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Higher-order integrable interactions for bosons in a multi–well potential

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Abstract

In this work we introduce an integrable family of Hamiltonians describing ultracold bosons behavior disposed in an optical lattice. The models are very general, taking into account the single and double tunnelings of the particles in multi-well potentials, the intra and inter site interactions between them, and also considering the correlated hopping between sites. The models are derived through the Quantum Inverse Scattering Method, and admit exact solution by applying an extended algebraic Bethe Ansatz, in which a new set of pseudovacua are required to obtain a complete set of Bethe states. This new family of models is noteworthy due to its generality, being capable of describing not only interesting and new behaviors, but also important models such as the Two-Site Bose-Hubbard model and, more recently, a three-site atomtronic switching device [1] can be obtained as a limiting case.

Resumo

Neste trabalho, introduzimos uma família de Hamiltonianos integráveis descrevendo o comportamento de bósons ultrafrios dispostos numa rede ótica. Os modelos apresentados são bastante gerais, e levam em consideração o tunelamento simples e duplo de particulas em poços múltiplos, as interações internas e externas de um determinado poço de potencial, e também o tunelamento correlacionado de partículas entre eles. Os modelos são derivados pelo Método de Espalhamento Quântico Inverso, e admitem solução exata através de uma extensão do Bethe Ansatz algébrico, no qual um novo conjunto de pseudovácuos se faz necessário para completar o conjunto de estados de Bethe. Esta nova família é notável por sua generalidade, sendo capaz não apenas de descrever comportamentos novos e interessantes para a literatura, como também modelos importantes como o Bose-Hubbard de dois sítios e, mais recentemente, um dispositivo atomotrônico de três sítios [1] pode ser obtido como caso limite.

Press Release:

We know that matter in our universe comes in many different states, such as for example gaseous, liquid, solid, plasma, Bose-Einstein condensate, and so on. The latter, which was theoretically predicted by Albert Einstein and Satyendra Nath Bose around 1925 [2], was first experimentally realised in 1995 [3] and is a very curious state of matter, in which a gas of particles called *bosons* lose their individual identity as they are cooled to temperatures very close to the absolute zero, becoming one single entity with quantum properties. In this work, we describe how these ultracold particles act when trapped within a set of "wells". Since it is a quantum phenomenon, we expect these ultracold particles to *tunnel* between wells, that is, to pass through the walls of their confinement. Our goal is to formulate a general mathematical description of a multi-wells setting, including high-order tunneling terms (that is, describing particles which tunnel twice or more before getting to their final destination, or particles which tunnel in pairs). It is also important that this description is exactly solvable, due to the possibility to better investigate the system's mathematical structures, from which we can obtain valuable physical information, also avoiding the use of computational methods, which are not practical in some cases. This type of study may have some applications in the development of quantum technologies.

Nota:

Sabemos que a matéria no universo pode assumir vários estados físicos diferentes, tais como por exemplo líquido, sólido, gasoso, plasma, condensado de Bose Einstein, entre outros. Este último, previsto teoricamente por Albert Einstein e Satyendra Nath Bose em meados de 1925 [2], foi realizado experimentalmente pela primeira vez em 1995 [3] e é um estado da matéria bastante curioso, no qual gases de partículas chamadas bósons, ao serem resfriados a temperaturas muito próximas do zero absoluto, perdem sua identidade individual, tornando-se uma só entidade com propriedades quânticas. Neste trabalho, nós descrevemos como essas partículas superfrias se comportam quando estão confinadas em vários "poços". Como se trata de um fenomeno quântico, nós esperamos que as particulas ultrafrias *tunelem* entre poços, isto é, atravessem as paredes de seu confinamento. Nosso objetivo é formular uma descrição matemática geral que descreva um sistema confinado em vários poços, e que inclua termos de tunelamento de alta ordem (isto é, que descrevam partículas que tunelem duas ou mais vezes até chegar a seu destino final, ou que tunelem em pares). Também é importante é que essa descrição seja exatamente solúvel, devido à possibilidade de melhor investigar as estruturas matemáticas do sistema, a partir das quais podemos obter informações físicas valiosas, e também para evitar de usar métodos computacionais, que não são práticos em alguns casos. Esse estudo pode ter aplicações no desenvolvimento de tecnologias quânticas.

Contents

Chapter 1

Introduction

Most physical systems present in nature display a chaotic behavior, meaning their immediate future is so highly sensitive to their initial conditions that what follows during time evolution can hardly be determined, leading to states of disorder and apparent "randomness", which then have to be dealt with stochastic methods [4]. Only very few, simple systems are "well-behaved" enough that we can mathematically describe their time evolution perfectly. Even though they are not as common in nature, these systems offer us the unique opportunity of investigating their fundamental aspects, and then be made into more complex systems, giving us insights about their inner workings. These systems are called "integrable", and they are important building blocks in our understanding of nature.

In classical mechanics, these highly symmetrical systems are characterized by the Liouville's Theorem, which states that, to be integrable, a system must display the same amount of conserved quantities as it's number of degrees of freedom. In this case, the word "integrability" refers to the possibility of finding the system's equation of motion by direct integration, and this feature translates into a well-ordered time evolution, constrained on a torus on the phase space [4].

But, despite being well defined and precisely understood in classical mechanics, there is still not a complete consensus on the concept of integrability for quantum systems [5–7]. What we have, then, are several different definitions of quantum integrability [5, 6], each more useful than the other in different contexts, sometimes overlapping. One such definition of quantum integrability (the one we will be using in this work) describes a quantum integrable system as a system where many-body interactions might be two-body reducible, or, equivalently, a system which presents non-diffractive scattering [8]. These integrable systems can be obtained through the Quantum Inverse Scattering Method (QISM), created in between the 70-80's by L. D. Fadeev and the Leningrad school [9, 10] as a quantum counterpart to the already established classical inverse scattering method [11], and which allows us to generate a set of conserved charges, from which we can then construct an integrable Hamiltonian.

Allied to this idea, there is the Algebraic Bethe Ansatz (ABA), also by Fadeev and his school, consisting in a generalization of the Bethe ansatz devised by Hans Bethe [12] as a way to solve the Heisenberg chain model. With this algebraic approach to the Bethe Ansatz [13–16], we are then able to analytically solve the QISM-obtained integrable Hamiltonian's eigenvalue problem. This can be particularly useful in some cases, where mean field and perturbation method approaches fail, and also where computational methods are not practical, such as when $N \to \infty$. They are also important when we want to acquire generalized results about the system, as well as information about the whole spectrum.

Armed with the Quantum Inverse Scattering Method and the Algebraic Bethe Ansatz, we are able to study fundamental aspects of quantum mechanics through the study of the quantum integrable systems and shed light into interesting phenomena, such as the Josephson tunnelling and self-trapping of Bose-Einstein condensates [17, 18], as well as the relaxation and thermalization of quantum systems [19]. But quantum integrable systems are not only useful as "toy models" to study new phenomena. Recent developments in experimental physics, such as the advent of laser cooling [20] and Bose-Einstein condensation [3] have paved the way to high-precision ultra cold atom manipulation [21–23], allowing the delicate quantum integrable systems to be experimentally feasible [24]. So now, not only we have a way to study the nuts and bolts of intricate quantum systems, we can also realize those systems experimentally to compare the results of theory and practice, and also build physical applications out of these mathematically beautiful models. More specifically, systems such as Bose-Einstein Condensates trapped in a three-well optical lattice can be manipulated in order to build atomtronic switches [1] and as a way to generate entangled states [25].

In this work we will be using the QISM and the Bethe ansatz to build a family of integrable models for Bose-Einstein condensates which take into account higher order interactions, further generalizing models in the literature [26] that, despite the absence of these high-order contributions, are already relevant as they have applications in atomtronics [1], in the generation of entangled states [25] and in a possible application to interferometry [27]. The main contribution of this work lies in the fact that this is done without compromising the integrability of the above models, meaning that our investigation can be carried out further than that made in the references cited above [1, 25, 27], and consequently opening up the possibility of exploring new physical effects. From a mathematical point of view, in this work we also show in the triple well case how to write the Bethe ansatz equations (a key ingredient in the construction) in an additive form (instead of the standard product form), opening new forms to treat the problem. The work is organized as follows:

- In Sec. (2.1) of this work, we first introduce a general, integrable model of bosons trapped in a three-site optical lattice. The model is unique in the sense it can describe high-order interactions between the bosons, and can be used, in a limiting case, to describe an atomtronic switching device [1].

- in Sec. (2.2) we briefly present the model's obtention through the Quantum Inverse Scattering Method, and then, using two different Bethe's ansatz, we analytically solve it's eigenvalue problem. In Sec. (2.3) the solutions found are written in a multiplicative form, being referred to as multiplicative Bethe Ansatz Equations (BAE), while in Sec. (2.4) they are written as a sum of terms, and thus are referred as additive BAE.

- A more theoretical approach is taken in Sec. (2.5) where we respectively discuss the conserved quantities of the system and the results obtained in both additive and multiplicative BAE's.

- In Sec. (3.1), we then turn to our second goal of this work, and further generalize our integrable high-order three-site model to a model of $n + m$ sites.

- In the following sections, Sec (3.2) and (3.3), we then discuss the QISM and the ABA for the more general case, obtaining the model's eigenvalues and eigenvectors and discussing it's conserved quantities.

The family of models obtained in this work, as well as the exact solutions, are new in the literature, and we expect to publish our results soon.

Chapter 2

Three-site model

In this chapter we will introduce an integrable triple-well model associated with complete bipartite graphs. This model generalizes previous results which were developed in [1, 26]. The interaction terms encountered in the integrable family of [1] can be classified as inter-well, intra-well, and tunneling types. However, there are many examples of physical boson systems where it has been argued that higher-order interactions will play a significant role. One notable example is due to pair tunneling processes [28]. Other examples include correlated tunneling terms [29] and very general three-body collisions [30]. In our work below we demonstrate, using the Quantum Inverse Scattering Method, how it is possible to incorporate some of these higher-order types of interactions into the triple-well setting, without violating integrability. Furthermore, we will analytically solve our model through two different types of Bethe's Ansatz, and compare the results to a numerical approach for a certain set of parameters.

2.1 Hamiltonian

The physical Hamiltonian that we will initially work with describes a system of N trapped bosons in a three-well optical lattice. This lattice constitutes a bi-partite system, with two of the three wells belonging to an energy class distinct to the remaining well. This Hamiltonian is unique because it describes correlated tunneling between bosons in different wells, while still being general enough to describe other systems already in literature [1]. We write

$$H = U(\tilde{N} - N_2)^2 + \mu(\tilde{N} - N_2) + t(A^\dagger a_2 + a_2^\dagger A)\ , \tag{2.1.1}$$

with operators $\tilde{N}$, A and $A^\dagger$ being defined as following

$$\begin{aligned} \tilde{N} &= \chi(N_1 + N_3) + \gamma\left(\alpha_1^2 N_1 + \alpha_3^2 N_3 + \alpha_1\alpha_3(a_1^\dagger a_3 + a_1 a_3^\dagger)\right) && (2.1.2)\\ A^\dagger &= \alpha_1 a_1^\dagger + \alpha_3 a_3^\dagger && (2.1.3)\\ A &= \alpha_1 a_1 + \alpha_3 a_3 && (2.1.4) \end{aligned}$$

where a_i and $a_i^\dagger$ are the annihilation and creation operators of a single particle in the $i-$th site, and $N_i = a_i^\dagger a_i$ is the number operator which counts the amount of particles in that site, satisfying the bosonic algebra

$$[a_i,\ a_j] = 0 = [a_i^\dagger,\ a_j^\dagger]; \tag{2.1.5}$$
$$[N_j, a_i^\dagger] = a_i^\dagger \delta_{ij}; \tag{2.1.6}$$
$$[a_i,\ a_j^\dagger] = \delta_{ij}. \tag{2.1.7}$$

Meanwhile, the operator set $\{\tilde{N}, A, A^\dagger\}$ satisfies the following commutation relations

$$\left[A,\ A^\dagger\right] = \alpha_1^2 + \alpha_3^2; \tag{2.1.8}$$
$$\left[\tilde{N},\ A^\dagger\right] = \left[\chi + \gamma(\alpha_1^2 + \alpha_3^2)\right] A^\dagger; \tag{2.1.9}$$
$$\left[\tilde{N},\ A\right] = -\left[\chi + \gamma(\alpha_1^2 + \alpha_3^2)\right] A, \tag{2.1.10}$$

matching the bosonic algebra under the conditions that

$$\begin{aligned} \alpha_1^2 + \alpha_3^2 &= 1 \\ \chi + \gamma &= 1. \end{aligned}$$

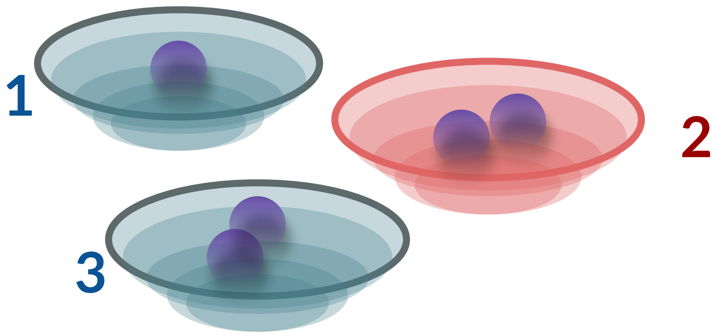

Figure 2.1: Schematic diagram representing three sites. The Hamiltonian in Eq.(2.1.1) describes a bi-partite system, here represented by colors blue, for sites 1 and 3, and red for site 2. Operators $a_1^\dagger$, a_1 and $a_3^\dagger$, a_3 act on the "blue" partition, creating/destroying a particle in well 1 and 3, retrospectively, while $a_2^\dagger$, a_2 creates/annihilates a particle in well 2. The operators $A^\dagger$, A as defined in Eq. (2.1.3, 2.1.4), creates/annihilates a particle in a superposition between wells 1 and 3, that is, somewhere in the "blue" partition. The purple spheres represent the bosons in each well.

