus K \tag{0.7}$$

assuming the weaker hypotheses on A

$$\begin{aligned} &i)\ \ A \in C\big([0,+\infty]\big)\,,\ \ A(0) > 0\,;\\ &ii)\ \ \delta \le A \le L\,,\ \text{for positive constants }\ \delta, L\,;\\ &iii)\ \ t \mapsto t^{p-1}A(t)\ \text{is strictly increasing for }\ t > 0\,. \end{aligned} \tag{0.8}$$

We prove that condition (0.5) is sufficient for the existence of the limit at infinity and, in case $p > n$, we show that, if $\ell = \lim_{|x|\to\infty} u(x) < \infty$, the convergence has a positive order β.

On the matter of the behavior of the solutions at infinity, we can assume with no loss of generality $K = \overline{B_1}$, as well as the validity of condition (0.5) for all $|x| \ge 1$. Our second theorem then reads as follows.

Theorem 2. For $p > 1$, let $u \in C^1(\mathbb{R}^n \setminus \overline{B_1})$ be a weak solution of (0.7) in $\mathbb{R}^n \setminus \overline{B_1}$, with A satisfying (0.8), and assume f satisfy (0.5). Then

i) If u is bounded from above or below, then $\ell = \lim\limits_{|x|\to\infty} u(x)$ exists, being possibly $\pm\,\infty$;

ii) In case $p > n$ and ℓ is finite, there exist positive constants C, β such that

$$|\;\ell - u(x)\,| < C\,|x|^{-\beta} \ \text{ for all }\ \ |x| \ \text{ large.} \tag{0.9}$$

More generally, any weak solution u satisfying either

$$\lim_{x\to\infty} \frac{|u(x)|}{|x|^\alpha} = 0 \quad \text{for } \alpha = \frac{p-n}{p-1} \ \text{ in case } \ \ p > n \tag{0.10}$$

or

$$\lim_{x\to\infty} \frac{|u(x)|}{\log|x|} = 0 \quad \text{in case } \ \ p = n \tag{0.11}$$

is bounded and, therefore by *i)*, converges to a finite limit at infinity, with (0.9) in case $p > n$.

The result of Theorem 2 is the best possible with respect to the exponent $-p-\epsilon < -p$ in (0.5). In fact, the theorem is false in case $\epsilon = 0$, for which a counterexample is given by the function

$$u(x) = \cos\left(\log\log|x|\right), \quad \text{for} \quad |x| > 1 .$$

Clearly u does not attain a limit at infinity but satisfies

$$\Delta_p u(x) = f$$

with f such that

$$|f(x)| \leq C\left(\log|x|\right)^{-p+1}|x|^{-p}, \quad \text{for all} \quad |x| \geq 2$$

for some positive constant C.

Corollary 1. The results of Theorem 2 can be readily extended for functions $f(x,u)$, under the assumption that f satisfies

$$f(x,t) \leq \frac{h(t)}{|x|^{p+\epsilon}}, \quad \text{for some} \quad h \in L^\infty_{loc}(\mathbb{R}) .$$

This includes, for instance, eigenvalue equations like

$$-\mathrm{div}\left(|\nabla u|^{p-2}A(|\nabla u|)\nabla u\right) = V(x)\,|u|^{p-2}u + g(x)$$

with V, g satisfying a decay rate as in (0.5).

In the second part of the work, we look at the Dirichlet problem on exterior domains for the fractional p-Laplacian equation $(-\Delta)^s_p\, u = 0$. Precisely, we consider for suitably fractional Sobolev functions the nonlinear nonlocal operator with differentiability order $s \in (0,1)$ and summability growth $p \in (1,+\infty)$ given by

$$(-\Delta)^s_p\, u(x) = \text{p.v.}\int_{\mathbb{R}^n} \frac{\left|\,u(x)-u(y)\,\right|^{p-2}\left(u(x)-u(y)\right)}{|\,x-y\,|^{n+sp}}\,dy . \tag{0.12}$$

Integro-differential equations have been a subject of intense research in recent years, finding applicability in many areas and posing problems of pure mathematical interest. The concerning literature is very wide and we refer to [41, 46, 6, 5, 54, 40, 31] and references therein for further treatments. Nowadays, the theory of nonlocal operators of fractional p-Laplacian type is in quite advanced stage of development. Several classical concepts and results from PDE, such as comparison principles, the Perron method, Wiener resolutivity [27, 34] and existence issues, Harnack inequalities and Hölder regularity [22, 20], to cite a few, have been successfully reformulated and applied in the nonlocal setting. The combined nonlocal and nonlinear nature of these operators impose challenging difficulties, making the use of some known tools from nonlocal theory impracticable; for instance, localization techniques as in Caffarelli and Silvestre [6] via extension problems does not seem to be adaptable for the nonlinear framework of $p \neq 2$. We refer to [46] for a survey on many recent results on nonlinear equations.

In our third theorem, we obtain the radially symmetric solutions in $\mathbb{R}^n$ of the fractional p-Laplacian equation, analogues to the radially symmetric (fundamental) solutions for the local p-Laplacian equations. We point out that, although the fundamental solutions to the fractional Laplacian are well known, given up to a constant by $|x|^{2s-n}$, the corresponding radial solutions of the fractional p-Laplacian, $p \neq 2$, seem to be unknown (or, otherwise, at least not been proven yet).

Theorem 3. Functions $|\cdot|^\alpha$, for $\alpha = \frac{sp-n}{p-1}$, $sp \neq n$, and $\log|\cdot|$, when $sp = n$, are (s,p)-harmonic in $\mathbb{R}^n \setminus \{0\}$.

This allows us to extend the existence result of Theorem 1 to the nonlocal setting, for the Dirichlet problem for the homogeneous equation for the fractional p-Laplacian on exterior domains.

Theorem 4. Let $K \subset \mathbb{R}^n$ be a compact set and $g \in C(K)$. Then, in case $sp > n$, there is a unique bounded weak solution $u \in C(\mathbb{R}^n) \cap W^{1,p}_{loc}(\mathbb{R}^n \setminus K)$ of

$$\begin{cases} (-\Delta)_p^s u = 0\,, & \text{in } \mathbb{R}^n \setminus K \\ u = g\,, & \text{in } K\,. \end{cases} \tag{0.13}$$

In addition, if g is α-Hölder continuous in K, with $\alpha = \frac{sp-n}{p-1}$, then $u \in C^\alpha(\mathbb{R}^n)$.

Chapter 1

PRELIMINARIES

This chapter is intended for a review on some concepts and results we use along the work.

1.1 On the degenerate operator

For some equations in divergence form, the existence of weak solutions for Dirichlet problems can be achieved by finding minimizers of certain functionals, since those are expected to solve the respective Euler-Lagrange equation in a weak sense. In fact, by a weak formulation, the equation is defined for functions lying in a suitable Sobolev Space, whose compactness properties are particularly useful for minimization methods. Once the existence of a weak solution is established, then some regularity of the solution can hopefully be obtained. The regularity results for the class of degenerate equations establish a priori $C^{1,\alpha}$ regularity for weak solutions and are due to DiBenedetto, Tolksdorf, Manfredi, and Lieberman [8, 64, 37, 33]. We explain in the following how these ideas apply in case of equation (0.6) and derive for it the classical existence results. The existence of weak solutions of the p-Laplacian equation can be found in [63].

Definition 1. A function $u \in W^{1,p}_{loc}(\Omega)$, defined on an open set $\Omega \subseteq \mathbb{R}^n$, is a weak solution of

$$-\mathrm{div}\Big(|\nabla u|^{p-2} A(|\nabla u|)\nabla u \Big) \;=\; f \quad \text{in } \Omega \tag{1.1}$$

if

$$\int_\Omega | \nabla u |^{p-2} A(|\nabla u|)\nabla u \cdot \nabla \eta \;=\; \int_\Omega f\eta \tag{1.2}$$

for all $\eta \in C_0^\infty(\Omega)$. We also call u a (sub)supersolution of (1.1) if

$$\int_\Omega | \nabla u |^{p-2} A(|\nabla u|)\nabla u \cdot \nabla \eta \,(\leq) \geq \int_\Omega f\eta \tag{1.3}$$

for all positive $\eta \in C_0^\infty(\Omega)$.

We make fundamental use of the Comparison Principle in the work. The following statement is a particular case of [52, Theorem 2.4.1], since the vector function $\mathbf{A}(\xi) = |\xi|^{p-2}A(\,|\xi|\,)\,\xi$ satisfies the monotonicity condition

$$\big(\,\mathbf{A}(\xi) - \mathbf{A}(\eta)\,\big)\cdot\big(\,\xi - \eta\,\big) \;>\; 0\,, \quad \text{for all} \quad \xi,\,\eta \in \mathbb{R}^n, \quad \xi \neq \eta$$

provided the function $\varphi(t) = t^{\,p-1}A(t)$ is increasing.

Comparison Principle. Let $u, v \in C^1(\Omega)$ on a bounded domain $\Omega \subset \mathbb{R}^n$ and assume A satisfies (0.4) or (0.8). If

$$-\mathrm{div}\big(\,|\nabla u|^{p-2}A(\,|\nabla u|\,)\nabla u\,\big) \;\leq\; -\mathrm{div}\big(\,|\nabla v|^{p-2}A(\,|\nabla v|\,)\nabla v\,\big) \tag{1.4}$$

in the weak sense in Ω and $u \leq v$ on $\partial\Omega$, then $u \leq v$ in Ω.

Here, $u \leq v$ on $\partial\Omega$ means that, for all $\epsilon > 0$,

$$u \leq v + \epsilon \quad \text{in some neighbourhood of } \; \partial\Omega\,. \tag{1.5}$$

We will often be applying the comparison principle for $u, v \in C(\overline{\Omega})$ satisfying (1.4), with $u \leq v$ on $\partial\Omega$ in the usual sense. In fact, in this case, (1.5) is satisfied due to the uniform continuity of the functions and the compacity of $\partial\Omega$.

Equation (1.1) is the Euler-Lagrange equation associated to the energy functional

$$I(u) \;=\; \int_\Omega \mathrm{L}(x, u(x), \nabla u(x))\,dx\,, \quad u \in W^{1,p}(\Omega) \tag{1.6}$$

for the lagrangian

$$\mathrm{L}(x,z,q) = \int_0^{|q|} \varphi(t)\,dt \;-\; z\,f(x)\,, \quad (x,z,q) \in \Omega \times \mathbb{R} \times \mathbb{R}^n \tag{1.7}$$

where

$$\varphi(t) = t^{\,p-1}A(t)\,. \tag{1.8}$$

L is of class C^2 in the variables z, $q \neq 0$, and convex in q. In fact, we have

$$\mathrm{L}_{q_i}(q) = \varphi(|q|)\,\frac{q_i}{|q|}\,, \qquad \mathrm{L}_{q_i\,q_j}(q) = \frac{\varphi(|q|)}{|q|}\,\delta_{ij} + \frac{q_iq_j}{|q|^2}\,\Big(\,\varphi'(|q|) - \frac{\varphi(|q|)}{|q|}\,\Big) \tag{1.9}$$

and it is easy to see the matrix $\mathrm{L}_{q_i\,q_j}$ has the eigenvalues $\varphi(|q|)/|q|$, of multiplicity $n-1$, and $\varphi'(|q|)$. By our assumptions on (0.4), the eigenvalues are bounded from below by

$$\frac{\varphi(|q|)}{|q|} \;=\; |q|^{\,p-2}\,A(|q|) \;\geq\; \delta\,|q|^{\,p-2}\,, \quad \varphi'(|q|) \;\geq\; \delta'\,|q|^{\,p-2}$$

from which follows

$$\mathrm{L}_{q_i\, q_j}(x, z, q)\, \xi_i\, \xi_j \;\geq\; \min\left\{\, \delta,\, \delta' \,\right\} |q|^{\,p-2}\, |\xi|^2, \quad \text{for all } \xi \in \mathbb{R}^n\,. \tag{1.10}$$

Now, let us assume $g \in W^{1,p}(\Omega)$ and restrict I to the class

$$\mathcal{A}_g \;=\; \{\, u \in W^{1,p}(\Omega) \mid u - g \in W_0^{1,p}(\Omega)\, \}$$

that is, the class of functions that coincide with g in $\partial\Omega$ in the trace sense. The strict convexity inequality (1.10) implies uniqueness of minimizers for I in $\mathcal{A}_g$ and consequently the uniqueness of solutions of (1.1) (See [11]). The uniqueness of solutions also follows from the comparion principle. To obtain the existence of minimizers, let us note first that I is bounded from below in $\mathcal{A}_g$ since

$$\int_\Omega \mathrm{L}(x, u, \nabla u)\, dx \;\geq\; \frac{\delta}{p}\, \|\nabla u\,\|^p_{L^p(\Omega)} - \int_\Omega u(x) f(x)\, dx$$

and using Hölder and Sobolev inequalities, we have for $u \in \mathcal{A}_g$,

$$\begin{aligned}
\int_\Omega |\, u(x) f(x)\,|\, dx \;&\leq \|f\|_{L^\infty} \Big(\int_\Omega u(x) - g(x)\, dx + \int_\Omega g(x)\, dx \Big) \\
&\leq \|f\|_{L^\infty}\, |\Omega|^{\frac{p-1}{p}} \Big(\|\, u - g\,\|_{L^p(\Omega)} + \|\, g\,\|_{L^p(\Omega)} \Big) \\
&\leq \|f\|_{L^\infty}\, |\Omega|^{\frac{p-1}{p}} \Big(C(|\Omega|) \|\, \nabla(u - g)\,\|_{\,L^p(\Omega)} + \|\, g\,\|_{L^p(\Omega)} \Big) \\
&\leq C\, \Big(\|\, \nabla u\,\|_{\,L^p(\Omega)} + \|\, g\,\|_{W^{1,p}(\Omega)} \Big)
\end{aligned}$$

for a positive constant C which does not depend on u. Hence,

$$\int_\Omega \mathrm{L}(x, u, \nabla u)\, dx \;\geq\; \frac{\delta}{p}\, \|\nabla u\,\|^p_{L^p(\Omega)} - C\Big(\|\, \nabla u\,\|_{\,L^p(\Omega)} + \|\, g\,\|_{W^{1,p}(\Omega)} \Big)$$

which is bounded below with respect to $\|\nabla u\,\|_{\,L^p(\Omega)}$. Next, considering a minimizing sequence $u_k \in \mathcal{A}_g$, by the inequalities above, it can be inferred a uniform bound on $\|u_k\|_{W^{1,p}(\Omega)}$ so that, by weak compactness, there is a subsequence of u_k that converges weakly in $W^{1,p}(\Omega)$ to a function u. Such u then belongs to $\mathcal{A}_g$, since $u_k - g \in W_0^{1,p}(\Omega)$ converges weakly to $u - g$ and $W_0^{1,p}(\Omega)$ is a weakly closed subspace, so that $u - g \in W_0^{1,p}(\Omega)$. Next, we should show u is actually a minimizer of I. For this, it is sufficient to show that I satisfies a weak lower semicontinuity property, *i.e.*, that for any sequence $u_k \in W^{1,p}(\Omega)$ converging weakly to some $u \in W^{1,p}(\Omega)$ there holds $I(u) \leq \liminf I(u_k)$. This can be done for the functional I with some small adaptations in the arguments in [32, 63].

To guarantee that the minimizer u is a weak solution of the Euler-Lagrange equation for L, a sufficient assumption is provided by the growth conditions

$$\left\{\begin{aligned}
\mathrm{L}(x, z, q) \;&\leq\; C(\, |z|^{p+1} + |q|^p\,) \\
D_z\mathrm{L}(x, z, q) \;&\leq\; C(\, 1 + |z|^{p-1} + |q|^{p-1}\,) \\
D_q\mathrm{L}(x, z, q) \;&\leq\; C(\, 1 + |z|^{p-1} + |q|^{p-1}\,)
\end{aligned}\right. \tag{1.11}$$

for some $C > 0$ and all $(x, z, q) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$ (see [32, 63]). Those conditions can be verified for the lagrangian (1.7) and we conclude the existence of a weak solution of (1.1) in $\mathcal{A}_g$. We summarize this exposition in the following.

Theorem 5. Let Ω be a bounded domain. For all $g \in W^{1,p}(\Omega)$, there exists a unique weak solution $u \in W^{1,p}(\Omega)$ of (1.1) satisfying $u = g$ in $\partial\Omega$ in the trace sense.