The parameter U refers to the intra-well interactions in each site and the inter-well interactions between adjacent sites, while μ refers to the potential difference due to population imbalance between wells and t is the energy involved in the tunnelling processes between adjacent wells. Differently from the physical parameters explained above, χ and γ are interpolation parameters, and are **purely mathematical**, and integrability requires that $\chi + \gamma = 1$. For $\gamma = 0$, we have a three-well model with open boundary conditions, as presented in the reference [1], and no high-order interactions, and for the case of $\gamma \neq 0$ additional, high-order interaction terms appear. This can be better seen when we write hamiltonian (2.1.1) explicitly[1]. For $\alpha_1 = \alpha_3$, we have:

$$\begin{aligned} H = U & \left[\chi(N_1^2 + N_3^2) + 2\chi^2 N_1 N_3 - (\chi+1)(N_1+N_3)N_2 + N_2^2\right] \\ & + \left(\mu\chi + \frac{\mu\gamma}{2}\right)(N_1+N_3) - \mu N_2 \\ & + \frac{\sqrt{2}}{2} t \left(a_1^\dagger a_2 + a_1 a_2^\dagger + a_3^\dagger a_2 + a_3 a_2^\dagger\right) + \frac{1}{2}\mu\gamma\left(a_1^\dagger a_3 + a_1 a_3^\dagger\right) \\ & + U\gamma(N_1+N_3-N_2)\left(a_1^\dagger a_3 + a_1 a_3^\dagger\right) \\ & + U\gamma^2 Q^2 \end{aligned} \tag{2.1.11}$$

In the equation above we can see that, when $\gamma \neq 0$, the following terms appear:

$$\frac{\mu\gamma}{2}(N_1+N_3) + \frac{1}{2}\mu\gamma\left(a_1^\dagger a_3 + a_1 a_3^\dagger\right) + U\gamma(N_1+N_3-N_2)\left(a_1^\dagger a_3 + a_1 a_3^\dagger\right) + U\gamma^2 Q^2$$

where $2Q = N_1 + N_3 - a_1^\dagger a_3 - a_3^\dagger a_1$ and

$$4Q^2 = N_1^2 + N_3^2 + 4N_1N_3 + N_1 + N_3 - 2(N_1+N_3)(a_1^\dagger a_3 + a_3^\dagger a_1) + (a_1^\dagger a_1^\dagger a_3 a_3 + a_3^\dagger a_3^\dagger a_1 a_1).$$

The term $\frac{1}{2}\mu\gamma\left(a_1^\dagger a_3 + a_1 a_3^\dagger\right)$ adds tunneling between wells 1 and 3 (non-adjacent wells), thus describing periodic boundary conditions in our system while maintaining it's integrability, and $U\gamma(N_1+N_3-N_2)\left(a_1^\dagger a_3 + a_1 a_3^\dagger\right)$ includes tunneling dependence on the imbalance population (or density) between wells 1 and 3 (the blue wells, depicted in Fig. (2.1))and well 2, which is often called *correlated tunneling* in the literature. These terms are not present in relation to the models [1, 31], which are obtained from our model when the interpolation parameter $\gamma = 0$, and we consider it a *higher-order interaction contribution*, added without interfering with the system's integrability.

Additionally, we see that the term $U\gamma^2 Q^2$ allows to add two higher-order contributions: one associated with the correlated tunneling between wells 1 and 3 and the other associated with a double tunnelling between wells 1 and 3, that is, the tunnelling of two particles at once from 1 to 3.

Now, for the next sections, we will show how we obtained our hamiltonian, (2.1.1), and the results regarding the additional conserved quantity Q as well.

2.2 Quantum Inverse Scattering Method

The model's integrability is ensured by the fact it can be obtained through the quantum inverse scattering method [5, 9]. This technique is already well established in the literature, being used to construct well known integrable and experimentally feasible models [21]. It's beautiful physical interpretation will be further discussed in Appendix B. For now it will be enough to follow through the procedure presented in [32], from which we start

[1] The obtention of this Hamiltonian is shown in Appendix (A)

from a given $su(2)$-invariant $R(u)$ matrix, that in our case is

$$R(u) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(u) & c(u) & 0 \\ 0 & c(u) & b(u) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.2.1}$$

where u is the spectral parameter, $b(u) = u/(u+\eta)$ and $c(u) = \eta/(u+\eta)$, with η a real number. Our Lax operators are defined analogously as presented in [26, 32, 33], being

$$\tilde{L}(u) = \begin{pmatrix} uI + \eta\tilde{N} & A \\ A^\dagger & \eta^{-1} \end{pmatrix}, \tag{2.2.2}$$

$$L(u) = \begin{pmatrix} uI + \eta N_2 & a_2 \\ a_2^\dagger & \eta^{-1} \end{pmatrix}. \tag{2.2.3}$$

where a_i, $a_i^\dagger$ are the annihilation/creation operators for a particle at the *i-th* site, and $N_i = a_i^\dagger a_i$ is the number operator for that site , satisfying the bosonic algebra.

Any Hamiltonians derived from the Lax operators in (2.2.2, 2.2.3) will be integrable as long as they satisfy the Yang-Baxter Equation,

$$R_{12}(u-v)L_1(u)L_2(v) = L_2(v)L_1(u)R_{12}(u-v)\,, \tag{2.2.4}$$

together with the $R(u)$ matrix[2]. This imposes conditions over the parameters on the $\tilde{N}$, A and $A^\dagger$ operators, namely that $\chi + \gamma = 1$ and $\alpha_1^2 + \alpha_2^2 = 1$. Given that these restrictions are satisfied, then (2.2.2, 2.2.3) along with the R matrix (2.2.1) obey the Yang-Baxter equation (2.2.4) and, by the co-multiplication property, we can define a monodromy matrix $\tilde{T}(u)$

$$\tilde{T}(u) = \tilde{L}(u+\omega)L(u-\omega) = \begin{pmatrix} \tilde{T}_{1,1}(u) & \tilde{T}_{1,2}(u) \\ \tilde{T}_{2,1}(u) & \tilde{T}_{2,2}(u) \end{pmatrix},$$

where

$$\begin{aligned} \tilde{T}_{1,1}(u) &= (u+\omega+\eta\tilde{N})(u-\omega+\eta N_2) + Aa_2^\dagger \\ \tilde{T}_{1,2}(u) &= (u+\omega+\eta\tilde{N})a_2 + \eta^{-1}A \\ \tilde{T}_{2,1}(u) &= A^\dagger(u-\omega+\eta N_2) + \eta^{-1}a_2^\dagger \\ \tilde{T}_{2,2}(u) &= A^\dagger a_2 + \eta^{-2}, \end{aligned}$$

which will also obey the Yang-Baxter relation (2.2.4) with the $R_{12}(u)$ matrix. Under the integrability condition, then, the trace of the monodromy matrix, $\tilde{\tau}(u) = \mathrm{Tr}\{\tilde{T}(u)\}$, which is the extended transfer matrix, will commute for different values of the spectral parameter,

$$[\tilde{\tau}(u), \tilde{\tau}(v)] = 0. \tag{2.2.5}$$

[2]The indexes in Eq. (2.2.4) refer to the vector spaces in which our operators are acting. In the remaining spaces, they act as the identity.

In general, the transfer matrix can be written as a power series of the spectral parameter as

$$\tilde{\tau}(u) = \sum_{l=0} \eta_l u^l,$$

In particular, we will write our transfer matrix as

$$\tilde{\tau}(u) = b_0 + b_1 u + b_2 u^2$$

with

$$\begin{aligned}
b_0 &= \tilde{\tau}(0) = -\omega^2 + \omega\eta N_2 - \omega\eta\tilde{N} + \eta^2 \tilde{N} N_2 + A a_2^\dagger + A^\dagger a_2 + \eta^{-2}, \\
b_1 &= \frac{d}{du}\tilde{\tau}(u)\bigg|_{u=0} = \eta(\tilde{N} + N_2), \\
b_2 &= \frac{1}{2}\frac{d^2}{du^2}\tilde{\tau}(u)\bigg|_{u=0} = 1.
\end{aligned}$$

Here we observe that the method above gives just two quantities conserved b_0 and b_1 (b_2 is the identity). However, the integrability for the three-mode model needs one more. In Sec. (2.5) we will show how to build up this extra conserved quantity.

Now, an integrable Hamiltonian will be given through the following relation

$$H = t\left(\tilde{\tau}(0) - \frac{1}{4}\eta^2[N_2 + \tilde{N}]^2 + \omega^2 - \eta^{-2}\right) \tag{2.2.6}$$

which, by doing $U = -\frac{t\eta^2}{4}$ and $\mu = -t\omega\eta$, gives us

$$H = U(\tilde{N} - N_2)^2 + \mu(\tilde{N} - N_2) + t(A^\dagger a_2 + a_2^\dagger A)\ , \tag{2.2.7}$$

which, written explicitly[3] in the case of $\alpha_1 = \alpha_3$, is then given by

$$\begin{aligned}
H = U & \left[\chi(N_1^2 + N_3^2) + 2\chi^2 N_1 N_3 - (\chi+1)(N_1 + N_3)N_2 + N_2^2\right] \\
& + \left(\mu\chi + \frac{\mu\gamma}{2}\right)(N_1 + N_3) - \mu N_2 \\
& + \frac{\sqrt{2}}{2} t\left(a_1^\dagger a_2 + a_1 a_2^\dagger + a_3^\dagger a_2 + a_3 a_2^\dagger\right) + \frac{1}{2}\mu\gamma\left(a_1^\dagger a_3 + a_1 a_3^\dagger\right) \\
& + U\gamma(N_1 + N_3 - N_2)\left(a_1^\dagger a_3 + a_1 a_3^\dagger\right) \\
& + U\gamma^2 Q^2
\end{aligned} \tag{2.2.8}$$

that is our Hamiltonian (2.1.11).

2.3 Algebraic Bethe Ansatz

Having obtained an integrable Hamiltonian, we are now interested in finding its solutions. To do so, we will use the Algebraic Bethe Ansatz, where we assume the eigenstates of the transfer matrix $\tilde{\tau}(u)$ can be generated

[3] This will be done in Appendix A

by applying the monodromy matrix element $\tilde{T}_{2,1}(u)$ as a creation operator in a suitable vacuum state. Since the Hamiltonian can be written as a function of the transfer matrix, its eigenvalues will also be written as a function of the transfer matrix's eigenvalues.

For our model, the Fock space vacuum alone is not enough, because the states that are generated over it do not span our entire vector space, which has dimension

$$dim = \frac{(N+2)!}{2!N!},$$

where N is the number of particles. In order to solve this problem we follow the approach of [26], defining operators

$$\Gamma = \alpha_3 a_1 - \alpha_1 a_3 \quad \text{and} \quad \Gamma^\dagger = \alpha_3 a_1^\dagger - \alpha_1 a_3^\dagger,$$

which we'll use to complete our basis.

Now we can define our set of pseudo-vacua as

$$\begin{aligned} |\phi_\ell\rangle &= (\Gamma^\dagger)^\ell |0\rangle = (\Gamma^\dagger)^\ell |0,0,0\rangle\ , \quad \ell = 0,1,2,...,N \\ &= (\alpha_3 a_1^\dagger - \alpha_1 a_3^\dagger)^l |0,0,0\rangle \\ &= \sum_{k=0}^{l} c_k |l-k,0,k\rangle\,, \end{aligned} \tag{2.3.1}$$

where $c_k = \frac{(-1)^k \alpha_1^{l-k} \alpha_3^l l!}{\sqrt{k!(l-k)!}}$,[4] and continue on with the Algebraic Bethe Ansatz method, using our "creation" operator $\tilde{T}_{2,1}(u)$ over our set of pseudo-vacua. We then define the Bethe States as:

$$|\psi_\ell\rangle = \begin{cases} \prod_{i=1}^{N-\ell} \tilde{T}_{2,1}(v_i)\,|\phi_\ell\rangle \text{ if } \ell < N, \\ |\phi_N\rangle \text{ if } \ell = N. \end{cases} \tag{2.3.2}$$

Our ansatz is that the Bethe States defined above will be the eigenvectors of the transfer matrix $\tilde{\tau}(u)$, albeit under some restrictions. To find out which restrictions are those, we now apply the transfer matrix $\tilde{\tau}(u) = \tilde{T}_{1,1}(u) + \tilde{T}_{2,2}(u)$ over the Bethe States $|\psi_\ell\rangle$. From the following commutation relations resulting from the Quantum Inverse Scattering Method [5], we obtain

$$\tilde{T}_{1,1}(u)\tilde{T}_{2,1}(v) = \frac{u-v+\eta}{u-v}\tilde{T}_{2,1}(v)\tilde{T}_{1,1}(u) - \frac{\eta}{u-v}\tilde{T}_{2,1}(u)\tilde{T}_{1,1}(v), \tag{2.3.3}$$

$$\tilde{T}_{2,2}(u)\tilde{T}_{2,1}(v) = \frac{u-v-\eta}{u-v}\tilde{T}_{2,1}(v)\tilde{T}_{2,2}(u) + \frac{\eta}{u-v}\tilde{T}_{2,1}(u)\tilde{T}_{2,2}(v), \tag{2.3.4}$$

and we have that

[4] The dimension Fock space of each pseudo-vacuum is $d_l = N - l + 1$ and the full dimension of the Hilbert space is $d = \sum_{l=0}^{N} d_l$.

[5] To be precise, the equation (2.2.4) generates a set of sixteen commutation relations, but the two presented in Eqs. (2.3.3, 2.3.4) are enough for finding the eigenvalues and eigenvectors of the integrable hamiltonian.

$$\tilde{\tau}(u)\,|\psi_\ell\rangle = \tilde{T}_{1,1}(u)\prod_{i=1}^{N-\ell}\tilde{T}_{2,1}(v_i)\,|\phi_\ell\rangle + \tilde{T}_{2,2}(u)\prod_{i=1}^{N-\ell}\tilde{T}_{2,1}(v_i)\,|\phi_\ell\rangle \tag{2.3.5}$$

$$= \left[\tilde{t}_{1,1}(u)\prod_{i=1}^{N-\ell}\left(\frac{u-v_i+\eta}{u-v_i}\right) + \tilde{t}_{2,2}(u)\prod_{i=1}^{N-\ell}\left(\frac{u-v_i-\eta}{u-v_i}\right)\right]|\psi_\ell\rangle \tag{2.3.6}$$

$$-\left[\sum_{i=1}^{N-\ell}\tilde{t}_{1,1}(v_i)\left(\frac{\eta}{u-v_i}\right)\prod_{j\neq i}^{N-\ell}\left(\frac{v_i-v_j+\eta}{v_i-v_j}\right)\right]\tilde{T}_{2,1}(u)\,|\phi_\ell\rangle \tag{2.3.7}$$

$$+\left[\sum_{i=1}^{N-\ell}\tilde{t}_{2,2}(v_i)\left(\frac{\eta}{u-v_i}\right)\prod_{j\neq i}^{N-\ell}\left(\frac{v_i-v_j-\eta}{v_i-v_j}\right)\right]\tilde{T}_{2,1}(u)\,|\phi_\ell\rangle\,. \tag{2.3.8}$$

We can identify in the equation above that the (2.3.6) portion would be the eigenvalues of that equation, if the (2.3.7) and (2.3.8) parts cancel each other out. Thus, under that set of conditions, which are known as the Bethe Ansatz Equations (BAE)[6]

$$\eta^2(v_i-\omega)\left[\chi(v_i+\omega+\eta\ell)+\gamma(v_i+\omega)\right] = \prod_{j\neq i}^{N-\ell}\left(\frac{v_i-v_j-\eta}{v_i-v_j+\eta}\right), \tag{2.3.9}$$

the eigenvalues of the transfer matrix are

$$\tilde{\Lambda}_\ell(u) = (u-\omega)\left[\chi(u+\omega+\eta\ell)+\gamma(u+\omega)\right]\prod_{i=1}^{N-\ell}\left(\frac{u-v_i+\eta}{u-v_i}\right) \tag{2.3.10}$$

$$+\eta^{-2}\prod_{i=1}^{N-\ell}\left(\frac{u-v_i-\eta}{u-v_i}\right). \tag{2.3.11}$$

Earlier we had determined our Hamiltonian to be obtained from the transfer matrix by the relation given in (2.2.6). Now that we have the transfer matrix eigenvalues $\tilde{\Lambda}_\ell$ in hand, we can write the Hamiltonian's eigenvalues in terms of the eigenvalues of the transfer matrix

$$H\,|\psi_\ell\rangle = t\left(\tilde{\Lambda}_\ell(0) - \frac{1}{4}\eta^2[N_2+\tilde{N}]^2+\omega^2-\eta^{-2}\right)|\psi_\ell\rangle\,, \tag{2.3.12}$$

thus analytically solving our model.

Thus, we demonstrate that the model's solutions are completely obtained by using an extension of the standard Bethe ansatz method. However, as we will show in the next section, the model solution can also be obtained through an alternative algebraic method using differential operators.

2.4 Additive form of the Bethe Ansatz Equations

In the previous section we showed how we can arrive at Bethe's equations using the algebraic method of Bethe's ansatz to arrive at Bethe's Equation in multiplicative form. In this section we will show that there is an alternative way, using differential operators, to obtain the Bethe Equation in the additive form.

[6]This is a multiplicative form of the BAE. We will present a second, additive form of the BAE further along this work, in Section 2.4

Recall that the Hamiltonian can be expressed as

$$\begin{aligned} H &= U(\tilde{N} - N_2)^2 + \mu(\tilde{N} - N_2) + t(A^\dagger a_2 + a_2^\dagger A) \\ &= U(2S^z + (1-\gamma)Q)^2 + \mu(2S^z + (1-\gamma)Q) + t(S^\dagger + S). \end{aligned} \tag{2.4.1}$$

Now, following reference [34, 35], we define the set of operators

$$S^+ = A^\dagger a_2, \qquad S^- = A a_2^\dagger, \qquad S^z = \frac{1}{2}(N_1 + N_3 - N_2)$$

which satisfy the su(2) algebra

$$\begin{aligned} [S^z, S^\pm] &= \pm S^\pm \\ [S^+, S^-] &= 2S^z. \end{aligned}$$

In terms of this realization, the above hamiltonian (Eq. (2.4.1)) reads

$$H = U(2S^z - \gamma Q)^2 + \mu(2S^z - \gamma Q) + t(S^+ + S^-). \tag{2.4.2}$$

Next, we use the differential operator realisation for the SU(2) algebra

$$\begin{aligned} S^z &= u\frac{\mathrm{d}}{\mathrm{d}u} - \frac{N-l}{2}, \\ S^\dagger &= (N-l)u - u^2\frac{\mathrm{d}}{\mathrm{d}u}, \\ S &= \frac{\mathrm{d}}{\mathrm{d}u}, \\ Q &= l, \end{aligned}$$

acting on the $(N-l+1)$-dimensional space of polynomials with basis $\{1, u, u^2, \ldots, u^{N-l}\}$. We can then equivalently represent (2.4.1) as the second-order differential operator

$$\begin{aligned} H &= U\left(4u^2\frac{\mathrm{d}^2}{\mathrm{d}u^2} + 4(1-N-(\gamma-2)l)u\frac{\mathrm{d}}{\mathrm{d}u} + (N+(\gamma-2)l)^2\right) \\ &\quad + \mu\left(2u\frac{\mathrm{d}}{\mathrm{d}u} - N - (\gamma-2)l\right) + t\left((N-l)u + (1-u^2)\frac{\mathrm{d}}{\mathrm{d}u}\right) \\ &= 4Uu^2\frac{\mathrm{d}^2}{\mathrm{d}u^2} + \left(4U(1-N-(\gamma-2)l) + 2\mu)u + t(1-u^2)\right)\frac{\mathrm{d}}{\mathrm{d}u} \\ &\quad + U(N+(\gamma-2)l)^2 - \mu(N+(\gamma-2)l) + t(N-l)u. \end{aligned} \tag{2.4.3}$$

Solving for the spectrum of the Hamiltonian (2.1.11) is then equivalent to solving the eigenvalue equation

$$HQ(u) = EQ(u), \tag{2.4.4}$$

where H is given by (2.4.3) and $Q(u)$ is a polynomial function of u of order $N-l$.

From this point, it is little effort to obtain an additive form of Bethe Ansatz solution for the Hamiltonian (2.1.11). First express $Q(u)$ and its derivatives in terms of its roots $\{v_j\}$:

$$\begin{aligned} Q(u) &= \prod_{j=1}^{N-l}(u-v_j) \\ \frac{d}{du}Q(u) &= Q(u)\sum_{j=1}^{N-l}\frac{1}{u-v_j} \\ \frac{d^2}{du^2}Q(u) &= Q(u)\sum_{j=1}^{N-l}\sum_{i=1}^{N-l}\frac{1}{u-v_j}\frac{1}{u-v_i}, \quad \text{with} \quad j\neq i. \end{aligned}$$

Evaluating (2.4.4) at $u = v_k$ for each k leads to the set of Bethe ansatz equations in the additive form

$$\frac{(4U(1-N-(\gamma-2)l)+2\mu)v_k + t(1-v_k^2)}{4Uv_k^2} = \sum_{j\neq k}^{N-l}\frac{2}{v_j-v_k}, \qquad k=1,\ldots,N-l. \tag{2.4.5}$$

and writing the asymptotic expansion $Q(u) \sim u^{N-l} - u^{N-l-1}\sum_{j=1}^{N-l} v_j$ and by considering the terms of order $N-l$ in (2.4.4), the energy eigenvalues are found to be

$$E = U(N-\gamma l)^2 + (N-\gamma l)\mu - t\sum_{j=1}^{N-l} v_j. \tag{2.4.6}$$

Notice that in the limit $\gamma = 0$ we recover the energy spectrum of the integrable triple well model [1].

The algebraic methods developed in the last two sections showed that the model in question is completely integrable, and the quantum inverse scattering method gives us two of the conserved quantities of the system. However, integrability in this case requires three independent constants. The total number of particles N and the energy H are two well-known independent constants of the model, so we must find the one missing. In the next session we will show how this additive independent constant can be obtained.

2.5 Conserved Quantities

As mentioned earlier, the algebraic method gives us apparently just two conserved quantities, but the model proposed here has three modes and therefore, to guarantee the integrability, it needs another conserved quantity. In what follows we will show how this new conserved quantity can be obtained.

As seen in Section 2.3, in order to use the Bethe Ansatz method to solve the systems we've been working on, we needed to extend our vector space, and have used the Γ operators,

$$\Gamma = \alpha_3 a_1 - \alpha_1 a_3 \quad \text{and} \quad \Gamma^\dagger = \alpha_3 a_1^\dagger - \alpha_1 a_3^\dagger,$$

to do so. These operators are defined such that when acting over the Fock space vacuum, $\Gamma^\dagger|0\rangle$ contributes with one vector orthogonal to the basis vectors $A^\dagger|0\rangle$ and $a_2^\dagger|0\rangle$, that is

$$\begin{aligned} \langle 0|A\Gamma^\dagger|0\rangle &= \langle 0|\Gamma^\dagger A|0\rangle = 0 \\ \langle 0|a_2\Gamma^\dagger|0\rangle &= \langle 0|\Gamma^\dagger a_2|0\rangle = 0. \end{aligned}$$

From this, we are able to define a basis of pseudo-vacua suitable for our vector space of dimension $dim = (N+2)!/2(N!)$, given by Eq. (2.3.1),

$$|\phi_\ell\rangle = (\Gamma^\dagger)^\ell |0\rangle \ , \qquad \ell = 0,1,2,...,N$$

where $\ell = 0$ recovers the Fock Space Vacuum. The $\Gamma^\dagger$ operator, then, creates **one**[7] particle in a state which is orthogonal to states created by the $A^\dagger$ or $a_2^\dagger$ operators, so $(\Gamma^\dagger)^\ell$ creates ℓ particles in this orthogonal subspace [8]. Our eigenstates then will be constructed by creating the remaining $N-\ell$ particles of the system using the $\tilde{T}_{21}(u)$ creation operator over the pseudo-vacuum in the following manner

$$|\psi_\ell\rangle = \begin{cases} \prod_{i=1}^{N-\ell} \tilde{T}_{21}(v_i)\,|\phi_\ell\rangle \text{ if } \ell < N, \\ \quad |\phi_N\rangle \qquad \text{if} \qquad \ell = N. \end{cases} \tag{2.5.1}$$

Naturally, the Γ operators obey[1] the bosonic algebra of creation and annihilation operators, as following

$$[\Gamma,\ \Gamma^\dagger] = \alpha_1^2 + \alpha_3^2 = 1, \tag{2.5.2}$$

so we can define an operator $Q = \Gamma^\dagger\Gamma$ associated with it, given by

$$Q = \alpha_3^2 N_1 + \alpha_1^2 N_3 - \alpha_1\alpha_3(a_1^\dagger a_3 + a_3^\dagger a_1) \tag{2.5.3}$$

which obeys the bosonic algebra of the number operator

$$[Q,\Gamma] = -\Gamma \tag{2.5.4}$$

$$[Q,\Gamma^\dagger] = \Gamma^\dagger. \tag{2.5.5}$$

This is the operator Q which appears in our Hamiltonian H, given in (2.1.11), where we used $\alpha_1 = \alpha_2 = \sqrt{2}/2$.

[7] Under the integrability condition that $\alpha_1^2 + \alpha_3^2 = 1$.