We wish now to apply the results due to Lieberman [33] and Tolksdorf [64] to improve the regularity of weak solutions up to the boundary. Let us henceforth assume $g \in C^{1,\alpha}(\overline{\Omega})$. [33, Theorem 1] ensures that, on a bounded domain Ω with $C^{1,\alpha}$ boundary, $\alpha > 0$, a bounded weak solution of a general quasilinear equation, with boundary values $g \in C^{1,\alpha}(\partial\Omega)$ in the trace sense, is in $C^{1,\beta}(\overline{\Omega})$, for some $\beta > 0$. This naturally requires some hypotheses on the coefficients of the equation, which in fact hold for equation (1.1) under assumption of (0.4). Among the hypotheses concerning the part under divergence, given in our case by $a_i(q) := |q|^{p-2} A(|q|)\, q_i = \mathrm{L}_{q_i}(q)$, it is required to hold an inequality as in (1.10) and a growth condition of the form

$$\sum_{i,j} \Big| \frac{\partial a_i}{\partial q_j} \Big| \le \Gamma \left(\kappa + |q| \right)^{p-2} \tag{1.12}$$

with constants $\Gamma > 0$ and $\kappa \ge 0$, for all $q \neq 0$. We have already shown (1.10) and to verify the inequality above, noting that $\frac{\partial a_i}{\partial q_j} = \mathrm{L}_{q_i q_j}$, we have by (1.9),

$$\sum_{i,j} |\, \mathrm{L}_{q_i q_j}(q) \,| \le n \frac{\varphi(|q|)}{|q|} + |q|^{-2} \sum_{i,j} |\, q_i q_j \,| \left(\varphi'(|q|) + \frac{\varphi(|q|)}{|q|} \right).$$

Then using that

$$\sum_{i,j} |\, q_i q_j \,| = \Big(\sum_i |q_i| \Big) \Big(\sum_j |q_j| \Big) = \Big(\sum_i |q_i| \Big)^2 \le n\, |q|^2$$

where the last inequality comes by the Cauchy-Schwarz inequality for the inner product of the vectors $(\, |q_1|, \dots, |q_n| \,)$, $(1, \dots, 1) \in \mathbb{R}^n$, it follows

$$\begin{aligned} \sum_{i,j} |\, \mathrm{L}_{q_i q_j}(q) \,| &\le n \frac{\varphi(|q|)}{|q|} + n \left(\varphi'(|q|) + \frac{\varphi(|q|)}{|q|} \right) \\ &\le 2\, n \left(\frac{\varphi(|q|)}{|q|} + \varphi'(|q|) \right) \le 2\, n\, (L + L')\, |q|^{p-2} \end{aligned}$$

where we have used condition (0.4), *ii*), *iii*). This shows the validity of (1.12) with $\Gamma = 2\, n\, (L + L')$, $\kappa = 0$. To apply [33, Theorem 1], it remains to ensure that the solutions are bounded. In case $p > n$, this is immediate from the Morrey's inequality (see [11]), which in fact guarantees α-Hölder continuity up to the boundary, $\alpha = 1 - n/p$, of functions in $W^{1,p}(\Omega)$, if $\partial\Omega$ is of class C^1. In case $p \le n$, a bound on solutions is obtained in [36, Theorem 3.12]. Assuming the solution u to be bounded by $|u| \leq M + \eta$, for a constant M and a function $\eta \in W_0^{1,p}(\Omega)$, it is shown that

$$\sup_{\Omega} |u| \leq C + M \tag{1.13}$$

with a constant C depending only on $n, p, |\Omega|$, and the parameters of the equation. In our case, since $u \in \mathcal{A}_g$, with $g \in C^1(\overline{\Omega})$, this assumption can be verified with $M = \sup |g| < \infty$, $\eta = u - g \in W_0^{1,p}(\Omega)$. Hence, (1.13) holds and the regularity result of [33, Theorem 1] can be applied, therefore, in all cases of $p > 1$. We can conclude the following existence result.

Theorem 6. Let Ω be a bounded domain of class $C^{1,\alpha}$, $\alpha > 0$. If $g \in C^{1,\alpha}(\overline{\Omega})$, there exists a unique weak solution $u \in C^{1,\beta}(\overline{\Omega})$, with $\beta > 0$, of (1.1) satisfying $u = g$ in $\partial\Omega$. The Hölder seminorm $|u|_{1+\beta}$ depends only on $|g|_{1+\alpha}$, α, n, p, $|\Omega|$, $\sup_{\Omega} |u|$, and the parameters of the equation.

Theorem 6 can be extended for the case of continuous boundary values as follows.

Theorem 7. Let Ω be a bounded domain of class $C^{1,\alpha}$, $\alpha > 0$. If $\phi \in C(\partial\Omega)$, there exists a unique weak solution $u \in C(\overline{\Omega}) \cap C^{1,\beta}_{loc}(\Omega)$, with $\beta > 0$, of (1.1) satisfying $u = \phi$ in $\partial\Omega$.

Proof. Let $\phi_k \in C^\infty(\overline{\Omega})$ be such that

$$\sup_{z \in \partial\Omega} |\, \phi(z) - \phi_k(z) \,| \to 0\,. \tag{1.14}$$

By Theorem 6, each of the problems

$$\begin{cases} -\mathrm{div}\big(|\,\nabla u\,|^{p-2} A(|\,\nabla u\,|)\nabla u\big) = f & \text{in } \Omega \\ \quad u = \phi_k & \text{in } \partial\Omega \end{cases} \tag{1.15}$$

has a weak solution $u_k \in C^{1,\beta}(\overline{\Omega})$, with $\beta > 0$. By estimate (1.13) provided by [36, Theorem 3.12], u_k is uniformly bounded on Ω. Moreover, by the local Hölder regularity result in [32, Theorem 1.1, p. 251], there exists a $\gamma > 0$ such that, for each compact V of Ω, there is a constant $C > 0$, depending only the parameters of the equation and V, such that

$$|\, u_k(x) - u_k(y) \,| \leq C |\, x - y \,|^\gamma\,, \ \text{ for all } x, y \in V\,.$$

Thus, u_k is also equicontinuous on V and, by Arzelá-Ascoli's Theorem, we can obtain a subsequence of u_k converging uniformly on V to some continuous function. Considering then a sequence of compact sets V_k such that $\Omega = \cup V_k$, using a standard diagonal argument, we can find a continuous function u on Ω and a subsequence of u_k that converges to u uniformly on compacts of Ω.

Now let $\epsilon > 0$. By (1.14), we have that for all sufficiently large integers k, l,

$$\phi_l < \phi_k + \epsilon \quad \text{in } \partial\Omega\,.$$

Hence, since $u_k = \phi_k$ in $\partial\Omega$ and u_k is uniformly continuous on $\overline{\Omega}$, we can conclude that

$$u_l < u_k + \epsilon \quad \text{in } \partial\Omega$$

holds in the sense of condition (1.5). We can then apply the comparison principle to extend this inequality to Ω. Then, sending $l \to \infty$, we obtain

$$u \leq u_k + \epsilon \quad \text{in } \overline{\Omega}\,.$$

Therefore, for all $y \in \partial\Omega$,

$$\limsup_{x \to y} u(x) \leq \phi_k(y) + \epsilon$$

and sending $k \to \infty$ it follows

$$\limsup_{x \to y} u(x) \leq \phi(y) + \epsilon\,.$$

An analogous inequality can be obtained for the lower limit, concluding the continuity of u up to the boundary, with boundary values ϕ.

To see that u is a weak solution in Ω, let $\eta \in C_0^1(\Omega)$. By [64, Theorem 1], there exist a $\beta > 0$ and a constant $C > 0$, which does not depend on u_k, such that

$$\begin{aligned} &|\,\nabla u_k(x)\,| \leq C \\ &|\,\nabla u_k(x) - \nabla u_k(y)\,| \leq C|\,x-y\,|^{\beta}\,, \quad \text{for all } x,\, y \in \mathit{supp}\,\eta\,. \end{aligned}$$

Then again, by the Arzelá-Ascoli's Theorem, up to a subsequence, ∇u_k converges uniformly to ∇u on $\mathit{supp}\,\eta$. Therefore, we obtain

$$\begin{aligned} \int_\Omega f\eta &= \int_\Omega |\nabla u_k|^{p-2} A\big(|\nabla u_k|\big) \nabla u_k \cdot \nabla\eta \\ &= \int_{\mathit{supp}\,\eta} |\nabla u_k|^{p-2} A\big(|\nabla u_k|\big) \nabla u_k \cdot \nabla\eta \\ &\to \int_{\mathit{supp}\,\eta} |\nabla u|^{p-2} A\big(|\nabla u|\big) \nabla u \cdot \nabla\eta \end{aligned}$$

hence showing u is a weak solution in Ω. □

1.2 On fractional Sobolev Spaces and the Fractional p-Laplacian

In this section, we review some basic facts of Fractional Sobolev Spaces and of the nonlocal operator we deal with.

Definition 2. Let $\Omega \subset \mathbb{R}^n$ be an open set. For each $s \in (0,1)$ and $p \in (1,\infty)$, the usual Fractional Sobolev Space $W^{s,p}(\Omega)$ is defined as

$$W^{s,p}(\Omega) = \left\{ u \in L^p(\Omega) \ ; \ \frac{|\, u(x) - u(y)\,|}{|\, x - y\,|^{\frac{n}{p}+s}} \in L^p\big(\Omega \times \Omega\big) \right\}.$$

The expression

$$\|\, u\,\|_{W^{s,p}(\Omega)} = \Big(\|\, u\,\|^p_{L^p(\Omega)} + [\, u\,]^p_{W^{s,p}(\Omega)} \Big)^{\frac{1}{p}}$$

where the term

$$[\, u\,]_{W^{s,p}(\Omega)} = \left(\int_\Omega \int_\Omega \frac{|\, u(x) - u(y)\,|^p}{|\, x - y\,|^{n+ps}}\, dxdy \right)^{\frac{1}{p}} \tag{1.16}$$

called the *Gagliardo seminorm* of u, defines a norm in $W^{s,p}(\Omega)$, for which $W^{s,p}(\Omega)$ is a Banach space (see [7, Proposition 4.24]). We denote by $W^{s,p}_0(\Omega)$ the closure of $C^\infty_0(\Omega)$ in $W^{s,p}(\mathbb{R}^n)$.

The following results on continuous embbedings for fractional Sobolev spaces are found in Propositions 2.1, 2.2 in [41].

Proposition 1. Let $p \in [1,+\infty)$ and $0 < s \le s' < 1$. Let Ω be an open set in $\mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$ be a measurable function. Then

$$\|u\|_{W^{s,p}(\Omega)} \le C\, \|u\|_{W^{s',p}(\Omega)}$$

for some suitable positive constant $C = C(n,s,p) \ge 1$. In particular,

$$W^{s',p}(\Omega) \subseteq W^{s,p}(\Omega)\,.$$

Proposition 2. Let $p \in [1,+\infty)$ and $s \in (0,1)$. Let Ω be an open set in $\mathbb{R}^n$ with Lipschitz bounded boundary and $u : \Omega \to \mathbb{R}$ be a measurable function. Then

$$\|u\|_{W^{s,p}(\Omega)} \le C\, \|u\|_{W^{1,p}(\Omega)}$$

for some suitable positive constant $C = C(n,s,p) \ge 1$. In particular,

$$W^{1,p}(\Omega) \subseteq W^{s,p}(\Omega)\,.$$

Next, we define a quantity called the *nonlocal tail*, which plays an important role in the study of nonlocal of operators.

Definition 3. For $s \in \big(0,1\big)$ and $p \in (\,1,+\infty\,)$, the *nonlocal tail* of a function u in the ball of radius $r > 0$ and center $z \in \mathbb{R}^n$ is defined as

$$\mathrm{Tail}(u;z,r) \;=\; \left(r^{sp} \int\limits_{\mathbb{R}^n\setminus B_r(z)} |u(x)|^{p-1}\,|\,x-z\,|^{-n-sp}\,dx \right)^{\frac{1}{p-1}}. \tag{1.17}$$

The *tail space* $L^{p-1}_{sp}(\mathbb{R}^n)$ is given by

$$L^{p-1}_{sp}(\mathbb{R}^n) \;=\; \left\{ u \in L^{p-1}_{Loc}(\mathbb{R}^n) \;:\; \mathrm{Tail}(u;0,1) < \infty \right\}. \tag{1.18}$$

One can show the inclusions $L^{\infty}(\mathbb{R}^n) \subset L^{p-1}_{sp}(\mathbb{R}^n)$ and $W^{s,p}(\mathbb{R}^n) \subset L^{p-1}_{sp}(\mathbb{R}^n)$.

Definition 4. We say that a function $u \in W^{s,p}(\Omega) \cap L^{p-1}_{sp}(\mathbb{R}^n)$ is a weak (sub)supersolution of

$$(-\Delta)^s_p\, u = 0 \quad \text{in } \Omega \tag{1.19}$$

if

$$\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \frac{|\,u(x)-u(y)\,|^{p-2}\,(\,u(x)-u(y)\,)\,(\,\eta(x)-\eta(y)\,)}{|\,x-y\,|^{n+sp}}\,dxdy \;(\le)\;\; \ge 0 \tag{1.20}$$

for all test functions $\eta \in C_0^{\infty}(\Omega)$ with $\eta \ge 0$. In addition, u is a weak solution to (1.19) if it is both a sub and a supersolution in the sense above, *i.e.*,

$$\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \frac{|\,u(x)-u(y)\,|^{p-2}\,(\,u(x)-u(y)\,)\,(\,\eta(x)-\eta(y)\,)}{|\,x-y\,|^{n+sp}}\,dxdy \;= 0 \tag{1.21}$$

for all $\eta \in C_0^{\infty}(\Omega)$.

For nonlocal operators, the Dirichlet boundary condition consists in assigning the values of u in the whole complement of Ω, rather than only on $\partial\Omega$. Hence, for an open set $\Omega \subset \mathbb{R}^n$ and $g : \mathbb{R}^n \longrightarrow \mathbb{R}$, we will be considering the problem

$$\begin{cases} (-\Delta)^s_p\, u \;=\; 0 \;\text{ in } \Omega \\ \qquad u \;=\; g \;\text{ in } \mathbb{R}^n \setminus \Omega \end{cases} \tag{1.22}$$

Throughout this work, we make significant use of the nonlocal comparison principle, as stated below. It requires the additional assumption that one function dominates the other, not only on the boundary of the domain but also on its complement. Even though this nonlocal version of the comparison principle is sufficient for many applications, its hypothesis turns to be quite restrictive and in some cases prevent the extension of successful ideas from the local setting. The following statement is found in [27, Lemma 6].

Lemma. (Comparison Principle) Let $s \in \big(0,1\big)$ and $p \in (\,1,+\infty\,)$. Let $\Omega \Subset \Omega'$ be bounded open sets of $\mathbb{R}^n$. Let $u \in W^{s,p}(\Omega')$ be a weak supersolution of (1.19) in Ω and $v \in W^{s,p}(\Omega')$ be a weak subsolution of (1.19) in Ω such that $u \ge v$ *a.e.* in $\mathbb{R}^n \setminus \Omega$. Then $u \ge v$ *a.e.* in Ω.

In Chapter 5, we make use of the following results of [10], which we state particularly for the p-Laplacian operator.

Theorem. (Theorem 3.1, [10]) Let $K \subset \mathbb{R}^n$ be a compact set and $u \in W^{s,p}_{loc}(\mathbb{R}^n \backslash K) \cap L^\infty(\mathbb{R}^n)$ a weak solution of $(-\Delta)^s_p u = 0$ in $\mathbb{R}^n \backslash K$. Suppose that $sp \geq n$. Then, for any open set $U \subset \mathbb{R}^n$ such that $K \subset U$ it holds

$$\left(\int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n \backslash U} \frac{|\, u(x) - u(y) \,|^p}{|\, x - y \,|^{n+sp}} \, dxdy \right)^{\frac{1}{p}} \leq C \sup |u| \tag{1.23}$$

with C depending on n, s, p, K and U.

The second result is a comparison principle for bounded solutions of the fractional p-Laplacian equation on exterior domains. This extends [3, Theorem 2] to nonlocal operators.

Theorem. (Theorem 3.3, [10]) Let K be a compact set of $\mathbb{R}^n$ and let $u, v \in C(\mathbb{R}^n) \cap W^{s,p}_{loc}(\mathbb{R}^n \backslash K)$ bounded solutions of $(-\Delta)^s_p u = 0$ in $\mathbb{R}^n \backslash K$. Suppose that $sp \geq n$. If $v \geq u$ on K then $v \geq u$ in $\mathbb{R}^n \backslash K$.

Chapter 2

AUXILIARY RESULTS

In this chapter we present some results to be used on the proofs of Theorems 1 and 2. We begin with the construction of radial barriers to the problem (0.6), assuming for A the weaker conditions in (0.8). Lemma 1 gives existence and estimates for local radially symmetric barriers on arbitrary balls; radially symmetric barriers globally defined are presented in Lemma 2.

Lemma 1. Let $f \in L^\infty(B_R(x_0))$, $x_0 \in \mathbb{R}^n$, $R > 0$. Then, for $p > n$, there exists a family of radially symmetric supersolutions $v_a = v_{a,x_0}$ of (0.6) in $B_R(x_0) \backslash \{x_0\}$ such that

$$\left(\frac{\|f\|_\infty}{nL} \right)^{\frac{1}{p-1}} a \, \frac{|\, x - x_0 \,|^\alpha}{\alpha} \;\leq\; v_a(x) \;\leq\; \left(\frac{\|f\|_\infty}{n\delta} \right)^{\frac{1}{p-1}} \big(\, a + R^{\frac{n}{p-1}} \, \big) \, \frac{|\, x - x_0 \,|^\alpha}{\alpha} \tag{2.1}$$

for $a \geq 0$, where δ, L are the constants in (0.8), $ii)$, associated to A.