[8] Since Eqs. (2.5.1) and (2.5.1) are true, $\langle 0|(A)^p(\Gamma^\dagger)^l|0\rangle = \langle 0|(a_2)^p(\Gamma^\dagger)^l|0\rangle = 0$ for any choice of p and l

It's action over the eigenstates $|\psi_\ell\rangle$ are given by

$$
\begin{aligned}
Q|\psi_\ell\rangle &= Q\prod_{i=1}^{N-\ell}\tilde{T}_{21}(v_i)\left(\Gamma^\dagger\right)^\ell|0\rangle \qquad \leftarrow [Q,\tilde{T}_{21}(v_i)]=0\\
&= \prod_{i=1}^{N-\ell}\tilde{T}_{21}(v_i)Q\left(\Gamma^\dagger\right)^\ell|0\rangle \qquad \leftarrow [Q,\Gamma^\dagger]=\Gamma^\dagger\\
&= \underbrace{\prod_{i=1}^{N-\ell}\tilde{T}_{21}(v_i)\left(\Gamma^\dagger\right)^\ell|0\rangle}_{|\psi_\ell\rangle}+\prod_{i=1}^{N-\ell}\tilde{T}_{21}(v_i)\Gamma^\dagger Q\left(\Gamma^\dagger\right)^{\ell-1}|0\rangle\\
&= |\psi_\ell\rangle+\prod_{i=1}^{N-\ell}\tilde{T}_{21}(v_i)\Gamma^\dagger Q\left(\Gamma^\dagger\right)^{\ell-1}|0\rangle \qquad \leftarrow [Q,\Gamma^\dagger]=\Gamma^\dagger\\
&= 2|\psi_\ell\rangle+\prod_{i=1}^{N-\ell}\tilde{T}_{21}(v_i)(\Gamma^\dagger)^2 Q\left(\Gamma^\dagger\right)^{\ell-2}|0\rangle \qquad \leftarrow [Q,\Gamma^\dagger]=\Gamma^\dagger\\
&= \cdots\\
&= \ell-1|\psi_\ell\rangle+\prod_{i=1}^{N-\ell}\tilde{T}_{21}(v_i)(\Gamma^\dagger)^{\ell-1} Q\left(\Gamma^\dagger\right)|0\rangle \qquad \leftarrow [Q,\Gamma^\dagger]=\Gamma^\dagger\\
&= \ell|\psi_\ell\rangle+\prod_{i=1}^{N-\ell}\tilde{T}_{21}(v_i)(\Gamma^\dagger)^{\ell}\cancelto{0}{Q|0\rangle}\\
Q|\psi_\ell\rangle &= \ell|\psi_\ell\rangle
\end{aligned}
$$

further confirming it's action as a number operator, counting the amount of particles created by $\Gamma^\dagger$ over the Fock space vacuum.

Now let's study the action of the $A^\dagger A+N_2$ operator on the Bethe vector. Given the following commutation relations

$$
\begin{aligned}
\left[A^\dagger A,\ \tilde{T}_{21}(u)\right] &= \left[A^\dagger A,\ (u-\omega+\eta N_2)A^\dagger+\cancelto{0}{\eta^{-1}a_2^\dagger}\right]\\
&= (u-\omega+\eta N_2)\left[A^\dagger A,\ A^\dagger\right]\\
&= (u-\omega+\eta N_2)A^\dagger \qquad (2.5.6)\\
\left[N_2,\ \tilde{T}_{21}(u)\right] &= \left[N_2,\ \cancelto{0}{(u-\omega+\eta N_2)A^\dagger}+\eta^{-1}a_2^\dagger\right]\\
&= \eta^{-1}\left[N_2,\ a_2^\dagger\right]\\
&= \eta^{-1}a_2^\dagger \qquad (2.5.7)
\end{aligned}
$$

we note that

$$
\begin{aligned}
\left[A^\dagger A+N_2,\ \tilde{T}_{21}(u)\right] &= (u-\omega+\eta N_2)A^\dagger+\eta^{-1}a_2^\dagger\\
&= \tilde{T}_{21}(u), \qquad (2.5.8)
\end{aligned}
$$

from which follows

$$
\begin{aligned}
(A^\dagger A + N_2)|\psi_l\rangle &= (A^\dagger A + N_2)\prod_{i=1}^{N-\ell}\tilde{T}_{21}(v_i)\left(\Gamma^\dagger\right)^\ell|0\rangle \\
&= |\psi_l\rangle + \tilde{T}_{21}(v_1)(A^\dagger A + N_2)\tilde{T}_{21}(v_2)\ldots\tilde{T}_{21}(v_{N-l})|\phi_\ell\rangle \\
&= 2|\psi_l\rangle + \tilde{T}_{21}(v_1)\tilde{T}_{21}(v_2)(A^\dagger A + N_2)\tilde{T}_{21}(v_3)\ldots\tilde{T}_{21}(v_{N-l})|\phi_\ell\rangle \\
&= 3|\psi_l\rangle + \tilde{T}_{21}(v_1)\tilde{T}_{21}(v_2)\tilde{T}_{21}(v_3)(A^\dagger A + N_2)\tilde{T}_{21}(v_4)\ldots\tilde{T}_{21}(v_{N-l})|\phi_\ell\rangle \\
&= \ldots \\
&= (N-l)|\psi_l\rangle + \prod_{i=1}^{N-\ell}\tilde{T}_{21}(v_i)\left(\Gamma^\dagger\right)^\ell(A^\dagger A + N_2)|0\rangle \\
&= (N-l)|\psi_l\rangle.
\end{aligned}
$$

Since $\Gamma^\dagger\Gamma$ commutes with both $A^\dagger A$ and N_2, with $\Gamma^\dagger\Gamma|\psi_\ell\rangle = \ell|\psi_\ell\rangle$, consequently, the $\Gamma^\dagger\Gamma + A^\dagger A + N_2$ operator acting on the Bethe vector counts the conserved number of particles in the system, that is, $(\Gamma^\dagger\Gamma + A^\dagger A + N_2)|\psi_l\rangle = N|\psi_l\rangle$. Since the Hamiltonian (2.1.11) is a three-mode model, integrability requires [5] that we have three conserved quantities. Through the Quantum Inverse Scattering Method, though, we find only two of them: $b_0 = -\omega^2 + \eta^{-2} + \eta^2(\tilde{N} + N_2) + H/t$ and $b_1 = \eta(\tilde{N} + N_2)$. Now we can observe that the constants $b_0,\ b_1$ can be written in function of the conserved quantity Q, that is, $\tilde{N} + N_2 = N - \gamma Q$. We now proceed to show that operator Q as defined above is the third conserved quantity.

Expressing the Hamiltonian as (2.2.7), and considering the following commutation relations

$$
\begin{aligned}
&[Q,\ \tilde{N}] = 0, \\
&[Q,\ N_2] = 0, \\
&[Q,\ A^\dagger] = 0 = [Q,\ A] \\
&[Q,\ a_2^\dagger] = 0 = [Q,\ a_2],
\end{aligned}
$$

we see that

$$
\begin{aligned}
[Q,\ H] &= U[Q,\ (\tilde{N} - N_2)^2] + \mu[Q,\ (\tilde{N} - N_2)] + t[Q,\ (A^\dagger a_2 + a_2^\dagger A)] \\
&= 0
\end{aligned}
$$

so Q is the additional conserved quantity for system. So, Q together with N and H are the three linearly independent conserved quantities for the model thereby ensuring the model's integrability.

2.6 Analytical solutions and exact diagonalization

In the previous sections we have shown that the exact solution of the model can be obtained through two algebraic methods, which give rise to Bethe's equations in the multiplicative and the additive form (eqs. (2.3.9) and (2.4.5)), as well as the expressions for the energy eigenvalues (eqs. (2.3) and (2.4.6)). In this section, we aim to make an analytical study for a small number of bosons ($N = 2$), solving the Bethe ansatz equations in both formats and obtaining the eigenvalues of energies for both cases. Furthermore, we make a comparison between the energy eigenvalues obtained analytically and those taken from the exact diagonalization.

2.6.1 Analytic solution for the multiplicative BAE

As an example to the Algebraic Bethe Ansatz developed in the section before, we will solve analytically a simple case of our system, for $N = 2$ bosons trapped in our optical lattice, with coupling parameters given by $\mu = 0,\ t = 0.5,\ U = 1$. In this case, then, the Hamiltonian (2.1.11) reads

$$\begin{aligned}H = {} & \chi(N_1^2 + N_3^2) + 2\chi^2 N_1 N_3 - (\chi+1)(N_1 + N_3)N_2 + N_2^2 \\ & + \frac{\sqrt{2}}{4}\left(a_1^\dagger a_2 + a_1 a_2^\dagger + a_3^\dagger a_2 + a_3 a_2^\dagger\right) \\ & + \gamma(N_1 + N_3 - N_2)\left(a_1^\dagger a_3 + a_1 a_3^\dagger\right) \\ & + \gamma^2 Q\end{aligned} \tag{2.6.1}$$

which we can be represented in the following matrix:

$$\begin{pmatrix} 4 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{\gamma^2}{4} + \left(\frac{\gamma}{2} + \chi - 1\right)^2 & \frac{1}{2} & \gamma\left(\frac{\gamma}{2} + \chi - 1\right) & 0.353553 & 0 \\ 0 & \frac{1}{2} & \frac{\gamma^2}{2} + (\gamma + 2\chi)^2 & 0 & \gamma(\gamma+2\chi)\sqrt{2} & \frac{\gamma^2}{2} \\ \frac{1}{2} & \gamma\left(\frac{\gamma}{2} + \chi - 1\right) & 0 & \frac{\gamma^2}{4} + \left(\frac{\gamma}{2} + \chi - 1\right)^2 & 0.353553 & \frac{1}{2} \\ 0 & 0.353553 & \gamma(\gamma+2\chi)\sqrt{2} & 0.353553 & \gamma^2 + (\gamma+2\chi)^2 & \gamma(\gamma+2\chi)\sqrt{2} \\ 0 & 0 & \frac{\gamma^2}{2} & \frac{1}{2} & \gamma(\gamma+2\chi)\sqrt{2} & \frac{\gamma^2}{2} + (\gamma+2\chi)^2 \end{pmatrix} \tag{2.6.2}$$

Now, for $\chi = 0.4$ and $\gamma = 0.6$, we can numerically diagonalize our matrix and obtain the following eigenvalues:

$$\begin{aligned} E_1 = {} & 4.23607, \\ E_2 = {} & 4.0, \\ E_3 = {} & 2.1034, \\ E_4 = {} & 0.64, \\ E_5 = {} & -0.236068, \\ E_6 = {} & 0.216602. \end{aligned}$$

For the parameters above defined ($\mu = 0,\ t = 0.5,\ U = 1,\ \chi = 0.4,\ \gamma = 0.6$), and reminding that we had defined $U = -\frac{t\eta^2}{4}$ and $\mu = -t\omega\eta$ in the QISM obtention of our Hamiltonian, we have $\eta = 2\sqrt{2}$ and $\omega = 0$, and our Bethe Ansatz Equations (2.3.9) will read

$$\begin{aligned} \eta^2(v_i - \omega)\left[\chi(v_i + \omega + \eta\ell) + \gamma(v_i + \omega)\right] = \prod_{j\neq i}^{N-\ell}\left(\frac{v_i - v_j - \eta}{v_i - v_j + \eta}\right) \rightarrow \\ 8(v_i)\left[0.4(v_i + 2\sqrt{2}\ell) + 0.6(v_i)\right] = \prod_{j\neq i}^{N-\ell}\left(\frac{v_i - v_j - 2\sqrt{2}}{v_i - v_j + 2\sqrt{2}}\right) \end{aligned}$$

for every $\ell \leq N$. The v_i, v_j values which satisfy the BAE above will be then the values of the spectral parameter u that satisfies the eigenvalue equation, given in Eq. (2.3.11). The energy of the system, given in Eq. (2.3), can then be obtained analytically, as displayed in Table (2.1), agreeing with the direct diagonalization results.

Table 2.1: Bethe Equations roots and corresponding Eigenvalues for: $\mu = 0,\ t = 0.5,\ U = 1$ and $\gamma = 0.6$, with $N = 2$

<table>
<tr><td>ℓ</td><td>BAE</td><td>BAE solutions</td><td>Energy</td></tr>
<tr><td rowspan="3">0</td><td rowspan="3">$\eta^2(v_1^2-\omega^2)=\frac{v1-v2-\eta}{v1-v2+\eta}$
$\eta^2(v_2^2-\omega^2)=\frac{v2-v1-\eta}{v2-v1+\eta}$</td><td>$v_1 \to -2.95302$
$v_2 \to -0.0423295$</td><td>4.2360679</td></tr>
<tr><td>$v_1 \to -2.87195$
$v_2 \to 0.0435244$</td><td>4.0</td></tr>
<tr><td>$v_1 \to 0.0834626 - 0.343561i$
$v_2 \to 0.0834626 + 0.343561i$</td><td>−0.2360679</td></tr>
<tr><td rowspan="2">1</td><td rowspan="2">$\eta^2(v_1-\omega)[\chi_A(v_1+\omega+\ell\eta)+\gamma_A(v_1+\omega)]=1$</td><td>$v_1 \to -1.23277$</td><td>2.103398</td></tr>
<tr><td>$v_1 \to 0.101398$</td><td>0.216601</td></tr>
<tr><td>2</td><td>$\nexists$</td><td>$\nexists$</td><td>0.64</td></tr>
</table>

2.6.2 Analytic solution for the additive BAE

Having encountered the additive form of the BAE, given in (2.4.5), we can take one step further and solve them analytically for the case $N = 2$ and $\mu = 0$. In order to do so, we take advantage of the discussion developed in [36], where it is established that a sign exchange of the parameter $\mu \leftrightarrow -\mu$ is equivalent to a similarity transformation, to which our energy expression must be invariant towards. As a consequence, our BAE equations must also be invariant, imposing a restriction over its $N - l + 1$ sets of solutions, $\{v_j,\ j = 1, \cdots, N - l\}$.

For now, let's start with the simpler cases, $l = 2$ and $l = 1$, which can be directly solved without any additional method. The $l = 2$ case is trivial, as there are in fact no BAE to be solved, and the energy takes the simple form

$$E = 4U(1-\gamma)^2. \tag{2.6.3}$$

When $l = 1$, we have two solutions, which can be found directly by solving the quadratic Bethe Ansatz Equation. We are left with as $k = 1$ and the sum on the right side of (2.4.5) goes to zero

$$\frac{4U(1-\gamma)v_1 + t(1-v_1^2)}{4Uv_1^2} = 0. \tag{2.6.4}$$

The energies then can be found by plugging the results of the expression above into (2.4.6).

When $l = 0$, though, such direct approach isn't fruitful, and we resort to the technique developed in [36]. There it is shown that, for $\mu = 0$, our sets of solutions to the Bethe Ansatz Equations must be symmetric towards inversion, such that if a set $A = \{v_j,\ j = 1, \cdots, N - l\}$ solves our BAE, then the set $B = \{w_j,\ j = 1, \cdots, N - l\}$, with $w_j = v_j^{-1}$, must also be a solution. With that in mind we can find, by inspection, that our first suitable set of solutions to (2.4.6) would be $v_1 = 1,\ v_2 = -1$. Since we will have $N - l + 1$ energy levels at any given moment, we still need to find two sets of BAE solutions. For that we can define a new Bethe Ansatz Equation, given by

$$\frac{t}{4U} + \frac{\kappa_2}{u_l - 1} + \frac{\kappa_1}{u_l + 1} = \sum_{j \neq l}^{M} \frac{1}{u_j - u_l}, \tag{2.6.5}$$

where $M = 2$ is the cardinality of our new set of roots u_l given by

$$u_l = \frac{v_l + v_l^{-1}}{2},$$

and κ_1 and κ_2 are parameters which, following the discussion in [36], will be $\kappa_1 = \kappa_2 = 3/4$. Since $l = 0$ and $N = 2$, we will have only two roots v_1 and v_2, leaving us with only one equation to solve for our new variable $u_1 = (v_1 + v_1^{-1})/2$

$$\frac{t}{4U} + \frac{\kappa_2}{u_l - 1} + \frac{\kappa_1}{u_l + 1} = 0. \tag{2.6.6}$$

Since (2.6.6) is a simple quadratic equation, we can recover the values of our original roots, v_1 and v_2, enabling us to find the energies of the system with (2.4.6), or, alternatively, straight from the new BAE's (2.6.5) associated energy

$$E = UN(1-\gamma)^2 - 2t(\kappa_1 - \kappa_2) - t\sum_{l=1}^{M} u_l. \tag{2.6.7}$$

For $U = 1$ and $t = -0.5$, we find the energies and results given in Table (2.2), the same results found by direct diagonalization.

Table 2.2: Additive Bethe Equations roots and corresponding Eigenvalues for: $\mu = 0$, $t = 0.5$, $U = 1$ and $\gamma = 0.6$, with $N = 2$

ℓ	BAE	BAE solutions	Energy
0	$\frac{-4Uv_1+t(1-v_1^2)}{4Uv_1^2} = \frac{2}{v_2-v_1}$	$v_1 \to -8.35241$ $v_2 \to -0.119726$	-0.236068
		$v_1 \to 1$ $v_2 \to -1$	4
	$\frac{-4Uv_2+t(1-v_2^2)}{4Uv_2^2} = \frac{2}{v_1-v_2}$	$v_1 \to 0.236068 - 0.971737i$ $v_2 \to 0.236068 + 0.971737i$	4.23607
1	$4U(1-\gamma)v_1 + t(1-v_1^2) = 0$	$v_1 \to -3.4868$	0.216602
		$v_1 \to 0.286796$	2.1034
2	$\nexists$	$\nexists$	0.64

As we saw in Sec. (2.6.1), we obtained the same eigenvalues assuming different kinds of ansatz for the eigenvectors of our system. Both methods agree with the result obtained by direct diagonalization of the system, as seen in Tables (2.2) and (2.1). We can see that, even though the energy resulting from the different methods is the same, there is a difference in the values of the roots of the Bethe Ansatz Equations, v_1 and v_2, for each different method.

Chapter 3

General n+m-sites model

In the previous chapter we presented an algebraic method to build an integrable model of a system containing a three mode structure. In what follows, we apply the generalization of this algebraic method to set up new integrable models with larger number of modes. The main objective of this generalization, in addition to building physical integrable models, is to allow the introduction of higher-order interactions in models that have a greater number of wells, such as in a model of four well [37–41], five well [42] and so on.

The family of integrable models that we present below allows, through the choice of parameters, to create more general models. Just to mention the choice of parameters $(n,m) = (1,1)$ generates the two well model well known in the literature [17]; the choice $(n,m) = (1,2)$ will generate the model discussed in the previous chapter, which is a generalization of a three-well model studied in the references [25]; the choice $(n,m) = (2,2)$ generates a generalization of the model presented in [43] and so on. In other words, the family of integrable models presented here will generalize the family of integrable models presented in the reference [26]. This flexibility of adding higher-order interaction to the models presented in [26] opens up the possibility of investigating new phenomena that are associated with higher-order excitations.

3.1 Hamiltonian

We begin this section by presenting the structure of the Hamiltonian, which includes higher-order interactions, used to describe a family of integrable models. The models proposed here generalizes a family of integrable models [26] already present in the literature.

The Hamiltonian (2.1.11), presented in the previous chapter, is but one in a family of general integrable systems with $n+m$ sites. The general model we work with is the following,

$$\tilde{H}_{n,m} = U(\tilde{N}_{A,n} - \tilde{N}_{B,m})^2 + \mu(\tilde{N}_{A,n} - \tilde{N}_{B,m}) + t(A_n^\dagger B_m + B_m^\dagger A_n). \tag{3.1.1}$$

The model describes a bipartite system of trapped bosons, which exhibits high-order tunneling of particles distributed between two different "classes": class A, with n sites, and class B, with m sites. Our parameter U mediates the intra and inter well interactions and μ is the potential difference due to the allocation of particles in different classes. The energy due to the tunnelling between classes (inter-well interactions) is given in parameter t.

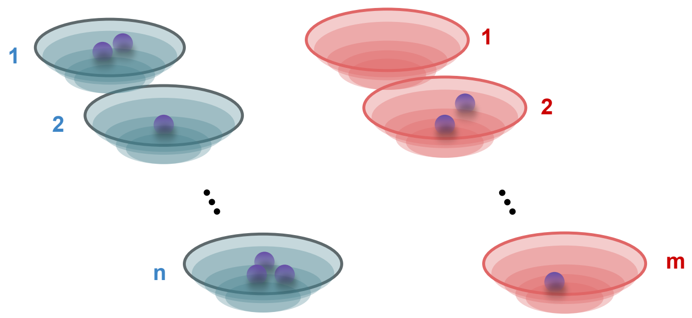

Figure 3.1: Schematic diagram representing two "classes" of wells, A (blue) with n sites and B (red) with m sites. The operators $A_n^\dagger$, A_n as defined in Eq. (3.1.2), creates/annihilates a particle in superposition somewhere between the n sites of class A (blue), while operators $B_m^\dagger$, B_m of Eq. (3.1.3) do the same for a particle in superposition somewhere between the m sites of class B (red). The purple spheres once again represent the bosons in each site.

The generality of the model, as well as the presence of high-order tunnelling between sites, can be attributed to our definition of the number, creation and annihilation operators acting in each class.

Our general creation and annihilation operators are defined as follows[1]

$$A_n = \sum_{i=1}^{n} \alpha_i a_i \qquad A_n^\dagger = \sum_{i=1}^{n} \alpha_i a_i^\dagger \tag{3.1.2}$$

$$B_m = \sum_{i=1}^{m} \beta_i b_i \qquad B_m^\dagger = \sum_{i=1}^{m} \beta_i b_i^\dagger \tag{3.1.3}$$

and each satisfy the bosonic algebra,

$$\left[A_n,\ A_n^\dagger\right] = 1 \qquad \left[A_n^\dagger,\ A_n\right] = -1 \qquad \left[A_n,\ A_n\right] = 0 = \left[A_n^\dagger,\ A_n^\dagger\right]$$

$$\left[B_m,\ B_m^\dagger\right] = 1 \qquad \left[B_m^\dagger\ B_m\right] = -1 \qquad \left[B_m,\ B_m\right] = 0 = \left[B_m^\dagger,\ B_m^\dagger\right]$$

under the condition that

$$\sum_{i=1}^{n} \alpha_i^2 = 1, \tag{3.1.4}$$

$$\sum_{j=1}^{m} \beta_j^2 = 1. \tag{3.1.5}$$

Under this condition, when acting over a suitable Fock space vacuum, $A_n^\dagger$ creates one boson in a superposition state over the n-sites of class A, while A_n annihilates this state. The same applies to class B and operators $B_m^\dagger$ and B_m.

Since A_n, $A_n^\dagger$ and B_m, $B_m^\dagger$ obey the bosonic algebra, we can define the operators

[1]The a_i, $a_i^\dagger$ operators are the annihilation/creation operators of a boson in the i-th site in class A, and the same stands for b_j, $b_j^\dagger$ and class B.

$$\begin{aligned} N'_{A,n} &= A_n^\dagger A_n \\ &= \sum_{i=1}^{n} \alpha_i^2 n_{a_i} + \sum_{i=1}^{n}\sum_{j>i}^{n} \alpha_i\alpha_j (a_i^\dagger a_j + a_j^\dagger a_i) \end{aligned} \tag{3.1.6}$$

$$\begin{aligned} N'_{B,m} &= B_m^\dagger B_m \\ &= \sum_{i=1}^{m} \beta_i^2 n_{b_i} + \sum_{i=1}^{m}\sum_{j>i}^{m} \beta_i\beta_j (b_i^\dagger b_j + b_j^\dagger b_i), \end{aligned} \tag{3.1.7}$$

which follow the "number" operator algebra,

$$\begin{aligned} \left[N'_{A,n},\ A_n^\dagger\right] &= A_n^\dagger \qquad & [N'_{A,n},\ A_n] &= -A_n; \\ \left[N'_{B,m},\ B_m^\dagger\right] &= B_m^\dagger \qquad & [N'_{B,m},\ B_m] &= -B_m, \end{aligned}$$

counting the amount of bosons created in superposition by the action of $A_n^\dagger$, $B_m^\dagger$ over their respective Fock space vacuum. This "extended" number operator, together with the usual, single site number operators $n_{a_i} = a_i^\dagger a_i$ and $n_{b_j} = b_j^\dagger b_j$, can be used to define the operators $\tilde{N}_{A,n}$, $\tilde{N}_{B,m}$ that are present in our Hamiltonian (3.1.1):

$$\begin{aligned} \tilde{N}_{A,n} &= \chi_A N_{A,n} + \gamma_A N'_{A,n} \\ &= \chi_A \sum_{i=1}^{n} n_{a_i} + \gamma_A \left(\sum_{i=1}^{n} \alpha_1^2 n_{a_i} + \sum_{\substack{i=1 \\ j<i}}^{n} \alpha_i\alpha_j (a_i^\dagger a_j + a_i a_j^\dagger) \right) \end{aligned} \tag{3.1.8}$$

$$\begin{aligned} \tilde{N}_{B,m} &= \chi_B N_{B,m} + \gamma_B N'_{B,m} \\ &= \chi_B \sum_{i=1}^{m} n_{b_i} + \gamma_B \left(\sum_{i=1}^{m} \alpha_1^2 n_{b_i} + \sum_{\substack{i=1 \\ j<i}}^{m} \beta_i\beta_j (b_i^\dagger b_j + b_i b_j^\dagger) \right). \end{aligned} \tag{3.1.9}$$

These two are more "general" number operators for classes A and B, given by a weighted sum of the single site number operators of a class ($N_{A,n} = \sum n_{a_i}$ and $N_{B,m} = \sum n_{b_j}$) and the "extended" number operators $N'_{A,n} = A_n^\dagger A_n$, and $N'_{B,m} = B_m^\dagger B_m$. The χ_i and γ_i parameters, then, correspond to the weights of the composition, with the index $i = A, B$ indicating which class it refers to. The operators $\tilde{N}_{A,n}$ and $\tilde{N}_{B,m}$ obey the bosonic number operator algebra as long as $\chi_i + \gamma_i = 1$, for $i = A, B$.