Proof. We start looking for radially symmetric solutions $v = v(r)$, $r = |x - x_0|$, of the equation

$$-\mathrm{div}(|\nabla v|^{p-2} A(|\nabla v|)\nabla v) = \|f\|_\infty$$

for $0 \leq r \leq R$. This leads to the following ODE

$$\frac{d}{dr} \left\{ \, |v'|^{\,p-2} A\big(|v'|\big) v' \, \right\} + \frac{n-1}{r} \, |v'|^{\,p-2} A\big(|v'|\big) v' = -\|f\|_\infty \, .$$

Multiplying this equation by the integrating factor r^{n-1}, we get

$$\frac{d}{dr} \left\{ \, |v'|^{\,p-2} A\big(|v'|\big) v' \, r^{n-1} \, \right\} = -\|f\|_\infty \, r^{n-1}$$

from which integrating from some $t_0 > 0$ to t, $0 < t \leq R$, comes

$$|v'(t)|^{\,p-2} A\big(\, |v'(t)| \, \big) \, v'(t) \, t^{n-1} \;\; = \;\; \frac{\|f\|_\infty}{n} \, (t_0^n - t^n) \; + \; |v'(t_0)|^{\,p-2} A\big(\, |v'(t_0)| \, \big) \, v'(t_0) \, t_0^{n-1} \, .$$

Assuming $v' \geq 0$ and taking

$$C \;=\; t_0^n \;+\; \frac{n}{\|f\|_\infty} \, v'(t_0)^{\,p-1} A(\, v'(t_0) \,) \, t_0^{n-1}$$

it follows

$$v'(t)^{p-1} A(\, v'(t)\,)\, t^{n-1} \;=\; \frac{\|f\|_\infty}{n}\,(C - t^n)$$

and then

$$v'(t)^{p-1} A(\, v'(t)\,) \;=\; \frac{\|f\|_\infty}{n}\,\frac{(C - t^n)}{t^{n-1}}\,.$$

Using the notation $\varphi(t) = t^{p-1}A(t)$, we can write

$$v'(t) = \varphi^{-1}\left(\frac{\|f\|_\infty}{n}\,(\,C - t^n\,)\, t^{-n+1} \right)$$

so that

$$v(r) = \int_0^r \varphi^{-1}\left(\frac{\|f\|_\infty}{n}\,(\,C - t^n\,)\, t^{-n+1} \right) dt\,, \;\; C \geq R^n \tag{2.2}$$

gives a family of supersolutions. We take then

$$C \;=\; R^n + a^{\,p-1}\,, \;\; a \;\geq\; 0 \tag{2.3}$$

and

$$v_a(r) := \int_0^r \varphi^{-1}\left(\frac{\|f\|_\infty}{n}\,(\,a^{p-1} + R^n - t^n\,)\, t^{-n+1} \right) dt\,, \;\; a \geq 0\,. \tag{2.4}$$

Note that by (0.8), *ii*), we have

$$\delta\, t^{p-1} \;\leq\; \varphi(t) \;\leq\; L\, t^{p-1}$$

so that, by the increasing monotonicity of φ^{-1},

$$\varphi^{-1}\big(\,\delta\, t^{p-1}\,\big) \;\leq\; t \;\leq\; \varphi^{-1}\big(\,L\, t^{p-1}\,\big)\,.$$

Hence, for a fixed $s > 0$, taking $t = \big(\,s/\delta\,\big)^{\frac{1}{p-1}}$ in the first inequality we get

$$\varphi^{-1}(s) \;\leq\; \big(\,s/\delta\,\big)^{\frac{1}{p-1}}\,;$$

Taking $t = \big(\,s/L\,\big)^{\frac{1}{p-1}}$ in second inequality, then

$$\big(\,s/L\,\big)^{\frac{1}{p-1}} \;\leq\; \varphi^{-1}(s)$$

so we obtain

$$\left(\frac{s}{L}\right)^{\frac{1}{p-1}} \leq \varphi^{-1}(s) \leq \left(\frac{s}{\delta}\right)^{\frac{1}{p-1}}\,, \quad \text{for } s > 0\,. \tag{2.5}$$

Using these inequalities to estimate (2), we have

$$v_a(r) \,\geq \left(\frac{\|f\|_\infty}{nL} \right)^{\frac{1}{p-1}} \int_0^r (\,a^{p-1} + R^n - t^n\,)^{\frac{1}{p-1}}\, t^{-\frac{n-1}{p-1}}\, dt$$

where noting that $t \leq R$, we obtain the lower bound

$$\begin{aligned} v(r) &\geq \left(\frac{\|f\|_\infty}{nL} \right)^{\frac{1}{p-1}} \int_0^r a\, t^{-\frac{n-1}{p-1}}\, dt \\ &= \left(\frac{\|f\|_\infty}{nL} \right)^{\frac{1}{p-1}} a\, \frac{r^\alpha}{\alpha}\,. \end{aligned}$$

For the upper bound we can estimate

$$\begin{aligned} v(r) &\leq \left(\frac{\|f\|_\infty}{n\delta} \right)^{\frac{1}{p-1}} \int_0^r (\, a^{p-1} + R^n - t^n\,)^{\frac{1}{p-1}}\, t^{-\frac{n-1}{p-1}}\, dt \\ &\leq \left(\frac{\|f\|_\infty}{n\delta} \right)^{\frac{1}{p-1}} \int_0^r \left(a^{p-1} + R^n \right)^{\frac{1}{p-1}} t^{-\frac{n-1}{p-1}}\, dt \\ &\leq \left(\frac{\|f\|_\infty}{n\delta} \right)^{\frac{1}{p-1}} \left(a + R^{\frac{n}{p-1}} \right) \int_0^r t^{-\frac{n-1}{p-1}}\, dt \;=\; \left(\frac{\|f\|_\infty}{n\delta} \right)^{\frac{1}{p-1}} \left(a + R^{\frac{n}{p-1}} \right) \frac{r^\alpha}{\alpha}\,. \end{aligned}$$

□

Lemma 2. In case $p > n$, for any $f \in L^\infty(\mathbb{R}^n)$ satisfying (0.5), there exists a family of radially symmetric supersolutions v_a of (0.6) in $\mathbb{R}^n \backslash \{\, 0 \,\}$ satisfying

i) $v_a(0) = 0$ and $v_a(r)$ is increasing in $(0, +\infty)$ for any $a \geq 0$;

ii) v_a is unbounded in $(0, +\infty)$ for $a > 0$; indeed, there exists a constant $c_0 = c_0(n, p, \epsilon, C_f, L) > 0$ such that

$$v_a(r) \;\geq\; c_0\, a\, r^\alpha \quad \text{for} \quad r \geq 0, \quad \text{where} \quad \alpha = \frac{p-n}{p-1};$$

iii) v_0 is bounded in $(0, +\infty)$; indeed, there exists a constant $C_0 = C_0(n, p, \epsilon, C_f, \delta) > 0$ such that

$$v_0(r) \leq C_0\,;$$

iv) $v_a(r) \to v_0(r)$ as $a \to 0$ for any $r \in (0, +\infty)$
for δ, L in (0.8), *ii*).

Proof. With no loss of generality, we can assume (0.5) holds for all $|x| \geq 1$, with $f \leq C_f$. Hence, to obtain the desired supersolution we consider

$$g(r) \;=\; \begin{cases} C_f & \text{for } r \leq 1 \\ C_f\, r^{-p-\epsilon} & \text{for } r \geq 1 \end{cases}$$

and look for radially symmetric solutions $v = v(r)$, $r = |x|$, of

$$-\mathrm{div}(|\nabla v|^{p-2} A(|\nabla v|) \nabla v) \;=\; g(r)$$

for $r > 0$. This leads to the ODE

$$\frac{d}{dr} \left\{\, |v'|^{p-2} A\big(|v'|\big) v'\, r^{n-1} \,\right\} = -g(r)\, r^{n-1}$$

which integrated from $r = 1$ to some $t > 0$ gives

$$|v'(t)|^{p-2} A\big(\,|v'(t)|\,\big)\, v'(t)\, t^{n-1} \;=\; -\int_1^t g(r)\, r^{n-1}\, dr \,+\, C$$

where

$$C \;=\; |v'(1)|^{p-2} A\big(\,|v'(1)|\,\big)\, v'(1)\,.$$

Assuming $v' \geq 0$ it follows

$$v'(t)^{\,p-1} A(\, v'(t)\,)\, t^{n-1} \;=\; -\int_1^t g(r)\, r^{n-1}\, dr \,+\, C$$

and then

$$v'(t)^{\,p-1} A(\, v'(t)\,) \;=\; \frac{-\int_1^t g(r)\, r^{n-1}\, dr + C}{t^{n-1}}\,.$$

Using $\varphi(t) = t^{p-1} A(t)$, we can write

$$v'(t) = \varphi^{-1}\left(\frac{-\int_1^t g(r)\, r^{n-1}\, dr + C}{t^{n-1}} \right)$$

so that

$$v(r) = \int_0^r \varphi^{-1}\left(\frac{-\int_1^t g(\tau)\, \tau^{n-1}\, d\tau + C}{t^{n-1}} \right) dt$$

gives a family of supersolutions. Recalling the definition of g, we have for $r \geq 1$, that

$$\begin{aligned}
v(r) \;&=\; \int_0^1 \varphi^{-1}\Big(\frac{-\int_1^t C_f\, \tau^{n-1}\, d\tau + C}{t^{n-1}} \Big)\, dt \\
&\quad+\; \int_1^r \varphi^{-1}\Big(\frac{-\int_1^t C_f\, \tau^{n-p-\epsilon-1}\, d\tau + C}{t^{n-1}} \Big)\, dt \\
&= \int_0^1 \varphi^{-1}\Big(\frac{\frac{C_f}{n}\,(\,1 - t^n\,) + C}{t^{n-1}} \Big)\, dt \\
&\quad+\; \int_1^r \varphi^{-1}\Big(\frac{\frac{C_f}{p-n+\epsilon}\,(\,t^{\,n-p-\epsilon} - 1\,) + C}{t^{n-1}} \Big)\, dt \\
&= \int_0^1 \varphi^{-1}\Big(\frac{\frac{C_f}{n}\,(\,1 - t^n\,) + C}{t^{n-1}} \Big)\, dt \\
&\quad+\; \int_1^r \varphi^{-1}\Big(\frac{C_f}{p-n+\epsilon}\, \frac{\big(\,t^{\,n-p-\epsilon} + C\,\frac{p-n+\epsilon}{C_f} - 1\,\big)}{t^{n-1}} \Big)\, dt\,.
\end{aligned}$$

By choosing

$$C = \frac{C_f}{p-n+\epsilon}\,(\,a^{\,p-1} + 1\,)\,, \;\; a \,\geq\, 0$$

it follows

$$\begin{aligned}
v_a(r) \;&=\; \int_0^1 \varphi^{-1}\Big(\frac{\frac{C_f}{n}\,(\,1 - t^n\,) + \frac{C_f}{p-n+\epsilon}\,(\,a^{\,p-1} + 1\,)}{t^{n-1}} \Big)\, dt \\
&\quad+\; \int_1^r \varphi^{-1}\Big(\frac{C_f}{p-n+\epsilon}\, \frac{\big(\,t^{\,n-p-\epsilon} + a^{\,p-1}\,\big)}{t^{n-1}} \Big)\, dt\,.
\end{aligned} \tag{2.6}$$

Using (2.5) we can estimate v_a from below as

$$
\begin{aligned}
v_a(r) \geq & \left(\frac{1}{L}\right)^{\frac{1}{p-1}} \int_0^1 \left(\frac{C_f}{n} (1 - t^n) + \frac{C_f}{p-n+\epsilon} (a^{p-1} + 1) \right)^{\frac{1}{p-1}} t^{-\frac{n-1}{p-1}}\, dt \\
& + \left(\frac{C_f}{(p-n+\epsilon)L}\right)^{\frac{1}{p-1}} \int_1^r \left(t^{n-p-\epsilon} + a^{p-1} \right)^{\frac{1}{p-1}} t^{-\frac{n-1}{p-1}}\, dt \\
\geq & \left(\frac{1}{L}\right)^{\frac{1}{p-1}} \int_0^1 \left(\frac{C_f}{p-n+\epsilon} (a^{p-1} + 1) \right)^{\frac{1}{p-1}} t^{-\frac{n-1}{p-1}}\, dt \\
& + \left(\frac{C_f}{(p-n+\epsilon)L}\right)^{\frac{1}{p-1}} \int_1^r a\, t^{-\frac{n-1}{p-1}}\, dt \\
\geq & \left(\frac{C_f}{(p-n+\epsilon)L}\right)^{\frac{1}{p-1}} a \left(\int_0^1 t^{-\frac{n-1}{p-1}}\, dt + \int_1^r t^{-\frac{n-1}{p-1}}\, dt \right) \\
\geq & \left(\frac{C_f}{(p-n+\epsilon)L}\right)^{\frac{1}{p-1}} a\, r^{\alpha}.
\end{aligned}
$$

For the upper bound, we can estimate from (2.6),

$$
\begin{aligned}
v_0(r) = & \int_0^1 \varphi^{-1}\left(\frac{\frac{C_f}{n} (1 - t^n) + \frac{C_f}{p-n+\epsilon}}{t^{n-1}} \right) dt \\
& + \int_1^r \varphi^{-1}\left(\frac{C_f}{p-n+\epsilon} \frac{t^{n-p-\epsilon}}{t^{n-1}} \right) dt \\
\leq & \left(\frac{1}{\delta}\right)^{\frac{1}{p-1}} \int_0^1 \left(\frac{C_f}{n} (1 - t^n) + \frac{C_f}{p-n+\epsilon} \right)^{\frac{1}{p-1}} t^{-\frac{n-1}{p-1}}\, dt \\
& + \left(\frac{1}{\delta}\right)^{\frac{1}{p-1}} \int_1^r \left(\frac{C_f}{p-n+\epsilon} \right)^{\frac{1}{p-1}} t^{\frac{-p-\epsilon+1}{p-1}}\, dt \\
\leq & \left(\frac{1}{\delta}\right)^{\frac{1}{p-1}} \int_0^1 \left(\frac{C_f}{p-n+\epsilon} (p+\epsilon) \right)^{\frac{1}{p-1}} t^{-\frac{n-1}{p-1}}\, dt \\
& + \left(\frac{1}{\delta}\right)^{\frac{1}{p-1}} \int_1^r \left(\frac{C_f}{p-n+\epsilon} \right)^{\frac{1}{p-1}} t^{\frac{-p-\epsilon+1}{p-1}}\, dt \\
\leq & \left(\frac{C_f (p+\epsilon)}{\delta (p-n+\epsilon)} \right)^{\frac{1}{p-1}} \left(\int_0^1 t^{-\frac{n-1}{p-1}}\, dt + \int_1^r t^{\frac{-p-\epsilon+1}{p-1}}\, dt \right) \\
\leq & \left(\frac{C_f (p+\epsilon)}{\delta (p-n+\epsilon)} \right)^{\frac{1}{p-1}} \left(\frac{1}{\alpha} + \frac{p-1}{\epsilon} \left(1 - r^{-\frac{\epsilon}{p-1}} \right) \right) \\
\leq & \left(\frac{C_f (p+\epsilon)}{\delta (p-n+\epsilon)} \right)^{\frac{1}{p-1}} \left(\frac{1}{\alpha} + \frac{p-1}{\epsilon} \right).
\end{aligned}
$$

$\square$

In Theorem 2 we use a Harnack inequality. For general quasilinear equations, the Harnack inequality is obtained in [56, Theorems 5, 6, 9] for the cases $p < n$, $p = n$ and $p > n$, respectively. For equation (0.7), these results yield the following Harnack inequality:

Theorem. Let u be a nonnegative weak solution of (0.7) on an open ball B_R. Assume that, in case $p \le n$, $f \in L^{\frac{n}{p-\theta}}(B_R)$, for some $\theta \in (0,1)$, and that, in case $p > n$, $f \in L^1(B_R)$. Then, for any $\sigma \in (0,1)$,

$$\sup_{B_{\sigma R}} u \le C\left(\inf_{B_{\sigma R}} u + K(R)\right) \tag{2.7}$$

where C depends on n, p, σ, δ, L and, in case $p \le n$, also on θ, and

$$K(R) = \left(R^{\theta} \, \| f \|_{L^{\frac{n}{p-\theta}}(B_R)} \right)^{\frac{1}{p-1}} \tag{2.8}$$

if $p \le n$, and

$$K(R) = \left(R^{p-n} \, \| f \|_{L^1(B_R)} \right)^{\frac{1}{p-1}} \tag{2.9}$$

if $p > n$.

The result above can be easily extended to arbitrary compact subsets. In fact, we can estate the corollary below, showing a Harnack inequality for solutions on exterior domains over spheres S_R, for all R large, with $C > 0$ taken independent of R.

Corollary 2. Let u be a non-negative weak solution of (0.7) on $\mathbb{R}^n \setminus \overline{B_1}$ and assume f satisfy condition (0.5). Then, for all $R \ge 4$,

$$\sup_{S_R} u \le C\left(\inf_{S_R} u + R^{-\frac{\epsilon}{p-1}}\right) \tag{2.10}$$

where C depends only on n, p, δ, L.