3.2 Integrability and the Quantum Inverse Scattering Method

As seen in Section 2.2, the integrability of the system described by (2.2.7) is ensured by the fact that it's obtainable through the Quantum Inverse Scattering Method (QISM) [5]. In this section, we will present the method again very briefly, with our operators generalised to the $n+m$-wells problem. For a more in-depth analysis of the QISM and it's meaning, see Appendix (B). We will begin taking two Lax Operators,

$$\tilde{L}_{A,n}(u) = \begin{pmatrix} u+\eta\tilde{N}_{A,n} & A_n \\ A_n^\dagger & \eta^{-1} \end{pmatrix} \tag{3.2.1}$$

$$\tilde{L}_{B,m}(u) = \begin{pmatrix} u+\eta\tilde{N}_{B,m} & B_m \\ B_m^\dagger & \eta^{-1} \end{pmatrix}. \tag{3.2.2}$$

and again building the monodromy matrix for our system, $\tilde{T}(u)$, defined in our case as

$$\tilde{T}(u) = \tilde{L}_{A,n}(u+\omega)\tilde{L}_{B,m}(u-\omega) = \begin{pmatrix} \tilde{T}_{1,1}(u) & \tilde{T}_{1,2}(u) \\ \tilde{T}_{2,1}(u) & \tilde{T}_{2,2}(u) \end{pmatrix}. \tag{3.2.3}$$

Computing the matrix multiplication above, we find that the elements on our monodromy matrix are

$$\begin{aligned}
\tilde{T}_{1,1}(u) &= (u+\omega+\eta\tilde{N}_{A,n})(u-\omega+\eta\tilde{N}_{B,m}) + A_nB_m^\dagger \\
\tilde{T}_{1,2}(u) &= (u+\omega+\eta\tilde{N}_{A,n})B_m + \eta^{-1}A_n \\
\tilde{T}_{2,1}(u) &= A_n^\dagger(u-\omega+\eta\tilde{N}_{B,m}) + \eta^{-1}B_m^\dagger \\
\tilde{T}_{2,2}(u) &= A_n^\dagger B_m + \eta^{-2},
\end{aligned}$$

from which, again, we will build our transfer matrix, $\tilde{\tau}(u) = \mathrm{Tr}\{\tilde{T}(u)\}$, given by

$$\begin{aligned}
\tilde{\tau}(u) &= \tilde{T}_{1,1}(u) + \tilde{T}_{2,2}(u) \\
&= (u^2-\omega^2+\eta^{-2}) + (u+\omega)\eta\tilde{N}_{B,m} + (u-\omega)\eta\tilde{N}_{A,n} + \eta^2\tilde{N}_{A,n}\tilde{N}_{B,m} + A_nB_m^\dagger + A_n^\dagger B_m,
\end{aligned}$$

Our goal is to ensure the transfer matrix $\tilde{\tau}(u)$ commutes with itself for every value of the spectral parameter (u), that is,

$$[\tilde{\tau}(u), \tilde{\tau}(v)] = 0.$$

When this condition is satisfied, we can build a Hamiltonian and it's conserved quantities directly from the transfer matrix, using it as our building blocks for an integrable system. In order to constrain our model to this condition, we need that our Lax operators $\tilde{L}(u)$ obey the Yang-Baxter Algebra's consistency relation with a $R(u)$ matrix

$$R_{12}(u-v)\tilde{L}_1(u)\tilde{L}_2(v) = \tilde{L}_2(v)\tilde{L}_1(u)R_{12}(u-v)\ , \tag{3.2.4}$$

where $R(u)$ is given by

$$R(u) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(u) & c(u) & 0 \\ 0 & c(u) & b(u) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.2.5}$$

with $b(u) = u/(u+\eta)$ and $c(u) = \eta/(u+\eta)$, and is a solution to the Yang-Baxter Equation[2],

$$R_{12}(u-v)R_{13}(u), R_{23}(v) = R_{23}(v)R_{13}(u), R_{12}(u-v). \tag{3.2.6}$$

One of the properties of the Yang-Baxter Algebra(YBA) is called co-multiplication, which means that if the Lax Operators obey the YBA, then the monodromy matrix $\tilde{T}(u) = \tilde{L}_{A,n}(u+\omega)\tilde{L}_{B,m}(u-\omega)$, built as a multiplication of Lax Operators, also obeys the same algebra. Then, we have the so-called RTT relations,

$$R_{12}(u-v)\tilde{T}_1(u)\tilde{T}_2(v) = \tilde{T}_2(v)\tilde{T}_1(u)R_{12}(u-v)\ , \tag{3.2.7}$$

which, when satisfied, ensures that a system built from the transfer matrix $\tilde{\tau}(u) = \mathrm{Tr}\{\tilde{T}(u)\}$ has as many conserved quantities as it has degrees of freedom, i.e., the system is integrable.

For our Lax Operators given in Eq. (3.2.1, 3.2.2), it turns out, from the QISM, that the condition for the relation (3.2.4) to be true is that the general number operators $\tilde{N}_{A,n}$, $\tilde{N}_{B,m}$, along with the tunnelling operators A_n, $A_n^\dagger$, B_m, $B_m^\dagger$, obey the bosonic algebra, so we need that

$$\begin{array}{lll} \left[A_n,\ A_n^\dagger\right] = 1 & \left[\tilde{N}_{A,n},\ A_n^\dagger\right] = A_n^\dagger & \left[\tilde{N}_{A,n},\ A_n\right] = -A_n \\ \left[B_m,\ B_m^\dagger\right] = 1 & \left[\tilde{N}_{B,m},\ B_m^\dagger\right] = B_m^\dagger & \left[\tilde{N}_{B,m},\ B_m\right] = -B_m. \end{array}$$

For this to be satisfied, there are two conditions: The first one is that $\sum_{i=1}^n \alpha_i^2 = 1$ and $\sum_{i=1}^m \beta_i^2 = 1$, and we also need that

$$\begin{aligned} \chi_A + \gamma_A &= 1 \\ \chi_B + \gamma_B &= 1\ , \end{aligned} \tag{3.2.8}$$

because

$$\begin{aligned} \left[\tilde{N}_{A,n},\ A_n^\dagger\right] &= \left[\left(\chi_A N_{a,n} + \gamma_A N'_{A,n}\right),\ A_n^\dagger\right] \\ &= \chi_A \left[N_{A,n},\ A_n^\dagger\right] + \gamma_A \left[N'_{A,n},\ A_n^\dagger\right] \\ &= \chi_A \left(\sum_{i=1}^n \alpha_i = 1\right) A_n^\dagger + \gamma_A \left(\sum_{i=1}^n \alpha_i = 1\right) A_n^\dagger \\ &= \left(\chi_A + \gamma_A\right) A_n^\dagger, \\ \left[\tilde{N}_{A,n},\ A_n\right] &= -\chi_A \left(\sum_{i=1}^n \alpha_i = 1\right) A_n - \gamma_A \left(\sum_{i=1}^n \alpha_i = 1\right) A_n \\ &= -\left(\chi_A + \gamma_A\right) A_n, \end{aligned}$$

and analogously for the "b" class operators. So, under both these conditions, we'll have that the Lax operators (3.2.1, 3.2.2) along with the R matrix (3.2.5) obey the Yang-Baxter equation (3.2.4). Since they do, by the co-multiplication property of the Yang-Baxter Algebra, our defined monodromy matrix $\tilde{T}(u)$ given by Eq. (3.2.3),

[2] The indexes in Eq. (3.2.4) refer to the vector spaces in which our operators are acting. In the remaining spaces, they act as the identity.

will also obey the Yang-Baxter relation. The resulting transfer matrix $\tilde{\tau}(u) = \mathrm{Tr}\{\tilde{T}(u)\}$ is then

$$\begin{aligned}\tilde{\tau}(u) &= \tilde{T}_{1,1}(u) + \tilde{T}_{2,2}(u) \\ &= (u^2 - \omega^2 + \eta^{-2}) + (u+\omega)\eta\tilde{N}_{B,m} + (u-\omega)\eta\tilde{N}_{A,n} + \eta^2\tilde{N}_{A,n}\tilde{N}_{B,m} + A_n B_m^\dagger + A_n^\dagger B_m\end{aligned}$$

and commutes for different values of the spectral parameter. Now then, through a series expansion around u, we can build an integrable Hamiltonian. So, $\tilde{\tau}(u) = b_0 + b_1 u + b_2 u^2$, with

$$\begin{aligned}b_0 &= \tilde{\tau}(0) = -\omega^2 + \omega\eta\tilde{N}_{B,m} - \omega\eta\tilde{N}_{A,n} + \eta^2\tilde{N}_{A,n}\tilde{N}_{B,m} + A_n B_m^\dagger + A_n^\dagger B_m + \eta^{-2}, \\ b_1 &= \frac{d}{du}\tilde{\tau}(u)\bigg|_{u=0} = \eta(\tilde{N}_{A,n} + \tilde{N}_{B,m}) = \eta\tilde{N}, \\ b_2 &= \frac{1}{2}\frac{d^2}{du^2}\tilde{\tau}(u)\bigg|_{u=0} = \mathbb{1}.\end{aligned}$$

where we defined $\tilde{N} = \tilde{N}_{A,n} + \tilde{N}_{B,m}$. Now, an integrable Hamiltonian will be given through the following relation

$$\begin{aligned}\tilde{H}_{n,m} &= t\tilde{\tau}(0) - \frac{t}{4}\left[\frac{d}{du}\tilde{\tau}(u)\bigg|_{u=0}\right]^2 + t\omega^2\mathbb{1} - t\eta^{-2}\mathbb{1} \qquad (3.2.9)\\ &= t\left(\tilde{\tau}(u) - u\left[\frac{d}{du}\tilde{\tau}(u)\bigg|_{u=0}\right] - \frac{1}{4}\left[\frac{d}{du}\tilde{\tau}(u)\bigg|_{u=0}\right]^2 - u^2\mathbb{1} - \eta^{-2}\mathbb{1} + \omega^2\mathbb{1}\right) \\ &= -\frac{t\eta^2}{4}(\tilde{N}_{A,n} - \tilde{N}_{B,m})^2 - t\omega\eta(\tilde{N}_{A,n} - \tilde{N}_{B,m}) + t(A_n^\dagger B_m + B_m^\dagger A_n)\end{aligned}$$

which, by doing $U = -\frac{t\eta^2}{4}$ and $\mu = -t\omega\eta$, gives us

$$\tilde{H}_{n,m} = U(\tilde{N}_{A,n} - \tilde{N}_{B,m})^2 + \mu(\tilde{N}_{A,n} - \tilde{N}_{B,m}) + t(A_n^\dagger B_m + B_m^\dagger A_n) \qquad (3.2.10)$$

which is the Hamiltonian we will be working with, given in Eq. (3.2.10).

3.3 Algebraic Bethe Ansatz

Naturally, as in Sec. (2.3), we now intend to solve our integrable Hamiltonian through the Algebraic Bethe Ansatz, albeit for the $n+m$ multi-well case. We will need, as before, to complete our Fock space basis, of dimension $n+m$, because it doesn't span our entire vector space, which has dimension

$$dim = \frac{(N+n+m-1)!}{(n+m-1)!N!}.$$

We will introduce, then, the generalized $\Gamma^\dagger$ and Γ operators,

$$\Gamma_i = \mu_{i_1}a_1 + \mu_{i_2}a_2 + ... + \mu_{i_n}a_n \;\text{ with }\; \sum_{l=1}^{n}\mu_{i_l}\alpha_l = 0, \qquad (3.3.1)$$

$$\bar{\Gamma}_j = \nu_{j_1}b_1 + \nu_{j_2}b_2 + +... + \nu_{j_m}b_m \;\text{ with }\; \sum_{k=1}^{m}\nu_{j_k}\beta_k = 0. \qquad (3.3.2)$$

These operators are defined in terms of the A_n, $A_n^\dagger$ and B_m, $B_m^\dagger$ operators such that each Γ_i or $\bar{\Gamma}_j$ contributes with one vector orthogonal to the basis vectors built by the action $A_n^\dagger|0\rangle$ and $B_m^\dagger|0\rangle$ over the Fock space vacuum, so that we will have $i = 1,...,n-1$ vectors orthogonal to $A_n^\dagger|0\rangle$ completing the A subspace, and $j = 1,...,m-1$ vectors orthogonal to $B_m^\dagger|0\rangle$ completing the B subspace.