Proof. We can cover S_R with a quantity N of balls $B_i = B_{R/2}(x_i)$ with centers x_i lying on S_R, with N not depending on R. Ordering these balls so that $B_i \cap B_{i+1} \neq \varnothing$, we have

$$\inf_{B_i} u \le \sup_{B_{i+1}} u\,. \tag{2.11}$$

Now we apply the previous theorem on each ball $B_{3R/4}(x_i) \subset \mathbb{R}^n \setminus \overline{B_1}$, with $\sigma = 2/3$. Using (0.5), a computation of the norms of f shows that, for any case, K can be estimated as

$$K(3R/4) \le C\, R^{-\frac{\epsilon}{p-1}}$$

for some constant C depending only on n, p, so we have by the theorem

$$\sup_{B_i} u \le C\left(\inf_{B_i} u + R^{-\frac{\epsilon}{p-1}}\right) \tag{2.12}$$

where C depends only on n, p, L and, in case $p \le n$, of a chosen $\theta \in (0,1)$. Then by combining inequalities (2.11) and (2.12) it follows, for all $i, j \in \{1, \dots, N\}$,

$$\sup_{B_i} u \le C\left(\inf_{B_j} u + R^{-\frac{\epsilon}{p-1}}\right)$$

after a proper redefinition of C depending only on N. This leads to (2.10). $\square$

Chapter 3

PROOF OF THEOREMS 1, 2

3.1 Proof of Theorem 1

The uniqueness of solutions is a direct consequence of the comparison principle in [3, Theorem 2], presented in Preliminaries. For the existence, we split the proof into three steps.

1. Construction of a bounded solution.

We consider a decreasing sequence of smooth compact sets K_m satisfying, for all m

i) $K \Subset K_{m+1} \Subset K_m$
ii) $dist(\partial K, \partial K_m) \to 0$.

Taking an increasing sequence of radii $R_m \to +\infty$, with $K_m \Subset B_{R_1}$, for all m, we continuously extend ϕ to the whole $\mathbb{R}^n$, keeping fixed $\sup|\phi|$ and setting $\phi = 0$ in $\mathbb{R}^n \setminus B_{R_1}$. We then look for the domains $\Omega_m := B_{R_m} \setminus K_m$ and the problems

$$\begin{cases} -\mathrm{div}\big(|\nabla u|^{p-2} A(|\nabla u|)\nabla u \big) = f & \text{in } \Omega_m \\ u = \phi & \text{in } \partial K_m \\ u = 0 & \text{in } \mathbb{R}^n \setminus B_{R_m}\,. \end{cases} \tag{3.1}$$

By Theorem 7, each of those problems has a weak solution $u_m \in C(\overline{\Omega}_m) \cap C^{1,\beta}_{loc}(\Omega_m)$.

Now let v_0 be the supersolution given by Lemma 2 and assume, with no loss of generality, that K contains the origin $0 \in \mathbb{R}^n$, so that $0 \notin \Omega_m$, for all m. Hence, the function $v_0 + \sup\phi$ is then a supersolution in Ω_m, with $u_m \le v_0 + \sup\phi$ on $\partial\Omega_m$, for all m. Since $v_0 + \sup\phi \le C_0 + \sup\phi$, we obtain by the comparison principle the uniform bound

$$\sup\, u_m \,\le\, C_0 + \sup\,\phi\,, \quad \text{for all } m\,. \tag{3.2}$$

Moreover, by the local Hölder regularity result in [32, Theorem 1.1, p. 251], there exists a $\gamma > 0$ such that, for each compact V of $\mathbb{R}^n \setminus K$, there is a constant $C > 0$, depending on the parameters of the equation and V, such that

$$|\, u_m(x) - u_m(y)\,| \,\leq\, C|\, x - y\,|^{\gamma}\,, \;\text{ for all }\; x, y \,\in\, V\,.$$

Thus, u_m is also equicontinuous on V and, by Arzelá-Ascoli's Theorem, we can obtain a subsequence of u_m converging uniformly on V to some continuous function. Considering then a sequence of compact sets V_k such that $B \setminus K = \bigcup V_k$, by means of a standard diagonal argument, we can find a continuous function u on $\mathbb{R}^n \setminus K$ and a subsequence of u_m that converges to u uniformly on any compact of $\mathbb{R}^n \setminus K$. By the same argument as in the proof of Theorem 7, using [64, Theorem 1], we can conclude u is a weak solution of (1.1) in $\mathbb{R}^n \setminus K$.

□

2. Continuity of u on the boundary.

Let $x_0 \in \partial K$, $\epsilon > 0$ and consider by Lemma 1 the supersolutions v_{a,x_0} on a large ball $B(x_0) \setminus \{x_0\}$. By the continuity of ϕ, there is some $R > 0$ such that

$$|\, \phi(x) - \phi(x_0)\,| \,<\, \epsilon\,, \quad \text{for } |x - x_0| < R$$

so that

$$\phi(x_0) + v_{a,x_0}(x) + \epsilon \,\geq\, \phi(x)\,, \quad \text{for } |x - x_0| \,<\, R\,,\;\; a \geq 0\,.$$

We then choose a sufficiently large in (2.1) to make

$$\phi(x_0) + v_{a,x_0}(x) + \epsilon \geq \sup\, \phi\,, \quad \text{for } |x - x_0| \,\geq\, R\,.$$

Therefore, the function

$$w^{+}_{a,x_0} \,:=\, \phi(x_0) + v_{a,x_0} + \epsilon$$

satisfies $w^{+}_{a,x_0} \,\geq\, \phi$, so that, in particular,

$$w^{+}_{a,x_0} \,\geq\, \phi \,=\, u_m \quad \text{ in }\; \partial K_m\,, \quad \text{for all } m\,.$$

By taking a larger if necessary, we can also make

$$w^{+}_{a,x_0} \,\geq\, u_m \quad \text{ in }\; \partial B\,, \quad \text{for all } m\,.$$

Then by applying the comparison principle on $B \setminus K_m$ we obtain

$$w^{+}_{a,x_0} \,\geq\, u_m \quad \text{ in }\; B \setminus K_m\,, \quad \text{for all } m$$

from which follows

$$w^+_{a,x_0} \geq u \quad \text{in } B \setminus K$$

since u_m converges to u on $B \setminus K$. Finally, this implies

$$\limsup_{x \to x_0} u(x) \leq \limsup_{x \to x_0} w^+_{a,x_0}(x) = \phi(x_0) + \epsilon$$

and by arbitrariness of ϵ we conclude

$$\limsup_{x \to x_0} u(x) \leq \phi(x_0)\,.$$

By an analogous argument with the subsolution $w^-_{a,x_0} := \phi(x_0) - v_{a,x_0} - \epsilon$ we can obtain the lower bound

$$\liminf_{x \to x_0} u(x) \geq \phi(x_0)$$

concluding the result.

□

3. Global Hölder Continuity of u.

Assume ϕ is α-Hölder continuous in K, with $\alpha = \frac{p-n}{p-1}$. We will show u is α-Hölder continuous in $\mathbb{R}^n$.

Let $y \in K$, $R > 0$ and $v_a = v_{a,y}$ a supersolution in $B_R(y) \setminus \{y\}$ as given in Lemma 1. We claim that for all a sufficiently large

$$\phi(y) - v_a \leq u \leq \phi(y) + v_a \;\; \text{in } B_R(y)$$

for all $y \in K$. For this, putting $C = |\phi|_\alpha$, the Höder seminorm of ϕ in K, we have by definition $|\,\phi(z) - \phi(y)\,| \leq C|\,z - y\,|^\alpha$, for all $z \in K$, hence

$$\phi(y) - C|\,z-y\,|^\alpha \leq \phi(z) \leq \phi(y) + C|\,z-y\,|^\alpha \quad \text{for all } z \in K\,.$$

Now by estimate 2.1 we see that for all a large enough v_a satisfies

$$C\,|\,x-y\,|^\alpha \leq v_a(x) \;\; \text{for all } \; x \in B_R(y)$$

so that from last inequality it follows

$$\phi(y) - v_a \leq \phi \leq \phi(y) + v_a \;\; \text{in } K \cap B_R(y)\,. \tag{3.3}$$

Now taking a larger if necessary, by estimate (2.1) we can also ensure that

$$|\,u - \phi(y)\,| \leq 2 \sup |u| \leq v_a \;\; \text{in } \partial B_R(y)$$

and so

$$\phi(y) - v_a \le u \le \phi(y) + v_a \text{ in } \partial B_R(y).$$

Along with (3.3), as $\phi = u$ in K, we see the inequality above holds on $\partial\big(B_R(y) \setminus K\big)$ so that by the comparison principle it extends to $B_R(y) \setminus K$. Notice the parameter a depends only on $|\phi|_\alpha$ and $\sup u$.

Now let $x_0 \in \mathbb{R}^n \setminus K$. It is enough to prove Hölder continuity on a neighbourhood of K so we may assume $d(x_0, K) < R$. By the claim we have, in particular for all $y \in B_R(x_0) \cap K$,

$$\phi(y) - v_{a,y}(x_0) \le u(x_0) \le \phi(y) + v_{a,y}(x_0)$$

This inequality gives

$$u(x_0) - v_{a,y}(x_0) \le \phi(y) \le u(x_0) + v_{a,y}(x_0)$$

and, as $\phi = u$ in K, we get

$$u(x_0) - v_{a,y}(x_0) \le u(y) \le u(x_0) + v_{a,y}(x_0) \quad \text{for all } y \in B_R(x_0) \cap K. \tag{3.4}$$

Using the upper estimate (2.1) we have for some constant C_1

$$v_{a,y}(x) \le C_1 |x - y|^\alpha \text{ for all } x \in B_R(y)$$

and, in particular,

$$v_{a,y}(x_0) \le C_1 |x_0 - y|^\alpha.$$

Now using the lower estimate in (2.1) for the supersolution v_{a,x_0} centered at x_0 we can obtain

$$C |x_0 - y|^\alpha \le v_{a,x_0}(y)$$

and so

$$v_{a,y}(x_0) \le \frac{C_1}{C} v_{a,x_0}(y).$$

From (3.4) it follows

$$u(x_0) - \frac{C_1}{C} v_{a,x_0}(y) \le u(y) \le u(x_0) + \frac{C_1}{C} v_{a,x_0}(y) \tag{3.5}$$

for all $y \in B_R(y) \cap K$. Provided that $C_1/C > 1$, we have that $\frac{C_1}{C} v_{a,x_0}$ is also a supersolution in $B_R(x_0)$ and by the previous choice of a, still $\frac{C_1}{C} v_{a,x_0} \ge 2 \sup |u|$ in $\partial B_R(x_0)$. Therefore, (3.5) holds for all $y \in \partial\big(B_R(x_0) \setminus K\big)$ and by the comparison principle it also holds on $B_R(x_0) \setminus K$, so we have

$$u(x_0) - \frac{C_1}{C} v_{a,x_0}(x) \le u(x) \le u(x_0) + \frac{C_1}{C} v_{a,x_0}(x)$$

for all $x \in B_R(x_0)$. Using again the upper estimate in (2.1) for v_{a,x_0} we get

$$v_{a,x_0}(x) \le C_1 \,|\, x - x_0 \,|^\alpha \quad \text{for all } \; x \in B_R(x_0)$$

which gives

$$u(x_0) - \frac{C_1^2}{C} \,|\, x - x_0 \,|^\alpha \le u(x) \le u(x_0) + \frac{C_1^2}{C} \,|\, x - x_0 \,|^\alpha$$

for all $x \in B_R(x_0)$, which is the Hölder continuity of u at x_0. This concludes the statement as x_0 is arbitrary and the Hölder seminorm of u is then bounded by C_1^2/C, independently of x_0.

□

3.2 Proof of Theorem 2

For the proof of Theorem 2, we need the following variants of Lemmas 1 and 2, for the case when f satisfies condition (0.5).

Lemma 1'. Assume $p > n$ and f satisfy the condition (0.5). Then, for any $x_0 \in S_{2R}$, $R > 1$, there exists a family of radially symmetric supersolutions $\{v_{a,x_0}\}_{a \ge 0}$ of (0.6) in $B_R(x_0) \setminus \{x_0\}$ satisfying

$$\left(\frac{C_f}{nL}\right)^{\frac{1}{p-1}} a \,\frac{|\, x - x_0 \,|^\alpha}{\alpha} \le v_a(x) \le \left(\frac{C_f}{n\delta}\right)^{\frac{1}{p-1}} \left(a + R^{-\frac{p-n+\epsilon}{p-1}} \right) \frac{|\, x - x_0 \,|^\alpha}{\alpha}, \tag{3.6}$$

for $a \ge 0$, with $\alpha = \frac{p-n}{p-1}$.

Proof. This comes by noting on Lemma 1 that, under hypothesis (0.5), $\|f\|_{L^\infty(B_R(x_0))} \le C_f\, R^{-p-\epsilon}$ and by redefining a to $aR^{-\frac{p+\epsilon}{p-1}}$. □

The second improvement concerns about supersolutions defined on the complement of large balls.

Lemma 2'. Assume $p \ge n$ and f satisfy the condition (0.5). Then, for all $R > 1$, there exists a family of radially symmetric supersolutions $\{v_a\}_{a \ge 0}$ of (0.6) in $\mathbb{R}^n \setminus B_R(0)$ satisfying

i) $v_a(R) = 0$ and $v_a(r)$ is increasing in $[R, +\infty)$ for any $a \ge 0$;

ii) v_a is unbounded in $[R, +\infty)$ for $a > 0$; indeed, there exists $c_0 = c_0(n, p, \epsilon, C_f, L) > 0$ such that

$$v_a(r) \ge c_0\, a\, (r^\alpha - R^\alpha) \quad \text{for} \quad r \ge R\,, \quad \text{if} \quad p > n$$

$$v_a(r) \ge c_0\, a\, (\log r - \log R) \quad \text{for} \quad r \ge R\,, \qquad \text{if} \quad p = n\,;$$

iii) v_0 is bounded in $[R,+\infty)$; indeed, there exists $C_0 = C_0(n,p,\epsilon,C_f,\delta) > 0$ such that

$$v_0(r) \,\leq\, C_0\,(R^{-\frac{\epsilon}{p-1}} - r^{-\frac{\epsilon}{p-1}}) \quad \text{for} \quad r \geq R\,;$$

iv) $v_a(r) \to v_0(r)$ as $a \to 0$ for any $r \in [R,+\infty)$.

Proof. This also follows the same lines of the proof of Lemma 2. In this case, the function g now can be taken as

$$g(r) \,=\, \frac{C_f}{r^{\,p+\epsilon}}\;,\quad r \,\geq\, R$$

and integrating from R onwards we obtain

$$v(r) = \int_R^r \varphi^{-1}\left(\frac{\frac{C_f}{p-n+\epsilon}\big(t^{n-p-\epsilon} - R^{n-p-\epsilon}\big) + C}{t^{n-1}}\right) dt$$

for $r \geq R$, with

$$C \,=\, |v'(R)|^{\,p-2} A\big(\,|v'(R)|\,\big)\, v'(R)\,.$$

Putting

$$C = \frac{C_f}{p-n+\epsilon}\,(\,a^{p-1} + R^{n-p-\epsilon}\,),\;\; a \,\geq\, 0$$

it follows

$$v_a(r) = \int_R^r \varphi^{-1}\left(\frac{\frac{C_f}{p-n+\epsilon}\big(t^{n-p-\epsilon} + a^{p-1}\big)}{t^{n-1}}\right) dt$$

Then using the estimates (2.5) for φ^{-1} we get

$$\begin{aligned} v_a(r) \,&\geq \left(\frac{C_f}{L(p-n+\epsilon)}\right)^{\frac{1}{p-1}} \int_R^r \big(\,t^{\,n-p-\epsilon} + a^{\,p-1}\,\big)^{\frac{1}{p-1}}\, t^{-\frac{n-1}{p-1}}\, dt \\ &\geq \left(\frac{C_f}{L(p-n+\epsilon)}\right)^{\frac{1}{p-1}} \int_R^r a\, t^{-\frac{n-1}{p-1}}\, dt \end{aligned}$$

from which follows

$$v_a(r) \,\geq\, \left(\frac{C_f}{L(p-n+\epsilon)}\right)^{\frac{1}{p-1}} a\,(\,r^{\,\alpha} - R^{\,\alpha}\,) \tag{3.7}$$

in case $p > n$ and

$$v_a(r) \,\geq \left(\frac{C_f}{L\,\epsilon}\right)^{\frac{1}{p-1}} a\,(\,\log r - \log R\,) \tag{3.8}$$

if $p = n$. For the upper bound for $a = 0$ we have for $p \geq n$

$$\begin{aligned} v_0(r) \,&\leq\, \left(\frac{C_f}{\delta(p-n+\epsilon)}\right)^{\frac{1}{p-1}} \int_R^r \left(\frac{t^{n-p-\epsilon}}{t^{n-1}}\right)^{\frac{1}{p-1}} dt \\ &\leq\, \left(\frac{C_f}{\delta(p-n+\epsilon)}\right)^{\frac{1}{p-1}} \Big(\frac{p-1}{\epsilon}\Big)\, R^{-\frac{\epsilon}{p-1}}\,. \end{aligned} \tag{3.9}$$

$\square$

To fix some notation, for each $R \geq 1$ we denote $M_R = \sup_{S_R} u$, $m_R = \inf_{S_R} u$, S_R being the sphere of radius R centered at the origin. The oscillation of u on S_R is defined as

$$\underset{S_R}{osc}\, u = M_R - m_R\,.$$

The next result is a kind of extension of estimates obtained in [61] (or Proposition 3 of [3]) for the nonhomogeneous case.