With that in hand, we can build our set of generalized pseudo-vacua for the $n+m$-well case, $|\phi_{\{l;k\}}\rangle$, which will be created by the successive action of the $\Gamma_i^\dagger$, $\bar{\Gamma}_j^\dagger$ operators over the Fock space vacuum

$$|\phi_{\{l;k\}}\rangle = \prod_{i=1}^{n-1}\left(\Gamma_i^\dagger\right)^{l_i}\prod_{j=1}^{m-1}\left(\bar{\Gamma}_j^\dagger\right)^{k_j}|0\rangle \quad \text{for every } l_i,\ k_j \text{ such that} \quad \sum_{i=1}^{n-1} l_i + \sum_{j=1}^{m-1} k_j \leq N \tag{3.3.3}$$

As discussed briefly in Sec. (2.2), the algebraic Bethe ansatz constitutes in building our eigenstates over a suitable pseudo-vacuum with the generator of our global algebra, represented by the monodromy matrix $\tilde{T}(u)$. For the following calculations, it's important that we know the commutation relations of $\Gamma^\dagger$ and the other operators introduced so far, so that we can compute how the monodromy matrix acts over the pseudovaccuum. These will be presented in Appendix C. Using the results there developed, we have that the action of the elements of the monodromy matrix $\tilde{T}(u)$ over $|\phi_{\{l;k\}}\rangle$ are

$$\begin{aligned}
\tilde{T_{1,1}}(u)|\phi_{\{l;k\}}\rangle &= \tilde{T_{1,1}}(u)\prod_{i=1}^{n-1}\left(\Gamma_i^\dagger\right)^{l_i}\prod_{j=1}^{m-1}\left(\bar{\Gamma}_j^\dagger\right)^{k_j}|0\rangle \\
&= [u^2-\omega^2+\eta(u+\omega)\chi_B\underbrace{\sum_{j=1}^{m-1}k_j}_{q}+\eta(u-\omega)\chi_A\underbrace{\sum_{i=1}^{n-1}l_i}_{p}]\,|\phi_{\{l;k\}}\rangle \\
&= [u^2-\omega^2+\eta(u+\omega)\chi_B q+\eta(u-\omega)\chi_A p]\,|\phi_{\{l;k\}}\rangle = \tilde{t}_{1,1}(u)|\phi_{\{l;k\}}\rangle
\end{aligned} \tag{3.3.4}$$

$$\begin{aligned}
\tilde{T}_{1,2}(u)|\phi_{\{l;k\}}\rangle &= \tilde{T}_{1,2}(u)\prod_{i=1}^{n-1}\left(\Gamma_i^\dagger\right)^{l_i}\prod_{j=1}^{m-1}\left(\bar{\Gamma}_j^\dagger\right)^{k_j}|0\rangle \\
&= (u+\omega+\eta\tilde{N}_{A,n})B_m+\eta^{-1}A_n|\phi_{\{l;k\}}\rangle \\
&= 0
\end{aligned} \tag{3.3.5}$$

$$\begin{aligned}
\tilde{T}_{2,1}(u)|\phi_{\{l;k\}}\rangle &= \tilde{T}_{2,1}(u)\prod_{i=1}^{n-1}\left(\Gamma_i^\dagger\right)^{l_i}\prod_{j=1}^{m-1}\left(\bar{\Gamma}_j^\dagger\right)^{k_j}|0\rangle \\
&= A_n^\dagger(u-\omega+\eta\tilde{N}_{B,m})+\eta^{-1}B_m^\dagger|\phi_{\{l;k\}}\rangle \\
&\neq 0
\end{aligned} \tag{3.3.6}$$

$$\begin{aligned}
\tilde{T}_{2,2}(u)|\phi_{\{l;k\}}\rangle &= \tilde{T}_{2,2}(u)\prod_{i=1}^{n-1}\left(\Gamma_i^\dagger\right)^{l_i}\prod_{j=1}^{m-1}\left(\bar{\Gamma}_j^\dagger\right)^{k_j}|0\rangle \\
&= \prod_{i=1}^{n-1}\left(\Gamma_i^\dagger\right)^{l_i}\prod_{j=1}^{m-1}\left(\bar{\Gamma}_j^\dagger\right)^{k_j}[\cancel{A_2^\dagger B_1}+\eta^{-2}]|0\rangle \\
&= \eta^{-2}|\phi_{\{l;k\}}\rangle = \tilde{t}_{2,2}(u)|\phi_{\{l;k\}}\rangle.
\end{aligned} \tag{3.3.7}$$

From these equations, we see that the $\tilde{T}_{2,1}$ element acts as a creation operator over our pseudo-vacua, and $\tilde{T}_{1,2}$ acts by annihilating them. Meanwhile, we see that the pseudo-vacua are eigenvectors of $\tilde{T}_{1,1}$ and $\tilde{T}_{2,2}$, with eigenvalues $\tilde{t}_{1,1}$ and $\tilde{t}_{2,2}$ respectively.

Our ansatz for the eigenstates of the system, then, are the states created by the generator of our algebra, $\tilde{T}_{2,1}$, over our pseudo-vacuum, for a few values of the spectral parameter $u = v_i$, $i = 1, \cdots, N-p-q$

$$|\psi_{\{l;k\}}\rangle = \prod_{i=1}^{N-p-q} \tilde{T}_{2,1}(v_i)\,|\phi_{\{l;k\}}\rangle \tag{3.3.8}$$

where we took $p = \sum_{i=1}^{n-1} l_i$ and $q = \sum_{j=1}^{m-1}$, and the index $\{l;k\}$ on $|\psi_{\{l;k\}}\rangle$ means we will have an eigenstate for every l_i, k_j such that $\sum_{i=1}^{n-1} l_i + \sum_{j=1}^{m-1} k_j \le N$.

The allowed values for the spectral parameter v_i will be given by the Bethe Ansatz Equations, which can be seen as the quantization rules for our system. In order to find them out, we will act on our ansatz with the transfer matrix $\tilde{\tau}(u)$, which, as we determined in Sec. (3.2), contains the central elements of our algebra and the information about the conserved charges of the system.

$$\begin{aligned}
\tilde{\tau}(u)\,|\psi_{\{l;k\}}\rangle &= \tilde{T}_{1,1}(u)\prod_{i=1}^{N-p-q}\tilde{T}_{2,1}(v_i)\,|\phi_\ell\rangle + \tilde{T}_{2,2}(u)\prod_{i=1}^{N-p-q}\tilde{T}_{2,1}(v_i)\,|\phi_{\{l;k\}}\rangle && (3.3.9)\\
&= \left[\tilde{t}_{1,1}(u)\prod_{i=1}^{N-p-q}\left(\frac{u-v_i+\eta}{u-v_i}\right) + \tilde{t}_{2,2}(u)\prod_{i=1}^{N-p-q}\left(\frac{u-v_i-\eta}{u-v_i}\right)\right]|\psi_{\{l;k\}}\rangle && (3.3.10)\\
&- \left[\sum_{i=1}^{N-p-q}\tilde{t}_{1,1}(v_i)\left(\frac{\eta}{u-v_i}\right)\prod_{j\neq i}^{N-p-q}\left(\frac{v_i-v_j+\eta}{v_i-v_j}\right) - \sum_{i=1}^{N-p-q}\tilde{t}_{2,2}(v_i)\left(\frac{\eta}{u-v_i}\right)\prod_{j\neq i}^{N-p-q}\left(\frac{v_i-v_j-\eta}{v_i-v_j}\right)\right]\tilde{T}_{2,1}(u)\,|\phi_{\{l;k\}}\rangle\,. && (3.3.11)
\end{aligned}$$

We can identify in the equation above that the (3.3.10) portion would be the eigenvalues of that equation, if the (3.3.11) part were zero. Thus, under that condition, which can be rewritten as

$$\eta^2\left[v_i^2-\omega^2+\chi_A p\eta(v_i-\omega)+\chi_B q\eta(v_i+\omega)\right] = \prod_{j\neq i}^{N-p-q}\left(\frac{v_i-v_j-\eta}{v_i-v_j+\eta}\right), \tag{3.3.12}$$

the eigenvalues of the transfer matrix are

$$\tilde{\Lambda}_{p;q}(u) = \left[u^2-\omega^2+\chi_A p\eta(u-\omega)+\chi_B q\eta(u+\omega)\right]\prod_{i=1}^{N-p-q}\left(\frac{u-v_i+\eta}{u-v_i}\right) + \eta^{-2}\prod_{i=1}^{N-p-q}\left(\frac{u-v_i-\eta}{u-v_i}\right). \tag{3.3.13}$$

Earlier we had determined our Hamiltonian to be obtained from the transfer matrix by the relation given in Eq. (3.2.9), which is

$$\begin{aligned}
\tilde{H}_{n,m} &= t\tilde{\tau}(0) - \frac{t}{4}\left[\frac{d}{du}\tilde{\tau}(u)\bigg|_{u=0}\right]^2 + t\omega^2\mathbb{1} - t\eta^{-2}\mathbb{1} && (3.3.14)\\
&= t\left(\tilde{\tau}(0) - \frac{1}{4}\eta^2[\tilde{N}_{B,m}+\tilde{N}_{A,n}]^2 + \omega^2 - \eta^{-2}\right) && (3.3.15)
\end{aligned}$$

Now that we have the transfer matrix eigenvalues $\tilde{\Lambda}_{p;q}$ in hand, we can write the Hamiltonian's eigenvalues in terms of the eigenvalues of the transfer matrix. For the $n = 2$, $m = 1$ case we'll have

$$\tilde{H}_{2,1}\left|\psi_{\{l;k\}}\right\rangle = t\left(\tilde{\Lambda}_{p;q}(0) - \frac{1}{4}\eta^2[\tilde{N}_{B,m} + \tilde{N}_{A,n}]^2 + \omega^2 - \eta^{-2}\right)\left|\psi_{\{l;k\}}\right\rangle \tag{3.3.16}$$

3.4 Conserved Quantities

In Sec. (), we needed to extend our vector space in order to obtain the solutions through the Bethe Ansatz, and we will do the same now for the general $n+m$ Hamiltonian, by defining a general Γ operator such that, given $A_n = \alpha_1 a_1 + \alpha_2 a_2 + ... + \alpha_n a_n$ and $B_m = \beta_1 b_1 + \beta_2 b_2 + ... + \beta_m b_m$, previously defined in Eqs. (3.1.2) and (3.1.3), then

$$\Gamma_i = \mu_{i_1} a_1 + \mu_{i_2} a_2 + ... + \mu_{i_n} a_n \;\text{ with }\; \sum_{l=1}^{n} \mu_{i_l}\alpha_l = 0, \tag{3.4.1}$$

$$\bar{\Gamma}_j = \nu_{j_1} b_1 + \nu_{j_2} b_2 + + ... + \nu_{j_m} b_m \;\text{ with }\; \sum_{k=1}^{m} \nu_{j_k}\beta_k = 0. \tag{3.4.2}$$

As stated in Sec.(), the Γ_i or $\bar{\Gamma}_j$ contributes with one vector orthogonal to the basis vectors $A_n^\dagger|0\rangle$ and $B_m^\dagger|0\rangle$, so we will have $i = 1,...,n-1$ vectors orthogonal to $A_n^\dagger|0\rangle$ completing the A subspace, and $j = 1,...,m-1$ vectors orthogonal to $B_m^\dagger|0\rangle$ completing the B subspace.

Given the similarity of A_n, B_m and Γ_i, $i = 1,...,n-1$ and $\bar{\Gamma}_j$, $j = 1,...,m-1$, we can define the $Q_i = \Gamma_i^\dagger\Gamma_i$ and $\bar{Q}_j = \bar{\Gamma}_j^\dagger\bar{\Gamma}_j$ quantities,

$$\begin{aligned}
Q_i &= \left(\sum_{k=1}^{n}\mu_{i_k}a_k^\dagger\right)\left(\sum_{j=1}^{n}\mu_{i_j}a_j\right) = \sum_{k=1}^{n}\mu_{i_k}^2 a_k^\dagger a_k + \sum_{k=1}^{n}\sum_{j<1}^{n}\mu_{i_k}\mu_{i_j}(a_k^\dagger a_j + a_j^\dagger a_k) && (3.4.3)\\
&= \sum_{k=1}^{n}\mu_{i_k}^2 n_{a_k} + \sum_{k=1}^{n}\sum_{j<1}^{n}\mu_{i_k}\mu_{i_j}(a_k^\dagger a_j + a_j^\dagger a_k), && (3.4.4)\\
\bar{Q}_i &= \sum_{k=1}^{m}\nu_{i_k}^2 b_k^\dagger b_k + \sum_{k=1}^{m}\sum_{j<1}^{m}\nu_{i_k}\nu_{i_j}(b_k^\dagger b_j + b_j^\dagger b_k) && (3.4.5)\\
&= \sum_{k=1}^{m}\nu_{i_k}^2 n_{b_k} + \sum_{k=1}^{m}\sum_{j<1}^{m}\nu_{i_k}\nu_{i_j}(b_k^\dagger b_j + b_j^\dagger b_k), && (3.4.6)
\end{aligned}$$

which bear a striking resemblance to the operators $N'_{A,n}$ and $N'_{B,m}$ defined in Eqs. (3.1.6) and (3.1.7). We also see that Γ_i and Q_i follows the same commutation relations as A_n and $N'_{A,n}$, which are analogous for the B class operators, $\bar{\Gamma}_i$ and $\bar{Q}_i$

$$
\begin{aligned}
[\Gamma_i,\, \Gamma_i^\dagger] &= \left(\sum_{k=1}^{n} \mu_{i_k} a_k\right)\left(\sum_{j=1}^{n} \mu_{i_j} a_j^\dagger\right) - \left(\sum_{j=1}^{n} \mu_{i_j} a_j^\dagger\right)\left(\sum_{k=1}^{n} \mu_{i_k} a_k\right) && (3.4.7)\\
&= \sum_{k=1}^{n}\sum_{j=1}^{n} \mu_{i_k}\mu_{i_j}\left(a_k a_j^\dagger - a_j^\dagger a_k\right) && (3.4.8)\\
&= \sum_{k=1}^{n}\sum_{j=1}^{n} \mu_{i_k}\mu_{i_j}\delta_{kj} = \sum_{k=1}^{n} \mu_{ik}^2 && (3.4.9)\\
\left[Q_i,\, \Gamma_i^\dagger\right] &= \Gamma_i^\dagger\Gamma_i\Gamma_i^\dagger - \Gamma_i^\dagger\Gamma_i^\dagger\Gamma_i && (3.4.10)\\
&= \Gamma_i^\dagger[\Gamma_i,\, \Gamma_i^\dagger] = \sum_{k=1}^{n} \mu_{ik}^2\Gamma_i^\dagger. && (3.4.11)
\end{aligned}
$$

As expected, we have that these operators obey the bosonic algebra if a condition similar to the one given for operators A_n and $N'_{A,n}$ is satisfied. Analyzing how these new quantities, Q_i and $\bar{Q}_j$, relates to our Hamiltonian, given in Eq. (2.1.11), we obtain, by using the commutation relations developed in Appendix C

$$
\begin{aligned}
[\tilde{H}_{n,m},\, Q_i] &= U([\tilde{N}_{A,n}^2,\, Q_i] + [\tilde{N}_{B,m}^2,\, Q_i] - 2[\tilde{N}_{A,n}\tilde{N}_{B,m},\, Q_i]) && (3.4.12)\\
&+ \mu([\tilde{N}_{A,n},\, Q_i] - [\tilde{N}_{B,m},\, Q_i]) + t([A_n^\dagger B_m,\, Q_i] + [B_m^\dagger A_n,\, Q_i]) && (3.4.13)\\
&= 0 && (3.4.14)
\end{aligned}
$$

What we can see is that, despite being very similar to the $N'_{A,n}$ and $N'_{B,m}$ operators introduced in the extended model, the operators Q_i and $\bar{Q}_i$ are conserved, while $N'_{A,n}$ and $N'_{B,m}$ are not.