Theorem 8. Let $u \in W^{1,p}_{loc}(\mathbb{R}^n \setminus \overline{B_1})$ a bounded weak solution of (0.6), with f satisfying condition (0.5). Then, in case $p \geq n$, for all $R \geq R_0$,

$$m_R - C_0 R^{-\frac{\epsilon}{p-1}} \leq u(x) \leq M_R + C_0 R^{-\frac{\epsilon}{p-1}} \quad \text{for} \quad x \in \mathbb{R}^n \setminus B_R, \tag{3.10}$$

where $C_0 = \left(\frac{C_f}{\delta(p-n+\epsilon)}\right)^{\frac{1}{p-1}} \left(\frac{p-1}{\epsilon}\right)$. In particular, if $R_0 = 1$, we have the following global bound for u:

$$\inf_{S_1} u - C_0 \leq u \leq \sup_{S_1} u + C_0.$$

Proof. Assume with no loss of generality that (0.5) holds for all $|x| \geq 1$. Suppose now that the weak solution u in question satisfies (0.10). For $R \geq 1$, consider the family of radially symmetric supersolutions $\{v_a\}_{a\geq 0}$ given by Lemma 2'. Hence the second property of v_a and since u is bounded(or satisfies the properties (0.10) or (0.11)), we obtain for each $a > 0$ a $R_a > R$ such that

$$M_R + v_a(|x|) \geq u(|x|) \quad \text{for all} \quad |x| \geq R_a\,.$$

Consequently, the function $w_a(r) := M_R + v_a(r)$, $r \geq R$, lies above u on the boundary of the annulus $B_{R_a} \setminus B_R$. Then, by the comparison principle, $w_a \geq u$ on $B_{R_a} \setminus B_R$, that is, $w_a \geq u$ on $\mathbb{R}^n \setminus B_R$. Then, for $x \in \mathbb{R}^n \setminus B_R$, the third and fourth properties of $\{v_a\}$ in Lemma 2' imply that

$$u(x) \leq \lim_{a\to 0} w_a(|x|) = M_R + v_0(|x|) < M_R + C_0 R^{-\frac{\epsilon}{p-1}}\,. \tag{3.11}$$

From (3.9), we can see that C_0 is given by

$$C_0 = \left(\frac{C_f}{\delta(p-n+\epsilon)}\right)^{\frac{1}{p-1}} \left(\frac{p-1}{\epsilon}\right).$$

Analogously, we can prove that $u(x) \geq m_R - C_0 R^{-\frac{\epsilon}{p-1}}$ for $x \in \mathbb{R}^n \setminus B_R$.

□

Corollary. Under the hypotheses of the previous Theorem, the limits

$$\lim_{x\to\infty} \max_{S_{|x|}} u, \quad \lim_{x\to\infty} \min_{S_{|x|}} u \tag{3.12}$$

are both finite.

Proof of Theorem 2.

1. Existence of the limit for nonnegative solutions.

Let u be a nonnegative weak solution of (0.7) on $\mathbb{R}^n \setminus \overline{B_1}$ and set $m = \liminf_{|x|\to\infty} u$. If $m = +\infty$, there is nothing to prove, so we assume $m < +\infty$. For a given $\varepsilon > 0$, there is some $R_0 > 0$ such that

$$u(x) \, > \, m - \varepsilon \; \text{ for all } x \text{ such that } |x| \geq R_0$$

so that the function

$$v \, = \, u - m + \varepsilon$$

is a positive solution on $\mathbb{R}^n \setminus B_{R_0}$. We pick up a sequence of points (x_k), with $|x_k| \to \infty$, $R_0 < |x_k| < |x_{k+1}|$, such that

$$u(x_k) \, \leq \, m + \epsilon$$

and, consequently,

$$v(x_k) \, \leq \, 2\epsilon \, . \tag{3.13}$$

Now let $R_k = |x_k|$, $S_{R_k} = \partial B_{R_k}(0)$. By applying the Corollary 2 to v we get

$$\sup_{S_{R_k}} v \, \leq \, C \left(\inf_{S_{R_k}} v + R_k^{-\frac{\epsilon}{p-1}} \right)$$

for a positive constant C independent of k. Hence, since $R^{-\frac{\epsilon}{p-1}} \to 0$ as $R \to \infty$, by (3.13) it follows that

$$\sup_{S_{R_k}} v \, \leq \, C\,\epsilon \, , \quad \text{for all } k \text{ sufficiently large}$$

and, consequently,

$$\sup_{\partial A(R_k, R_{k+1})} v \quad \leq \, C\,\epsilon \, , \quad \text{for all } k \text{ sufficiently large.} \tag{3.14}$$

We then proceed to bound v on the interior of each annuli with the use of barriers. By Lemma 2', *iii*), for each k, we have a positive supersolution $v_0 = v_{0,\,k}$ in $\mathbb{R}^n \setminus \overline{B_{R_k}}$ satisfying

$$v_{0,\,k} \leq C_0 \, R_k^{-\frac{\epsilon}{p-1}} \, .$$

Hence, the function

$$w_k(x) \, := \, C\,\varepsilon \, + \, v_{0,\,k}(x)$$

where C is the constant from (3.13), is such that, for any natural $l > k$, $w_k \geq v$ in $\partial A(R_k, R_l)$. The comparison principle then gives $w_k \geq v$ in $A(R_k, R_l)$, from which follows the bound

$$v \, \leq \, w_k \, \leq \, C\,\varepsilon \, + \, R_k^{-\frac{\epsilon}{p-1}} \quad \text{ in } \; \mathbb{R}^n \setminus \overline{B_{R_k}}$$

and, by redefining the constant C,

$$v(x) \le C\,\epsilon\,, \quad \text{for all } |x| \text{ sufficiently large.}$$

Then, by definition of v, we have

$$u(x) - m \le C\,\epsilon\,, \quad \text{for all } |x| \text{ sufficiently large,}$$

and by arbitrariness of ϵ it follows

$$\limsup_{|x|\to\infty} u \le m$$

which proves $\lim_{|x|\to\infty} u(x) = m\,.$

$\square$

Now we turn to the statement on the convergence rate estimate for the case $p > n$. The following Lemma establishes some control of the oscillation of u.

Lemma 3. Let $u \in C^1(\mathbb{R}^n \setminus B_1)$ a bounded solution of (0.7) in $\mathbb{R}^n \setminus B_1$ and assume f satisfy condition (0.5) and $p > n$. Then, there are constants $0 < C < 1$ and $K \ge 0$, such that

$$\underset{S_{2R}}{osc}\, u \le C\left(\underset{S_R}{osc}\, u + K.R^{-\frac{\epsilon}{p-1}} \right) \tag{3.15}$$

for all $R \ge 1$.

Proof. Given $R \ge 1$, let $x_1 \in S_{2R}$ such that $u(x_1) = m_{2R}$. For $x' \in S_{2R}$, let $\gamma \subset S_{2R}$ be an arc of circle joining x_1 to x'. By a recursive process starting at x_1, we obtain estimates for u on successive balls with centers in γ, up to x'.

In the first step, we set $u_1 = u(x_1)$ and define for $x \in B_R(x_1)$

$$w_1(x) = w_1(r) = u_1 + v_{a_1,x_1}(r), \;\; r = |x - x_1| \le R,$$

where v_{a_1,x_1} is a supersolution in $B_R(x_1)$ given by Lemma 3.2. We will chose a_1 so that

$$w_1(R) \ge M_R + C_0 R^{-\frac{\epsilon}{p-1}}.$$

For this, using the lower estimate for v_{a_1,x_1} in Lemma 3.2, it is sufficient to require

$$u_1 + \left(\frac{C_f}{nL}\right)^{\frac{1}{p-1}} a_1 \frac{R^\alpha}{\alpha} \ge M_R + C_0 R^{-\frac{\epsilon}{p-1}}$$

where solving for a_1 we get

$$a_1 \ge \alpha R^{-\alpha} \left(\frac{C_f}{nL}\right)^{-\frac{1}{p-1}} \left(M_R + C_0 R^{-\frac{\epsilon}{p-1}} - u_1\right)\,.$$

Hence, putting

$$a_1 \,=\, \alpha R^{-\alpha} \left(\frac{C_f}{nL}\right)^{-\frac{1}{p-1}} \left(M_R + C_0 R^{-\frac{\epsilon}{p-1}} - u_1\right) \tag{3.16}$$

we have $w_1 \geq u$ on $\partial B_R(x_1)$ so that, by the comparison principle,

$$w_1 \geq u \quad \text{on } B_R(x_1)\,. \tag{3.17}$$

Next, we wish to find some radius $R_1 \leq R$ such that

$$w_1(r) \,\leq\, M_R + C_0 R^{-\frac{\epsilon}{p-1}} + \frac{1}{\alpha}\left(\frac{C_f}{nL}\right)^{\frac{1}{p-1}} R^{-\frac{\epsilon}{p-1}} \quad \text{for all } r \,\leq\, R_1\,.$$

In view of the upper estimate in Lemma 1' we have

$$w_1(r) = u_1 + v_{a_1,x_1}(r) \leq u_1 + \left(\frac{C_f}{n\delta}\right)^{\frac{1}{p-1}} \left(\, a_1 + R^{-\frac{p-n+\epsilon}{p-1}}\,\right) \frac{r^\alpha}{\alpha}\,. \tag{3.18}$$

Hence, it is enough to find $R_1 \leq R$ such that

$$\begin{aligned} &u_1 + \left(\frac{C_f}{n\delta}\right)^{\frac{1}{p-1}} \left(\, a_1 + R^{-\frac{p-n+\epsilon}{p-1}}\,\right) \frac{R_1^\alpha}{\alpha} \\ &\leq M_R + C_0 R^{-\frac{\epsilon}{p-1}} + \frac{1}{\alpha}\left(\frac{C_f}{nL}\right)^{\frac{1}{p-1}} R^{-\frac{\epsilon}{p-1}}\,. \end{aligned} \tag{3.19}$$

Substituting the expression of a_1 and solving for R_1 gives

$$R_1 < \left(\frac{\delta}{L}\right)^{\frac{1}{(p-1)\alpha}} R = \left(\frac{\delta}{L}\right)^{\frac{1}{p-n}} R$$

so we take

$$R_1 = \lambda R, \quad \lambda = \frac{1}{2}\left(\frac{\delta}{L}\right)^{\frac{1}{p-n}}\,. \tag{3.20}$$

To the next step, motivated by (3.18), we define

$$u_2 \,=\, u_1 + \left(\frac{C_f}{n\delta}\right)^{\frac{1}{p-1}} \left(\, a_1 + R^{-\frac{p-n+\epsilon}{p-1}}\,\right) \frac{(\,\lambda R\,)^\alpha}{\alpha}$$

which is the upper bound for w_1 in $B_{\lambda R}(x_1)$. We then take

$$x_2 \in \gamma \cap \partial B_{\lambda R}(x_1)$$

the closest point to x' in this intersection and define as before

$$w_2(r) = u_2 + v_{a_2,x_2}(r), \quad \text{for } r = |x - x_2| \leq R$$

with v_{a_2,x_2} being the supersolution in $B_R(x_2)$ given in Lemma 3.2. Analogously to the previous step, the choice

$$a_2 \,=\, \alpha R^{-\alpha} \left(\frac{C_f}{nL}\right)^{-\frac{1}{p-1}} \left(M_R + C_0 R^{-\frac{\epsilon}{p-1}} - u_2\right)$$

ensures that

$$w_2 \geq u \;\text{ on } B_R(x_2)\,.$$

Also, the same calculation carried out in the first step shows

$$w_2(r) < M_R + C_0 R^{-\frac{\epsilon}{p-1}} + \frac{1}{\alpha}\left(\frac{C_f}{nL}\right)^{\frac{1}{p-1}} R^{-\frac{\epsilon}{p-1}}, \quad \text{for } r \leq \lambda R$$

for λ already defined in (3.20). Next we take

$$u_3 = u_2 + \left(\frac{C_f}{n\delta}\right)^{\frac{1}{p-1}} \left(a_2 + R^{-\frac{p-n+\epsilon}{p-1}} \right) \frac{(\lambda R)^\alpha}{\alpha}$$

and

$$x_3 \in \gamma \cap \partial B_{\lambda R}(x_2)$$

the closest point to x' in this intersection, and repeat the procedure. After $k-1$ steps, we reach at some point $x_k \in \gamma$, having defined

$$u_k = u_{k-1} + \left(\frac{C_f}{n\delta}\right)^{\frac{1}{p-1}} \left(a_{k-1} + R^{-\frac{p-n+\epsilon}{p-1}} \right) \frac{(\lambda R)^\alpha}{\alpha} \tag{3.21}$$

with

$$a_{k-1} = \alpha R^{-\alpha} \left(\frac{C_f}{nL}\right)^{-\frac{1}{p-1}} \left(M_R + C_0 R^{-\frac{\epsilon}{p-1}} - u_{k-1} \right) \tag{3.22}$$

and

$$u \leq u_k < M_R + C_0 R^{-\frac{\epsilon}{p-1}} + \frac{1}{\alpha}\left(\frac{C_f}{nL}\right)^{\frac{1}{p-1}} R^{-\frac{\epsilon}{p-1}} \quad \text{in } B_{\lambda R}(x_k)\,.$$

From (3.21), (3.22), we obtain the recurrence

$$\begin{aligned} u_k &= u_{k-1}\left(1 - \lambda^\alpha \left(\frac{L}{\delta}\right)^{\frac{1}{p-1}}\right) \quad + \\ &+ \quad \lambda^\alpha \left(\frac{L}{\delta}\right)^{\frac{1}{p-1}} \left(M_R + C_0\, R^{-\frac{\epsilon}{p-1}} + \frac{1}{\alpha}\left(\frac{C_f}{nL}\right)^{\frac{1}{p-1}} R^{-\frac{\epsilon}{p-1}} \right), \\ u_1 &= m_{2R} \end{aligned}$$

from which we determine

$$\begin{aligned} u_k &= M_R + C_b R^{-\frac{\epsilon}{p-1}} + \frac{1}{\alpha}\left(\frac{C_f}{nL}\right)^{\frac{1}{p-1}} R^{-\frac{\epsilon}{p-1}} \\ &- \left(M_R + C_b R^{-\frac{\epsilon}{p-1}} - m_{2R} + \frac{1}{\alpha}\left(\frac{C_f}{nL}\right)^{\frac{1}{p-1}} R^{-\frac{\epsilon}{p-1}} \right) \left(1 - \lambda^\alpha \left(\frac{L}{\delta}\right)^{\frac{1}{p-1}} \right)^{k-1} \end{aligned}$$

(Recall the solution to the recurrence relation

$$a\, u_k + b\, u_{k-1} + c \;=\; 0$$

is given by

$$u_k \;=\; -\frac{c}{a+b} + \Big(u_1 + \frac{c}{a+b}\Big)\Big(-\frac{b}{a}\Big)^{k-1}.)$$

We stop the process when γ is fully covered by the balls $B_{\lambda R}(x_k)$, which happens when the point x_k reaches a distance to x' less than λR. As the length of γ is less than $2R\pi$ and each ball covers a segment over γ with length greater than λR, we see the number l of balls needed to cover γ is independent of R and always less than $2\pi/\lambda+1$. Now, since $x' \in B_{\lambda R}(x_l)$ and $u \le w_l \le u_{l+1}$ in $B_{\lambda R}(x_l)$ it follows that

$$\begin{aligned} u(x') &\le u_{l+1} \\ &= M_R + C_0 R^{-\frac{\epsilon}{p-1}} + \frac{1}{\alpha}\left(\frac{C_f}{nL}\right)^{\frac{1}{p-1}} R^{-\frac{\epsilon}{p-1}} + \\ &\quad -\left(M_R + C_0 R^{-\frac{\epsilon}{p-1}} - m_{2R} + \frac{1}{\alpha}\left(\frac{C_f}{nL}\right)^{\frac{1}{p-1}} R^{-\frac{\epsilon}{p-1}} \right) c \end{aligned}$$

for

$$c = \left(1 - \lambda^\alpha \left(\frac{L}{\delta}\right)^{\frac{1}{p-1}}\right)^l = \Big(\frac{1}{2}\Big)^l < 1\,.$$

Being x' arbitrary, we have $M_{2R} \le u_{l+1}$ we have

$$\begin{aligned} &M_{2R} - m_{2R} \\ \le &\left(M_R + C_0 R^{-\frac{\epsilon}{p-1}} - m_{2R} + \frac{1}{\alpha}\left(\frac{C_f}{nL}\right)^{\frac{1}{p-1}} R^{-\frac{\epsilon}{p-1}} \right)(1-c) \end{aligned}$$

Then using that $m_{2R} \ge m_R - C_0 R^{-\frac{\epsilon}{p-1}}$, by Proposition 8, it comes

$$\begin{aligned} &M_{2R} - m_{2R} \\ \le &\left(M_R - m_R + 2C_0 R^{-\frac{\epsilon}{p-1}} + \frac{1}{\alpha}\left(\frac{C_f}{nL}\right)^{\frac{1}{p-1}} R^{-\frac{\epsilon}{p-1}} \right)(1-c) \end{aligned}$$

that is

$$\operatorname*{osc}_{S_{2R}} u \;\le\; C\,\Big(\, \operatorname*{osc}_{S_R} u + K R^{-\frac{\epsilon}{p-1}} \Big) \tag{3.23}$$

for

$$C \;=\; 1-c\,, \qquad K = 2C_0 + \frac{1}{\alpha}\left(\frac{C_f}{nL}\right)^{\frac{1}{p-1}}\,.$$

□

2. Proof of (0.9).

By iteration of inequality (3.23) we obtain

$$\operatorname*{osc}_{S_{2^k R}} u \;\le\; C^k \Big(\, \operatorname*{osc}_{S_R} u + K R^{-\frac{\epsilon}{p-1}} \sum_{j=1}^{k} \left(\frac{2^{-\frac{\epsilon}{p-1}}}{C}\right)^j \Big)\,.$$

Here we admit $C > 2^{-\frac{\epsilon}{p-1}}$, redefining C if this is not the case. Then we have

$$\sum_{j=1}^{k} \left(\frac{2^{-\frac{\epsilon}{p-1}}}{C} \right)^j \;\leq\; \frac{1}{1 - \frac{2^{-\frac{\epsilon}{p-1}}}{C}} \;\leq\; \frac{1}{C - 2^{-\frac{\epsilon}{p-1}}}$$

and we get

$$\underset{S_{2^k R}}{osc}\, u \;\leq\; C^k \left(\underset{S_R}{osc}\, u \,+\, \frac{K\, R^{-\frac{\epsilon}{p-1}}}{C - 2^{-\frac{\epsilon}{p-1}}} \right) \quad \text{for all} \quad R \geq 1\,. \tag{3.24}$$

In particular,

$$\underset{S_{2^k}}{osc}\, u \;\leq\; C^k \left(\underset{S_1}{osc}\, u \,+\, \frac{K}{C - 2^{-\frac{\epsilon}{p-1}}} \right). \tag{3.25}$$

Now, let $x \in \mathbb{R}^n \backslash B_1$ and let k the integer such that

$$2^k \leq |x| \leq 2^{k+1}\,. \tag{3.26}$$

From Theorem 8 and our assumption that $C > 2^{-\frac{\epsilon}{p-1}}$, we obtain

$$\begin{aligned} \underset{S_{|x|}}{osc}\, u \;&\leq\; \underset{S_{2^k}}{osc}\, u + 2C_0 \left(2^k\right)^{-\frac{\epsilon}{p-1}} \\ &\leq\; \underset{S_{2^k}}{osc}\, u + 2C_0 C^k \end{aligned}$$

and then, by (3.25),

$$\underset{S_{|x|}}{osc}\, u \leq \; \left(\underset{S_1}{osc}\, u + \frac{2K}{C - 2^{-\frac{\epsilon}{p-1}}} + 2C_0 \right) \, C^k\,. \tag{3.27}$$

Now (3.26) also gives

$$\log|x| \;\leq\; (k+1)\log 2\,, \quad \text{hence} \quad k \;\geq\; \frac{\log|x|}{\log 2} - 1\,.$$

Therefore, as $C < 1$, we have

$$C^k \;\leq\; \frac{C^{\frac{\log|x|}{\log 2}}}{C} \;=\; \frac{\left(e^{\log C}\right)^{\frac{\log|x|}{\log 2}}}{C} \;=\; \frac{|x|^{\frac{\log C}{\log 2}}}{C}$$

and then, by (3.2), if follows

$$\underset{S_{|x|}}{osc}\, u \;\leq\; \frac{1}{C} \left(\underset{S_1}{osc}\, u + \frac{2K}{C - 2^{-\frac{\epsilon}{p-1}}} + 2C_0 \right) \, |x|^{\frac{\log C}{\log 2}}$$

We rewrite this inequality as

$$\underset{S_{|x|}}{osc}\, u \;\leq\; C\,|x|^{-\tilde{\beta}} \tag{3.28}$$

where we have redefined the constant C and taken $\tilde{\beta} = -\frac{\log C}{\log 2}$.

Finally, we can conclude (0.9) by noting that, by Theorem 8, for any $x \in \mathbb{R}^n \backslash B_1$,

$$m_{|x|} - C_0\,|x|^{-\frac{\epsilon}{p-1}} \;\leq\; \ell \;\leq\; M_{|x|} + C_0\,|x|^{-\frac{\epsilon}{p-1}}$$

which implies

$$|\,u(x) - \ell\,| \;\leq\; \underset{S_{|x|}}{osc}\, u \;+ C_0\,|x|^{-\frac{\epsilon}{p-1}}\,. \tag{3.29}$$

Hence, using (3.28), it follows the existence of constants $C > 0$, $\beta = \min\{\,\tilde{\beta}, \frac{\epsilon}{p-1}\,\} > 0$ such that (0.9) holds.

$\square$

Chapter 4

RADIALLY SYMMETRIC SOLUTIONS OF THE FRACTIONAL p-LAPLACIAN EQUATION

In this chapter we study how the (s,p)-laplacian acts on the functions $x \mapsto |x|^\alpha$, $\alpha \neq 0$, and $x \mapsto \log|x|$, for $x \in \mathbb{R}^n \backslash \{0\}$.

Recall we have

$$(-\Delta)_p^s |\cdot|^\alpha(x) := \int_{\mathbb{R}^n} \frac{\big| |x|^\alpha - |y|^\alpha \big|^{p-2} \big(|x|^\alpha - |y|^\alpha \big)}{|x-y|^{n+sp}} \, dy \tag{4.1}$$

where $s \in (0,1)$ and $p \in (1,+\infty)$. At infinity, this integrand has an order of growth of $|y|^{\alpha(p-1)-n-sp}$, being integrable at infinity only if $\alpha < \frac{sp}{p-1}$. Besides the natural singularity at x, when $\alpha < 0$, the integrand has near the origin an order of growth of $|y|^{\alpha(p-1)}$, hence being integrable at 0 only if $\alpha > -\frac{n}{p-1}$. Therefore, the (s,p)-laplacian of the function $|\cdot|^\alpha$ is well defined if, and only if, $\alpha \in \big(-\frac{n}{p-1}, \frac{sp}{p-1}\big)$.

Proposition 3. $(-\Delta)_p^s |\cdot|^\alpha$ and $\log|\cdot|$ are radially symmetric functions and satisfy, for all $\lambda > 0$, $x \in \mathbb{R}^n \backslash \{0\}$,

$$(-\Delta)_p^s |\cdot|^\alpha(\lambda x) = \lambda^{\alpha(p-1)-sp} (-\Delta)_p^s |\cdot|^\alpha(x) \tag{4.2}$$

$$(-\Delta)_p^s \log|\cdot|(\lambda x) = \lambda^{-sp} (-\Delta)_p^s \log|\cdot|(x). \tag{4.3}$$

Proof. For the radial symmetry, we have for any rotation $R \in SO(n)$,

$$
\begin{aligned}
(-\Delta)_p^s |\cdot|^\alpha (Rx) &= \int\limits_{\mathbb{R}^n} \frac{\big| |Rx|^\alpha - |y|^\alpha \big|^{p-2} \big(|Rx|^\alpha - |y|^\alpha \big)}{|Rx - y|^{n+sp}}\, dy \\
&= \int\limits_{\mathbb{R}^n} \frac{\big| |Rx|^\alpha - |Rw|^\alpha \big|^{p-2} \big(|Rx|^\alpha - |Rw|^\alpha \big)}{|Rx - Rw|^{n+sp}}\, dw \\
&= \int\limits_{\mathbb{R}^n} \frac{\big| |x|^\alpha - |w|^\alpha \big|^{p-2} \big(|x|^\alpha - |w|^\alpha \big)}{|x - w|^{n+sp}}\, dw \\
&= (-\Delta)_p^s |\cdot|^\alpha (x)
\end{aligned}
$$

where we have used the change o variables $y = Rw$, $dy = dw$. An analogous calculation can be done with $\log|\cdot|$. For (4.2), we have

$$
\begin{aligned}
(-\Delta)_p^s |\cdot|^\alpha (\lambda x) &= \int\limits_{\mathbb{R}^n} \frac{\big| |\lambda x|^\alpha - |y|^\alpha \big|^{p-2} \big(|\lambda x|^\alpha - |y|^\alpha \big)}{|\lambda x - y|^{n+sp}}\, dy \\
&= \int\limits_{\mathbb{R}^n} \frac{\big| |\lambda x|^\alpha - |\lambda w|^\alpha \big|^{p-2} \big(|\lambda x|^\alpha - |\lambda w|^\alpha \big)}{|\lambda x - \lambda w|^{n+sp}}\, \lambda^n\, dw \\
&= \int\limits_{\mathbb{R}^n} \lambda^{\alpha(p-1)-sp} \frac{\big| |x|^\alpha - |w|^\alpha \big|^{p-2} \big(|x|^\alpha - |w|^\alpha \big)}{|x - w|^{n+sp}}\, dw \\
&= \lambda^{\alpha(p-1)-sp} (-\Delta)_p^s |\cdot|^\alpha (x)
\end{aligned}
\tag{4.4}
$$

where the second equality comes by the change of variables $y = \lambda w$, $dy = \lambda^n w$. The same way we find

$$
\begin{aligned}
(-\Delta)_p^s \log|\cdot| (\lambda x) &= \int\limits_{\mathbb{R}^n} \frac{\big| \log|\lambda x| - \log|y| \big|^{p-2} \big(\log|\lambda x| - \log|y| \big)}{|\lambda x - y|^{n+sp}}\, dy \\
&= \int\limits_{\mathbb{R}^n} \frac{\big| \log|x| - \log|w| \big|^{p-2} \big(\log|x| - \log|w| \big)}{\lambda^{n+sp} |x - w|^{n+sp}}\, \lambda^n\, dw \\
&= \lambda^{-sp} (-\Delta)_p^s \log|\cdot| (x)\,.
\end{aligned}
\tag{4.5}
$$

□

In particular, Proposition 3 gives for all $x \in \mathbb{R}^n \backslash \{0\}$

$$
\begin{aligned}
(-\Delta)_p^s |\cdot|^\alpha (x) &= |x|^{\alpha(p-1)-sp} (-\Delta)_p^s |\cdot|^\alpha \Big(\frac{x}{|x|} \Big) = |x|^{\alpha(p-1)-sp} (-\Delta)_p^s |\cdot|^\alpha (e_1) \\
(-\Delta)_p^s \log|\cdot|(x) &= |x|^{-sp} (-\Delta)_p^s \log|\cdot|(e_1)
\end{aligned}
\tag{4.6}
$$

where we have taken by choice the unit vector $e_1 \in \mathbb{R}^n$.

Proposition 4. Let $e_1 \in \mathbb{R}^n$ the unit vector. Then

$$(-\Delta)_p^s |\cdot|^\alpha(e_1) = \int\limits_{B_1} \left(1-|y|^{-n+sp-\alpha(p-1)}\right) \frac{\left|1-|y|^\alpha\right|^{p-2}\left(1-|y|^\alpha\right)}{|e_1-y|^{n+sp}}\,dy \tag{4.7}$$

Proof. Let $\mathcal{C}B_1$ the complement of B_1 in $\mathbb{R}^n$. We have

$$\begin{aligned}(-\Delta)_p^s |\cdot|^\alpha(e_1) &= \lim_{\epsilon\to 0} \int\limits_{\mathbb{R}^n\setminus B_\epsilon(e_1)} \frac{\left|1-|y|^\alpha\right|^{p-2}\left(1-|y|^\alpha\right)}{|e_1-y|^{n+sp}}\,dy \\ &= \lim_{\epsilon\to 0}\left(\int\limits_{B_1\setminus B_\epsilon(e_1)} \frac{\left|1-|y|^\alpha\right|^{p-2}\left(1-|y|^\alpha\right)}{|e_1-y|^{n+sp}}\,dy \; + \right. \\ &\qquad \left. + \int\limits_{\mathcal{C}B_1\setminus B_\epsilon(e_1)} \frac{\left|1-|y|^\alpha\right|^{p-2}\left(1-|y|^\alpha\right)}{|e_1-y|^{n+sp}}\,dy \right).\end{aligned} \tag{4.8}$$

We plan to change variables in the last integral using the inversion through ∂B_1 in $\mathbb{R}^n$, namely, the mapping $T : \mathbb{R}^n\setminus\{0\} \to \mathbb{R}^n\setminus\{0\}$ defined by $T(w) := |w|^{-2}w$. T is actually an involution that maps the interior of $B_1\setminus\{0\}$ onto $\mathcal{C}B_1$ while keeps fixed the points of ∂B_1.

Making the change of variables $y = T(w) = |w|^{-2}w$, $dy = |w|^{-2n}dw$, for $w \in T^{-1}\left(\mathcal{C}B_1\setminus B_\epsilon(e_1)\right) = T\left(\mathcal{C}B_1\setminus B_\epsilon(e_1)\right)$, we get

$$\begin{aligned}&\int\limits_{\mathcal{C}B_1\setminus B_\epsilon(e_1)} \frac{\left|1-|y|^\alpha\right|^{p-2}\left(1-|y|^\alpha\right)}{|e_1-y|^{n+sp}}\,dy \\ &= \int\limits_{T\left(\mathcal{C}B_1\setminus B_\epsilon(e_1)\right)} \frac{\left|1-|w|^{-\alpha}\right|^{p-2}\left(1-|w|^{-\alpha}\right)}{\left|e_1-|w|^{-2}w\right|^{n+sp}}\,|w|^{-2n}dw \\ &= \int\limits_{T\left(\mathcal{C}B_1\setminus B_\epsilon(e_1)\right)} \frac{|w|^{-\alpha(p-2)}\left||w|^\alpha-1\right|^{p-2}|w|^{-\alpha}\left(|w|^\alpha-1\right)}{|w|^{-(n+sp)}\left||w|e_1-|w|^{-1}w\right|^{n+sp}}\,|w|^{-2n}dw \\ &= \int\limits_{T\left(\mathcal{C}B_1\setminus B_\epsilon(e_1)\right)} |w|^{-n+sp-\alpha(p-1)} \frac{\left||w|^\alpha-1\right|^{p-2}\left(|w|^\alpha-1\right)}{|w-e_1|^{n+sp}}\,dw\,,\end{aligned} \tag{4.9}$$

where for the last equality we have used the identity

$$\left||w|e_1-|w|^{-1}w\right|^2 = |w|^2-2\langle |w|e_1, |w|^{-1}w\rangle+1 = |w|^2-2\langle e_1, w\rangle+1 = \left|e_1-w\right|^2. \tag{4.10}$$

Now about $T\left(\mathcal{C}B_1\setminus B_\epsilon(e_1)\right)$ we claim

$$\left(B_1\setminus\{0\}\right)\setminus B_\epsilon(e_1) \subseteq T\left(\mathcal{C}B_1\setminus B_\epsilon(e_1)\right) \subseteq \left(B_1\setminus\{0\}\right)\setminus B_{\frac{\epsilon}{1+\epsilon}}(e_1)\,. \tag{4.11}$$

To confirm this, note that, for any $y \neq 0$,

$$\begin{aligned} |\,Ty - e_1\,| \;&=\; \big|\, |y|^{-2}y - e_1 \,\big| \;=\; |y|^{-1}\, \big|\, |y|^{-1}y - |y|e_1 \,\big| \\ &=\; |Ty|\, |\,y - e_1\,|\,, \end{aligned} \tag{4.12}$$

by using $|y|^{-1} = |Ty|$ and (4.10). Then, as we always have $|Ty| \geq 1 - |\,Ty - e_1\,|$, it follows that

$$|\,Ty - e_1\,| \;\geq\; \frac{|\,y - e_1\,|}{1 + |\,y - e_1\,|}\,. \tag{4.13}$$

In particular, for $y \in \mathcal{C}B_1 \backslash B_\epsilon(e_1)$ we get

$$|\,Ty - e_1\,| \;\geq\; \frac{\epsilon}{1+\epsilon}\,, \tag{4.14}$$

which proves the second inclusion in (4.11). Still, for $w \in \big(\,B_1\backslash\{0\}\,\big) \setminus B_\epsilon(e_1)$, (4.12) gives

$$|\,Tw - e_1\,| \;=\; |w|^{-1}\,|\,w - e_1\,| \;\geq\; \epsilon\,. \tag{4.15}$$

Hence

$$T\big(\,\big(\,B_1\backslash\{0\}\,\big) \setminus B_\epsilon(e_1)\big) \;\subseteq\; \mathcal{C}B_1\backslash B_\epsilon(e_1)\,,$$

which is equivalent to the first inclusion of (4.11) and proves the claim.

Now putting

$$\Gamma_\epsilon \;:=\; T\,\big(\,\mathcal{C}B_1\backslash B_\epsilon(e_1)\,\big) \,\cap\, B_\epsilon(e_1) \tag{4.16}$$

by the claim, we have the disjoint union

$$T\,\big(\,\mathcal{C}B_1\backslash B_\epsilon(e_1)\,\big) \;=\; \Big(\,\big(\,B_1\backslash\{0\}\,\big) \setminus B_\epsilon(e_1)\Big) \,\cup\, \Gamma_\epsilon\,.$$

Hence, (4.9) writes

$$\begin{aligned} &\int\limits_{\mathcal{C}B_1\backslash B_\epsilon(e_1)} \frac{\big|\,1 - |\,y\,|^\alpha\,\big|^{p-2}\,\big(\,1 - |\,y\,|^\alpha\,\big)}{|\,e_1 - y\,|^{n+sp}}\,dy \\ &= \int\limits_{B_1\backslash B_\epsilon(e_1)} |w|^{-n+sp-\alpha(p-1)}\, \frac{\big|\,|w|^\alpha - 1\,\big|^{p-2}\,\big(\,|w|^\alpha - 1\,\big)}{|\,w - e_1\,|^{n+sp}}\,dw \;+ \\ &\quad+ \int\limits_{\Gamma_\epsilon} |w|^{-n+sp-\alpha(p-1)}\, \frac{\big|\,|w|^\alpha - 1\,\big|^{p-2}\,\big(\,|w|^\alpha - 1\,\big)}{|\,w - e_1\,|^{n+sp}}\,dw \end{aligned} \tag{4.17}$$

which taken to (4.8) yields

$$\begin{aligned} &(-\Delta\,)_p^s\,|\cdot|^\alpha\,(e_1) \\ &= \lim_{\epsilon\to 0} \Bigg(\int\limits_{B_1\backslash B_\epsilon(e_1)} \big(\,1 - |y|^{-n+sp-\alpha(p-1)}\,\big)\, \frac{\big|\,1 - |\,y\,|^\alpha\,\big|^{p-2}\,\big(\,1 - |\,y\,|^\alpha\,\big)}{|\,e_1 - y\,|^{n+sp}}\,dy \;+ \\ &\quad+ \int\limits_{\Gamma_\epsilon} |y|^{-n+sp-\alpha(p-1)}\, \frac{\big|\,1 - |\,y\,|^\alpha\,\big|^{p-2}\,\big(\,|\,y\,|^\alpha - 1\,\big)}{|\,e_1 - y\,|^{n+sp}}\,dy \Bigg)\,. \end{aligned} \tag{4.18}$$

We claim

$$\lim_{\epsilon\to 0} \int\limits_{\Gamma_\epsilon} |y|^{-n+sp-\alpha(p-1)} \frac{\big|1-|\,y\,|^{\alpha}\,\big|^{p-2}\big(\,|\,y\,|^{\alpha}-1\,\big)}{|\,e_1-y\,|^{n+sp}}\,dy \;=\; 0\,. \tag{4.19}$$

For this, notice that, by (4.11),

$$\Gamma_\epsilon \;\subseteq\; A\Big(\frac{\epsilon}{1+\epsilon},\epsilon\Big)(e_1) \tag{4.20}$$

so we have

$$\begin{aligned} &\left|\;\int\limits_{\Gamma_\epsilon} |y|^{-n+sp-\alpha(p-1)} \frac{\big|1-|\,y\,|^{\alpha}\,\big|^{p-2}\big(\,|\,y\,|^{\alpha}-1\,\big)}{|\,e_1-y\,|^{n+sp}}\,dy\;\right| \\ \leq &\int\limits_{A\left(\frac{\epsilon}{1+\epsilon},\epsilon\right)(e_1)} |y|^{-n+sp-\alpha(p-1)} \frac{\big|1-|\,y\,|^{\alpha}\,\big|^{p-1}}{|\,e_1-y\,|^{n+sp}}\,dy\,. \end{aligned} \tag{4.21}$$

As Γ_ϵ is far from the origin, we can clearly bound $|y|^{-n+sp-\alpha(p-1)}$ by a constant. Moreover, using that $x\mapsto|x|^\alpha$ is a locally Lipschitz function in $\mathbb{R}^n\backslash\{0\}$, we can estimate for some constant $C>0$ independent of ϵ

$$\big|\,|\,y\,|^{\alpha}-1\,\big| \;\leq\; C\,|\,y-e_1\,|\;,\;\; \text{for all }\; y\in B_\epsilon(e_1)\,.$$

Hence it follows

$$\begin{aligned} \int\limits_{A\left(\frac{\epsilon}{1+\epsilon},\epsilon\right)(e_1)} |y|^{-n+sp-\alpha(p-1)} \frac{\big|1-|\,y\,|^{\alpha}\,\big|^{p-1}}{|\,e_1-y\,|^{n+sp}}\,dy \;&\leq C \int\limits_{A\left(\frac{\epsilon}{1+\epsilon},\epsilon\right)(e_1)} |\,e_1-y\,|^{p-1-n-sp}\,dy \\ &\leq C\int_{\frac{\epsilon}{1+\epsilon}}^{\epsilon} r^{(1-s)p-2}\,dr \end{aligned} \tag{4.22}$$

with the constant C being appropriately redefined. Now in case $(1-s)p-2=-1$,

$$\int_{\frac{\epsilon}{1+\epsilon}}^{\epsilon} r^{(1-s)p-2}\,dr \;=\; \log(1+\epsilon)\to 0 \;\;\text{with}\;\; \epsilon\;\to\;0\,.$$

If $(1-s)p-2>-1$,

$$\int_{\frac{\epsilon}{1+\epsilon}}^{\epsilon} r^{(1-s)p-2}\,dr \;=\; C\,\Big(\,\epsilon^{(1-s)p-1}-\Big(\frac{\epsilon}{1+\epsilon}\Big)^{(1-s)p-1}\,\Big) \;\to 0\;\;\text{with}\;\;\epsilon\to 0$$

and, if $(1-s)p-2<-1$,

$$\int_{\frac{\epsilon}{1+\epsilon}}^{\epsilon} r^{(1-s)p-2}\,dr \;=\; C\,\frac{(1+\epsilon)^{-(1-s)p+1}-1}{\epsilon^{-(1-s)p+1}} \;\leq\; C\,\epsilon^{(1-s)p}\;\to 0\;\;\text{with}\;\;\epsilon\;\to\;0$$

proving the claim.

We then have from (4.18)

$$(-\Delta\,)^s_p\,|\cdot|^\alpha\,(e_1) \;=\; \lim_{\epsilon\to 0}\int\limits_{B_1\setminus B_\epsilon(e_1)} \big(\,1-|y|^{-n+sp-\alpha(p-1)}\,\big)\,\frac{\big|\,1-|\,y\,|^\alpha\,\big|^{p-2}\big(\,1-|\,y\,|^\alpha\,\big)}{|\,e_1-y\,|^{n+sp}}\,dy$$

and to finish the proof we note again that by the Lipschitz property on a neighbourhood of e_1,

$$\Big|\,\big(\,1-|y|^{-n+sp-\alpha(p-1)}\,\big)\,\frac{\big|\,1-|\,y\,|^\alpha\,\big|^{p-2}\big(\,1-|\,y\,|^\alpha\,\big)}{|\,e_1-y\,|^{n+sp}}\,\chi_{B_1\setminus B_\epsilon(e_1)}(y)\,\Big| \;\le\; C\,|\,e_1-y\,|^{(1-s)p-n}$$

for some constant C independent of ϵ. Now $y\mapsto\,|\,e_1-y\,|^{(1-s)p-n}$ is integrable in a neighbourhood of e_1, hence the limit is finite, proving (4.7).

$\square$

Theorem 3 can now be easily derived from Propositions 3 and 4, in case $sp\neq n$. In case $sp=n$, a similar calculation as carried out in Proposition 4, with $\log|\cdot|$, shows it solves the equation.

We remark that for other exponents $\alpha\neq 0$, (4.7) shows the sign of $(-\Delta\,)^s_p\,|\cdot|^\alpha$ is determined by the sign of the product

$$\big(\,1-|y|^{-n+sp-\alpha(p-1)}\,\big)\,\big(\,1-|\,y\,|^\alpha\,\big)\,.$$

By inspection on each case of α we can further state:

Theorem 3'.

$$\text{If } sp<n\,,\;\;(-\Delta\,)^s_p\,|\cdot|^\alpha\;\begin{cases} <\,0\,, & \text{for }\;\alpha<\frac{sp-n}{p-1}<0\;\text{ or }\;\alpha>0\\ =\,0\,, & \text{for }\;\alpha=0\;\text{ or }\;\alpha=\frac{sp-n}{p-1}\\ >\,0\,, & \text{for }\;\frac{sp-n}{p-1}<\alpha<0\,.\end{cases} \tag{4.23}$$

$$\text{If } sp>n\,,\;\;(-\Delta\,)^s_p\,|\cdot|^\alpha\;\begin{cases} <\,0\,, & \text{for }\;\alpha<0\;\text{ or }\;0<\frac{sp-n}{p-1}<\alpha\\ =\,0\,, & \text{for }\;\alpha=0\;\text{ or }\;\alpha=\frac{sp-n}{p-1}\\ >\,0\,, & \text{for }\;0<\alpha<\frac{sp-n}{p-1}\,.\end{cases} \tag{4.24}$$

Chapter 5

EXISTENCE THEOREM FOR THE EXTERIOR DIRICHLET PROBLEM FOR THE FRACTIONAL p-LAPLACIAN

The existence of solutions of the Dirichlet problem on bounded domains for operators of fractional p-laplacian type was addressed with great generality in [27, Theorem 17], from where we can state a result as follows.

Theorem. Let $\Omega \Subset \Omega'$ bounded open sets in $\mathbb{R}^n$ and assume Ω has Lipschitz regularity. Suppose $g \subset C(\Omega') \cap L^{p-1}_{sp}(\mathbb{R}^n)$. Then there is a unique weak solution of $(-\Delta)^s_p u = 0$ in Ω, which is continuous in Ω' and has boundary values g on $\mathbb{R}^n \setminus \Omega$.

On the original theorem, the assumption concerning the regularity of Ω is in fact weaker than Lipschitz regularity. It is assumed a measure density condition on $\mathbb{R}^n \setminus \Omega$, requiring the existence of $r_0 > 0$ and $\delta_\Omega \in (0, 1)$ such that, for every $x_0 \in \partial\Omega$,

$$\inf_{0<r<r_0} \frac{|\, (\mathbb{R}^n \backslash \Omega) \cap B_r(x_0)\, |}{|\, B_r(x_0)\, |} \geq \delta_\Omega\,. \tag{5.1}$$

We make use of this result to build the solution.

Proof of Theorem 4.

The uniqueness of bounded solutions is a direct consequence of the nonlocal comparison principle in [10, Theorem 3.3], presented in Preliminaries. For the existence part, we split the proof in the following steps.

1. Construction of a bounded solution u.

We start by considering a decreasing sequence of smooth compact sets K_m satisfying
$i)$ $K \Subset K_{m+1} \Subset K_m$
$ii)$ $dist(\partial K, \partial K_m) \to 0$.

Taking some $R_0 > 0$ such that $K_m \Subset B_{R_0}$, we continuously extend g to the whole $\mathbb{R}^n$, keeping fixed $\sup_K g$ and making $g = 0$ in $\mathbb{R}^n \setminus B_{R_0}$. Then taking an increasing sequence of radii $R_m \to +\infty$, we look for the domains $\Omega := B_{R_m} \setminus K_m$ and the problems

$$P_m : \begin{cases} (-\Delta)_p^s u_m = 0 & \text{in } \Omega \\ u_m = g & \text{in } K_m \\ u_m = 0 & \text{in } \mathbb{R}^n \setminus B_{R_m} . \end{cases} \tag{5.2}$$

By the existence theorem above, there is a solution $u_m \in W^{s,p}(\Omega) \cap C(\mathbb{R}^n)$ of each of the problems P_m. Next, by verifying the hypotheses of the Arzelá-Ascoli's theorem, we show a subsequence of u_m converges a.e. in $\mathbb{R}^n$. The equilimitation of u_m is get through the comparison principle, which gives

$$\sup |u_m| \leq \sup_{\mathbb{R}^n} |g| .$$

For the equicontinuity of u_m, we use the interior Hölder continuity result in [9, Theorem 1.2], which we can state in particular as follows.

Theorem. Let $s \in (0,1)$ and $p \in (1, \infty)$. Let $u \in W^{s,p}(\Omega) \cap L^{p-1}_{sp}(\mathbb{R}^n)$ be a weak solution of $(-\Delta)_p^s u = 0$. Then u is locally Hölder continuous in Ω. In particular, there are positive constants $\alpha < sp/(p-1)$ and c, both depending only on n, p, s, such that if $B_{2r}(x_0) \subset \Omega$, then

$$\underset{B_\rho(x_0)}{osc\, u} \leq c \left(\frac{\rho}{r}\right)^{\alpha} \left(\mathrm{Tail}\,(u; x_0, r) + \left(\fint_{B_{2r}(x_0)} |u|^p \, dx \right)^{\frac{1}{p}} \right)$$

holds whenever $\rho \in (0, r]$.

By applying the theorem to u_m, since the quantities $\mathrm{Tail}(u_m; x_0, r)$ and $\fint_{B_{2r}(x_0)} |u_m|^p \, dx$ are uniformly bounded on m, we obtain uniform Hölder continuity of u_m on balls of Ω, and it can easily be extended to compacts sets of $\mathbb{R}^n \setminus K$. Hence, u_m is also equicontinuous on compacts and, therefore, by using the Arzelá-Ascoli's Theorem and a continuous function u such that a subsequence of u_m converges to u $a.e.$ in $\mathbb{R}^n$.

To prove that u is a weak solution to the original problem (0.13), let $\eta \in C_0^\infty(\mathbb{R}^n \setminus K)$

a test function. We will show that

$$\begin{aligned}
&\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \frac{|\,u(x)-u(y)\,|^{p-2}}{|\,x-y\,|^{n+sp}} \big(\,u(x)-u(y)\,\big)\,\big(\,\eta(x)-\eta(y)\,\big)\,dxdy \\
= \;&\lim_{m\to\infty} \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \frac{|\,u_m(x)-u_m(y)\,|^{p-2}}{|\,x-y\,|^{n+sp}} \big(\,u_m(x)-u_m(y)\,\big)\,\big(\,\eta(x)-\eta(y)\,\big)\,dxdy
\end{aligned} \tag{5.3}$$

and since the integrals under the limit vanish, being u_m a weak solution to P_m, we conclude u is a weak solution of (0.13).

We obtain the convergence of the integrals by means of the Vitali Convergence Theorem, as in [53]. Before stating it, let us recall some definitions.

Let $(X, \mathcal{M}, \mu)$ be a general measure space and f_m a sequence of integrable functions on X. The sequence f_m is said to be *uniformly integrable* over X provided that, for each $\epsilon > 0$, there is a $\delta > 0$ such that, for any measurable subset E of X,

$$\mu(E) < \delta \quad \text{implies} \quad \sup_m \int\limits_E |f_m|\, d\mu < \epsilon\,. \tag{5.4}$$

The sequence f_m is said to be *tight* over X provided that, for each $\epsilon > 0$, there is a subset F of X, with finite measure, such that

$$\sup_m \int\limits_{X\setminus F} |f_m|\, d\mu < \epsilon\,. \tag{5.5}$$

Theorem. (Vitali Convergence Theorem) Let $(X, \mathcal{M}, \mu)$ be a measure space and f_m a sequence of functions on X that is uniformly integrable and tight over X. Assume that $f_m \to f$ *a.e.* on X and that the function f is integrable over X. Then

$$\lim_{m\to\infty} \int\limits_X f_m\, d\mu = \int\limits_X f\, d\mu\,.$$

We plan to apply Vitali's theorem to show (5.3). Observe that from the *a.e.* convergence $u_m \to u$ in $\mathbb{R}^n$, we readily obtain the convergence *a.e.* in $\mathbb{R}^n \times \mathbb{R}^n$ of the integrands

$$\begin{aligned}
&\lim_{m\to\infty} \frac{|\,u_m(x)-u_m(y)\,|^{p-2}}{|\,x-y\,|^{n+sp}} \big(\,u_m(x)-u_m(y)\,\big)\,\big(\,\eta(x)-\eta(y)\,\big) \\
= \;& \frac{|\,u(x)-u(y)\,|^{p-2}}{|\,x-y\,|^{n+sp}} \big(\,u(x)-u(y)\,\big)\,\big(\,\eta(x)-\eta(y)\,\big)\,.
\end{aligned} \tag{5.6}$$

To match the remaining hypotheses of the theorem, let U be an open set containing K such that $supp\,\eta \cap U = \varnothing$. Then, given any measurable set $E \subseteq \mathbb{R}^n \times \mathbb{R}^n$, by applying the Hölder's inequality, we can write

$$
\begin{aligned}
&\iint\limits_{E} \frac{|\, u_m(x) - u_m(y)\,|^{p-1}}{|\, x-y\,|^{n+sp}} \,\big|\, \eta(x) - \eta(y) \,\big|\, dxdy \\
&= \iint\limits_{E \setminus U \times U} \frac{|\, u_m(x) - u_m(y)\,|^{p-1}}{|\, x-y\,|^{n+sp}} \,\big|\, \eta(x) - \eta(y) \,\big|\, dxdy \\
&\leq \Bigg(\iint\limits_{E \setminus U \times U} \frac{|\, u_m(x) - u_m(y)\,|^{p}}{|\, x-y\,|^{n+sp}}\, dxdy \Bigg)^{\frac{p-1}{p}} \Bigg(\iint\limits_{E \setminus U \times U} \frac{|\, \eta(x) - \eta(y)\,|^{p}}{|\, x-y\,|^{n+sp}}\, dxdy \Bigg)^{\frac{1}{p}} \qquad (5.7) \\
&\leq \Bigg(\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n \setminus U} \frac{|\, u_m(x) - u_m(y)\,|^{p}}{|\, x-y\,|^{n+sp}}\, dxdy \Bigg)^{\frac{p-1}{p}} \Bigg(\iint\limits_{E} \frac{|\, \eta(x) - \eta(y)\,|^{p}}{|\, x-y\,|^{n+sp}}\, dxdy \Bigg)^{\frac{1}{p}}
\end{aligned}
$$

where for last inequality we have simply used that $E \setminus (U \times U) \subset \mathbb{R}^n \times (\mathbb{R}^n \setminus U)$ and also $E \setminus U \times U \subset E$. Now we apply [10, Theorem 3.1], shown in the preliminaries, to uniformly bound

$$
\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n \setminus U} \frac{|\, u_m(x) - u_m(y)\,|^{p}}{|\, x-y\,|^{n+sp}}\, dxdy \,\leq\, C \qquad (5.8)
$$

for some constant C that does not depend on u_m. We should notice that, although the theorem requires u_m to be a solution in the whole $\mathbb{R}^n \backslash K$, which is not the case here, the argument there still applies with the choice of a test function of the form $\psi^p\, u_m \in W_0^{s,p}(\Omega)$, with $\psi \in C^1(\mathbb{R}^n)$ satisfying $\psi = 0$ in K and $\psi = 1$ in $\mathbb{R}^n \backslash U$.

We have then obtained from (5.7), (5.8),

$$
\iint\limits_{E} \frac{|\, u_m(x) - u_m(y)\,|^{p-1}}{|\, x-y\,|^{n+sp}} \,\big|\, \eta(x) - \eta(y) \,\big|\, dxdy \,\leq\, C \Bigg(\iint\limits_{E} \frac{|\, \eta(x) - \eta(y)\,|^{p}}{|\, x-y\,|^{n+sp}}\, dxdy \Bigg)^{\frac{1}{p}}. \qquad (5.9)
$$

Therefore, the sequence of integrands is uniformly integrable in $\mathbb{R}^n \times \mathbb{R}^n$ since, given any $\epsilon > 0$, there is a $\delta > 0$ such that

$$
\iint\limits_{E} \frac{|\, \eta(x) - \eta(y)\,|^{p}}{|\, x-y\,|^{n+sp}}\, dxdy \,<\, \left(\frac{\epsilon}{C}\right)^p
$$

whenever $|E| < \delta$. Moreover, (5.9) also gives the tightness of the sequence since, for all $\epsilon > 0$, it is clear that there is a subset $F \subset \mathbb{R}^n \times \mathbb{R}^n$ with finite measure such that

$$
\iint\limits_{\mathbb{R}^n \times \mathbb{R}^n \setminus F} \frac{|\, \eta(x) - \eta(y)\,|^{p}}{|\, x-y\,|^{n+sp}}\, dxdy \,<\, \left(\frac{\epsilon}{C}\right)^p.
$$

Still, by taking $E = \mathbb{R}^n \times \mathbb{R}^n$, (5.9) gives

$$
\begin{aligned}
&\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \frac{|\, u_m(x) - u_m(y)\,|^{p-1}}{|\, x-y\,|^{n+sp}} \,\big|\, \eta(x) - \eta(y) \,\big|\, dxdy \\
&\qquad \leq\, C \Bigg(\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \frac{|\, \eta(x) - \eta(y)\,|^{p}}{|\, x-y\,|^{n+sp}}\, dxdy \Bigg)^{\frac{1}{p}} \qquad (5.10)
\end{aligned}
$$

so that, by Fatou's Lemma,

$$\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \frac{|\, u(x) - u(y)\,|^{p-1}}{|\, x-y\,|^{n+sp}} \,|\, \eta(x) - \eta(y)\,|\, dxdy \tag{5.11}$$
$$\leq \; C \left(\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \frac{|\, \eta(x) - \eta(y)\,|^{p}}{|\, x-y\,|^{n+sp}} \, dxdy \right)^{\frac{1}{p}} < \infty$$

hence showing the limit function is integrable. Therefore, by Vitalli's Theorem, this concludes that u is a weak solution.

□

2. Continuity up to the boundary

To establish next the continuity of u on ∂K, let $x_0 \in \partial K$ and $\epsilon > 0$. We consider the function

$$w(x) \;=\; w_C(x) \;:=\; C\,|\, x - x_0\,|^{\alpha} + g(x_0) + \epsilon$$

where $|\, x - x_0\,|^{\alpha}$ is the radial solution given in Theorem 3, with $\alpha = \frac{sp-n}{p-1} > 0$. We claim that $w_C \geq u$ in $\mathbb{R}^n$, for a sufficiently large constant C. First, by the continuity of g, there is some $R > 0$ such that

$$g(x_0) + \epsilon \;\geq\; g(x)\,, \quad \text{for} \;\; |x - x_0| < R$$

so clearly

$$w_C \;\geq\; g \;\; \text{in} \;\; B_R(x_0)$$

for all $C > 0$. We then choose C sufficiently large to make

$$w_C(x) \;=\; g(x_0) + C\,|\, x - x_0\,|^{\alpha} + \epsilon \;>\; \sup_{\mathbb{R}^n} \,|g|\,, \quad \text{for} \;\; |x - x_0| \geq R\,.$$

Now let u_m be a solution to the problem (5.2) and assume m is large enough so that $\partial K_m \cap B_R(x_0) \;\neq\; \varnothing$. Observe that $w > u_m$ outisde $B_R(x_0)$, since $w > \sup|g|$ outside $B_R(x_0)$ and $\sup u_m = \sup|g|$ by the comparison principle. Then, as $u_m = g$ in K_m and clearly $w > g$ in $\partial K_m \cap B_R(x_0)$, we can find a neighbourhood $\tilde{A}_m$ of K_m such that $w \geq u_m$ in $\tilde{A}_m$, so we have

$$w \geq u_m \quad \text{in} \;\; \tilde{A}_m \cup \big(\mathbb{R}^n \backslash B_R(y)\big).$$

Then, we can apply the comparison principle for w and u_m on the domain $B_R(y)\backslash\tilde{A}_m$, noting that $w, u_m \in W^{s,p}_{loc}(B_{\tilde{R}}(y)\backslash K_m)$, where we take a suitable $\tilde{R} > R$, with $B_R(x_0)\backslash\tilde{A}_m \Subset B_{\tilde{R}}(x_0)\backslash K_m$. This gives

$$w \;\geq\; u_m \;\;\; \text{in} \;\; B_R(x_0)$$

and consequently

$$w \geq u_m \quad \text{in } \ \mathbb{R}^n\,, \ \text{ for all } m$$

from which follows

$$w \geq u \quad \text{in } \ \mathbb{R}^n\,. \tag{5.12}$$

Finally, (5.12) implies

$$\limsup_{x \to x_0} u(x) \ \leq \ \limsup_{x \to x_0} w(x) \ = \ g(x_0) + \epsilon$$

and by arbitrariness of ϵ we conclude

$$\limsup_{x \to x_0} u(x) \ \leq \ g(x_0)\,.$$

By an analogous argument, we can also obtain

$$\liminf_{x \to x_0} u(x) \ \geq \ g(x_0) \tag{5.13}$$

and this finishes the argument.

$\square$

3. Hölder continuity up to the boundary

Being assumed in addition that g is α-Hölder continuous in K, with $\alpha = \frac{sp-n}{p-1}$, we show u is α-Hölder continuous in $\mathbb{R}^n$.

We claim that, if $C = |g|_\alpha$, the Hölder seminorm of g in K, then, for all $y \in K$,

$$g(y) - C\,|\,x - y\,|^\alpha \ \leq \ u(x) \ \leq \ g(y) + C\,|\,x - y\,|^\alpha\,, \quad \text{for all } x \in \mathbb{R}^n.$$

In fact, let $y \in K$. We have by definition of C

$$g(y) - C|\,z - y\,|^\alpha \ \leq \ g(z) \ \leq \ g(y) + C|\,z - y\,|^\alpha\,, \quad \text{for all } z \in K. \tag{5.14}$$

Now it is clear there is some $R > 0$ such that

$$C|\,x - y\,|^\alpha \ \geq \ 2 \sup |g|\,, \quad \text{for } \ |\,x - y\,| \ \geq \ R\,.$$

Since $|u| \leq \sup |g|$, this inequality gives

$$g(y) - C|\,x - y\,|^\alpha \ \leq \ u(x) \ \leq \ g(y) + C|\,x - y\,|^\alpha\,, \quad \text{for all } x \in \mathbb{R}^n \setminus B_R(y)\,. \tag{5.15}$$

Then using (5.14) with $g(z) = u(z)$, we see the inequality above holds on $K \cup \big(\mathbb{R}^n \setminus B_R(y)\big)$. Hence, for a fixed $\epsilon > 0$, the strict inequality

$$g(y) - C|\,x - y\,|^\alpha - \epsilon \ < \ u(x) \ < \ g(y) + C|\,x - y\,|^\alpha + \epsilon \tag{5.16}$$

holds, for all x in $K \cup \big(\mathbb{R}^n \setminus B_R(y)\big)$. By the continuity of u, we can find an open set $A \Supset K$ such that (5.16) holds for $x \in A \cup \big(\mathbb{R}^n \setminus B_R(y)\big)$. Thus, by applying the comparison principle in $B_R(y) \setminus A$, we obtain

$$g(y) - C|\, x - y\,|^{\alpha} - \epsilon \,\le\, u(x) \,\le\, g(y) + C|\, x - y\,|^{\alpha} + \epsilon \tag{5.17}$$

for $x \in B_R(y) \setminus A$ and, therefore, for all $x \in \mathbb{R}^n$, which concludes the claim by the arbitrariness of ϵ.

Now let $x \in \mathbb{R}^n \setminus K$. By the claim we have, for all $y \in K$,

$$u(y) - C\,|\, x - y\,|^{\alpha} \,\le\, u(x) \,\le\, u(y) + C\,|\, x - y\,|^{\alpha}$$

and these inequalities give

$$u(x) - C\,|\, x - y\,|^{\alpha} \,\le\, u(y) \,\le\, u(x) + C\,|\, x - y\,|^{\alpha}$$

for all $y \in K$. Then, taking some $\epsilon > 0$,

$$u(x) - C\,|\, x - y\,|^{\alpha} - \epsilon \,<\, u(y) \,<\, u(x) + C\,|\, x - y\,|^{\alpha} + \epsilon$$

holds for all y in some open set $A \Supset K$. As the inequalities above clearly hold on the complement of a large ball, say B_R, we apply the comparison principle in $B_R \setminus A$ to extend it to all $y \in \mathbb{R}^n$. By the arbitrariness of ϵ, this concludes the Hölder continuity of u at x. $\square$

Corollary. Assume $sp > n$, $g \in C^{\alpha}(K)$, with $\alpha = \frac{sp-n}{p-1}$, and $f \in L^{\infty}(\mathbb{R}^n)$ satisfying, for positive constants C, ϵ,

$$|f(x)| \,\le\, C_f\, |x|^{-n-\epsilon}\,, \quad \text{for all } \ |x| \,\ge\, 1\,.$$

Then any bounded weak solution $u \in C(\mathbb{R}^n) \cap W^{s,p}_{loc}(\mathbb{R}^n \setminus K)$ of

$$\begin{cases} (-\Delta)^s_p\, u \,=\, f & \text{in } \ \mathbb{R}^n \setminus K \\ u = g & \text{in } \ K \end{cases}$$

satisfies $u \in C^{\beta}(\mathbb{R}^n)$, for all $\beta < \frac{sp-n}{p-1}$, with the Hölder seminorm $|u|_{\beta}$ depending on $s, p, n, \|g\|_{C^{0,\beta}}, C$, and $\max\{\, R\,,\, 1\,\}$, where $R = \sup\{\, |x|\,;\, x \in K\,\}$.

Proof. The proof follows the same lines as in step *3* above, with a proper modification only in the barrier argument. We can assume with no loss of generality that ϵ is small enough so that $sp - n - \epsilon > 0$ and it is sufficient to prove the Hölder regularity for $\frac{sp-n-\epsilon}{p-1} \le \beta < \frac{sp-n}{p-1}$.

For a fixed $y \in K$ and a constant $C > 0$ we consider the functions

$$w^{\pm}_{C,y}(x) \;=\; g(y) \pm C\,|\,x-y\,|^{\beta}\,.$$

We will show that for some $C \geq |g|_\beta$ sufficiently large

$$(-\Delta)^s_p\, w^-_{C,y} \;\leq\; f \;\leq (-\Delta)^s_p\, w^+_{C,y} \quad \text{in} \;\; \mathbb{R}^n \backslash K$$

(We restrict C to values grater than $|g|_\beta$ since it is necessary for the argument of step *3*.) In fact, observing that, for any $x \neq y$,

$$(-\Delta)^s_p\,|\,\cdot - y\,|^{\beta}(x) \;=\; (-\Delta)^s_p\,|\cdot|^{\beta}(x-y)$$

by (4.6) we have

$$(-\Delta)^s_p\,|\,\cdot - y\,|^{\beta}(x) \;=\; (-\Delta)^s_p\,|\cdot|^{\beta}(e_1)\,|\,x-y\,|^{\beta(p-1)-sp}$$

with $(-\Delta)^s_p\,|\cdot|^{\beta}(e_1) > 0$ by Theorem 3', since $0 < \beta < \alpha$. Next, we define

$$M \;=\; \max\left\{\,|\,g\,|^{p-1}_{\beta}\,,\; \|\,f\,\|_{L^\infty}\,,\; C_f\,\right\}$$

and

$$\tilde{M} \;=\; \max\left\{\frac{M}{(-\Delta)^s_p\,|\cdot|^{\beta}(e_1)},\; M\,,\; 1\right\}.$$

Now let $R \geq 1$ such that $K \subset B_R(0)$. We claim

$$C \;=\; \left(\,\tilde{M}\,(2R)^{-\beta(p-1)+sp}\,\right)^{\frac{1}{p-1}}$$

is the desired constant. In fact, for such a choice of C, we have

$$\begin{aligned}(-\Delta)^s_p\, w^+_{C,y}(x) \;&=\; \tilde{M}\,(2R)^{-\beta(p-1)+sp}\,(-\Delta)^s_p\,|\,\cdot - y\,|^{\beta}(x)\\ &\geq\; M\,(2R)^{-\beta(p-1)+sp}\,|\,x-y\,|^{\beta(p-1)-sp}\end{aligned}$$

hence, if $|x| \leq R$, so that $|\,x-y\,| \leq 2R$, it follows

$$(-\Delta)^s_p\, w^+_{C,y}(x) \geq\; M \;\geq\; \|\,f\,\|_{L^\infty}\,.$$

If $|x| > R$, note that

$$\frac{|\,x-y\,|^{-\beta(p-1)+sp}}{|\,x\,|^{-\beta(p-1)+sp}} \;\leq\; \left(1 + \frac{|\,y\,|}{|\,x\,|}\,\right)^{-\beta(p-1)+sp} \;\leq\; 2^{-\beta(p-1)+sp}$$

and then

$$2^{-\beta(p-1)+sp}\,|\,x-y\,|^{\beta(p-1)-sp} \;\geq\; |x|^{\beta(p-1)-sp}\,.$$

Therefore, since $\beta(p-1) - sp \geq -n-\epsilon\,,$ $|x| \geq 1$, we have

$$(-\Delta)^s_p\, w^+_{C,y}(x) \;\geq\; M\,|x|^{\beta(p-1)-sp} \;\geq\; C_f\,|x|^{-n-\epsilon} \;\geq\; f(x)$$

Thus we have shown $(-\Delta)^s_p\, w^+_{C,y} \;\geq\; f$ in $\mathbb{R}^n \setminus K$ and, analogously, one can obtain the reversed inequality for $w^-_{C,y}$. The result then follows by the argument of step *3* in the proof of Theorem 4. □

This result also holds for solutions on bounded sets $\Omega \subset \mathbb{R}^n$, *i.e.*, for weak solutions $u \in C(\Omega) \cap W^{s,p}_{loc}(\Omega)$ of $(-\Delta)^s_p u = f$ in Ω, $u = g$ in $\mathbb{R}^n \backslash \Omega$. It is sufficient to assume in this case $g \in C^\alpha(\tilde{\Omega}) \cap L^\infty(\mathbb{R}^n)$, $\alpha = \frac{sp-n}{p-1}$, for some open neighbourhood $\tilde{\Omega}$ of $\partial\Omega$, and only assume $f \in L^\infty(\Omega)$. Then $u \in C^\beta(\Omega)$, for all $\beta < \frac{sp-n}{p-1}$ and the Hölder seminorm $|u|_\beta$ depends on $s, p, n, |g|_\alpha, \|g\|_\infty, \|f\|_\infty$.

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