Chapter 4

Conclusions and Outlook

In this work, we first developed a general integrable model describing bosons trapped in a three-site optical lattice. The model, presented in Ch. 2.1, besides the single particle tunneling, intra well and inter well interactions, also includes higher-order interactions as double particle tunneling and imbalance population tunneling dependence. The relevance of this model relies on its generality, being capable to recover models that already exist in literature, such as [1,26] for a certain choice of parameters, as well as to provide new ones. The integrability was demonstrated through the QISM, in Ch. 2.2. The method pointed out the existence of one more independent conserved quantity, besides the energy and the total number of particles, as shown in Ch. 2.5. This extra conserved quantity has a very important role in adding higher-order interactions to the model. Then we obtained the exact solution of this model in two different ways: Firstly in Ch. 2.3 by using the generalized Bethe Ansatz method to get the Bethe Ansatz Equation in the standard multiplicative form. Secondly, we took an alternative route using differential operators, and we obtained the so called Bethe Ansatz Equation in the additive form, which was done in Ch. 2.4. Then, analytic and numeric calculations were carried out for a small number of particles. We then compared the results for the energies obtained by solving the multiplicative and the additive forms of the BAE, and found, as expected that these results coincide.

Next, in Ch. 3.1, we went further and developed an even more general model, similar to the first except that, this time, we were able to vary the amount of sites in our lattice for a $n+m$-site system of trapped Bose-Einstein Condensates. Again the integrability is shown through the QISM (Ch. 3.2) and the Bethe Ansatz Method needs a set of pseudovacuum. The way how these pseudovacua are obtained is presented in details in Ch. 3.3. As it is required by the integrability, one then needs $n+m$ independent conserved quantities, however the algebraic method gives just two of these quantities. In Ch. 3.4 we have shown how we can complete our set, generating $n+m-2$ new independent conserved quantities. The generalized Hamiltonian can be seen as a new family of integrable models, obtained by varying the choice of the n and m parameters. In the case of $(n,m) = (2,1)$, we get the model presented in details in Ch. 2.1, which is also the model presented in reference [1] with the addition of some higher-order tunneling terms. On the other hand, the choice $(n,m) = (2,2)$ is the high-order excitation model for the four well model presented in reference [25], and so on. In other words, the family of integrable models presented in this work is a generalization of the family of integrable models presented in [26].

To sum up, the main contribution of this work is related to the introduction of higher order interactions (beyond intra and interwell interactions and single particle tunneling) into multi-well models (higher than two wells). The models are derived through the Quantum Inverse Scattering Method, and they admit exact solution by applying an extended algebraic Bethe Ansatz, in which a new set of pseudovacua are required to obtain a complete set of Bethe states. The models can be used to describe bosonic systems loaded in multiwell optical lattices. The models presented in this dissertation are generalizations of models that have recently appeared in the literature [26, 31, 43], and the introduction of higher-order excitations opens up the possibility for future investigations in many directions:

1. It will allow to explore effects related to the new excitations introduced in existing models in the literature, going beyond the usual single-band approximation treatment. To relate higher-order interactions, in the sense of this dissertation, with contributions from higher band interactions, see reference [44].

2. Another extension of this work will be the study of quantum dynamics to better understand what is the role played by the high-order interaction in the integrable models proposed in this dissertation;

3. It was presented in this work that the exact solution of the three-well model can be obtained via two different algebraic methods by which we arrive at two different sets of equations: the multiplicative form and the additive form of the Bethe equation. The mapping between these equations is unknown [45] and will be a topic for future research.

Appendix A

Three-well Hamiltonian Obtention

In this Appendix, we will show how to derive the three-well Hamiltonian Eq. (2.1.11) from the Hamiltonian obtained through the QISM, given by Eq. (2.2.7).

Our Eq. (2.2.7) reads

$$H = U(\tilde{N} - N_2)^2 + \mu(\tilde{N} - N_2) + t(A^\dagger a_2 + a_2^\dagger A)\,,$$

and if we open up the operators $\tilde{N}$ and $A^\dagger,\ A$ to their definition

$$\begin{aligned}
\tilde{N} &= \chi(N_1 + N_3) + \gamma\left(\alpha_1^2 N_1 + \alpha_3^2 N_3 + \alpha_1\alpha_3(a_1^\dagger a_3 + a_1 a_3^\dagger)\right) \\
A^\dagger &= \alpha_1 a_1^\dagger + \alpha_3 a_3^\dagger \\
A &= \alpha_1 a_1 + \alpha_3 a_3
\end{aligned}$$

we can rewrite the Hamiltonian (2.2.7) as

$$\begin{aligned}
H = \frac{U}{4}&\left[(4\chi^2 + \gamma^2)(N_1 + N_3)^2 - (8\chi + 4\gamma)(N_1 + N_3)N_2 + 4N_2^2\right] \\
&+ U\chi\gamma(N_1^2 + N_3^2 + 2N_1N_3) + \frac{U\gamma^2}{2}N_1N_3 \\
&+ \left(\mu\chi + \frac{\mu\gamma}{2} + \frac{U\gamma^2}{4}\right)(N_1 + N_3) - \mu N_2 \\
&+ \frac{\sqrt{2}}{2}t\left(a_1^\dagger a_2 + a_1 a_2^\dagger + a_3^\dagger a_2 + a_3 a_2^\dagger\right) + \frac{1}{2}\mu\gamma\left(a_1^\dagger a_3 + a_1 a_3^\dagger\right) \\
&+ U\gamma\left(\frac{\gamma}{2} + \chi\right)(N_1 + N_3)\left(a_1^\dagger a_3 + a_1 a_3^\dagger\right) - U\gamma(a_1^\dagger a_3 + a_1 a_3^\dagger)N_2 \\
&+ \frac{1}{4}U\gamma^2\left(a_1^\dagger a_1^\dagger a_3 a_3 + a_3^\dagger a_3^\dagger a_1 a_1\right)
\end{aligned}$$

We will further manipulate the equation above in order to simplify a few terms, which will be colored below for a better visualization

$$
\begin{aligned}
H = {} & U\left[\chi^2(N_1+N_3)^2-(2\chi+\gamma)(N_1+N_3)N_2+N_2^2\right]+\frac{U}{4}\gamma^2(N_1+N_3)^2 \\
& +U\chi\gamma(N_1+N_3)^2+\frac{U\gamma^2}{2}N_1N_3 \\
& +\left(\mu\chi+\frac{\mu\gamma}{2}\right)(N_1+N_3)+\frac{U\gamma^2}{4}(N_1+N_3)-\mu N_2 \\
& +\frac{U}{2}\gamma^2(N_1+N_3)\left(a_1^\dagger a_3+a_1a_3^\dagger\right)+U\gamma\chi(N_1+N_3)\left(a_1^\dagger a_3+a_1a_3^\dagger\right)-U\gamma(a_1^\dagger a_3+a_1a_3^\dagger)N_2 \\
& +\frac{1}{4}U\gamma^2\left(a_1^\dagger a_1^\dagger a_3a_3+a_3^\dagger a_3^\dagger a_1a_1\right) \\
& +\frac{\sqrt{2}}{2}t\left(a_1^\dagger a_2+a_1a_2^\dagger+a_3^\dagger a_2+a_3a_2^\dagger\right)+\frac{1}{2}\mu\gamma\left(a_1^\dagger a_3+a_1a_3^\dagger\right)
\end{aligned} \tag{A.0.1}
$$

We can simplify the terms in red using the fact that $\chi+\gamma=1$, and $\chi(\chi+\gamma)=\chi$:

$$\chi^2(N_1+N_3)^2+\chi\gamma(N_1+N_3)^2=\chi(N_1+N_3)^2$$

Also noticing that:

$$
\begin{aligned}
2Q &= N_1+N_3-a_1^\dagger a_3-a_3^\dagger a_1 \\
4Q^2 &= (N_1+N_3)^2-2(N_1+N_3)(a_1^\dagger a_3+a_3^\dagger a_1)+a_1^\dagger a_1^\dagger a_3a_3+a_3^\dagger a_3^\dagger a_1a_1+N_1+N_3+2N_1N_3 \\
U\gamma^2Q^2 &= \frac{U\gamma^2}{4}(N_1+N_3)^2-\frac{U\gamma^2}{2}(N_1+N_3)(a_1^\dagger a_3+a_3^\dagger a_1)+\frac{U\gamma^2}{4}\left(a_1^\dagger a_1^\dagger a_3a_3+a_3^\dagger a_3^\dagger a_1a_1\right) \\
& \quad +\frac{U\gamma^2}{4}(N_1+N_3)+\frac{U\gamma^2}{2}N_1N_3
\end{aligned} \tag{A.0.2}
$$

The blue terms can be written as:

$$
\begin{aligned}
U\gamma^2Q^2+U\gamma^2(N_1+N_3)(a_1^\dagger a_3+a_3^\dagger a_1) = {} & \frac{U\gamma^2}{4}(N_1+N_3)^2+\frac{U\gamma^2}{2}(N_1+N_3)(a_1^\dagger a_3+a_3^\dagger a_1) \\
& +\frac{U\gamma^2}{4}\left(a_1^\dagger a_1^\dagger a_3a_3+a_3^\dagger a_3^\dagger a_1a_1\right) \\
& +\frac{U\gamma^2}{4}(N_1+N_3)+\frac{U\gamma^2}{2}N_1N_3
\end{aligned}
$$

Then we will be left with the equation below,

$$
\begin{aligned}
H = {} & U\left[\chi(N_1+N_3)^2-(2\chi+\gamma)(N_1+N_3)N_2+N_2^2\right] \\
& +\left(\mu\chi+\frac{\mu\gamma}{2}\right)(N_1+N_3)-\mu N_2 \\
& +U\gamma\chi(N_1+N_3)\left(a_1^\dagger a_3+a_1a_3^\dagger\right)-U\gamma(a_1^\dagger a_3+a_1a_3^\dagger)N_2 \\
& +\frac{\sqrt{2}}{2}t\left(a_1^\dagger a_2+a_1a_2^\dagger+a_3^\dagger a_2+a_3a_2^\dagger\right)+\frac{1}{2}\mu\gamma\left(a_1^\dagger a_3+a_1a_3^\dagger\right) \\
& +U\gamma^2Q^2+U\gamma^2(N_1+N_3)(a_1^\dagger a_3+a_3^\dagger a_1)
\end{aligned} \tag{A.0.3}
$$

We can further simplify the equation above by writing off the terms in green as

$$U\gamma^2(N_1+N_3)(a_1^\dagger a_3+a_3^\dagger a_1)+U\gamma\chi(N_1+N_3)\left(a_1^\dagger a_3+a_1 a_3^\dagger\right)=U\gamma(N_1+N_3)(a_1^\dagger a_3+a_3^\dagger a_1)$$

Then we will finally be left with

$$\begin{aligned}H = U&\left[\chi(N_1+N_3)^2-(2\chi+\gamma)(N_1+N_3)N_2+N_2^2\right]\\&+\left(\mu\chi+\frac{\mu\gamma}{2}\right)(N_1+N_3)-\mu N_2\\&+U\gamma(N_1+N_3-N_2)\left(a_1^\dagger a_3+a_1 a_3^\dagger\right)\\&+\frac{\sqrt{2}}{2}t\left(a_1^\dagger a_2+a_1 a_2^\dagger+a_3^\dagger a_2+a_3 a_2^\dagger\right)+\frac{1}{2}\mu\gamma\left(a_1^\dagger a_3+a_1 a_3^\dagger\right)\\&+U\gamma^2Q^2\end{aligned}\tag{A.0.4}$$

Which is exactly

$$\begin{aligned}H=\mathscr{H}+U\gamma^2Q^2 = U&\left[\chi(N_1^2+N_3^2)+2\chi^2N_1N_3-(\chi+1)(N_1+N_3)N_2+N_2^2\right]\\&+\left(\mu\chi+\frac{\mu\gamma}{2}\right)(N_1+N_3)-\mu N_2\\&+\frac{\sqrt{2}}{2}t\left(a_1^\dagger a_2+a_1 a_2^\dagger+a_3^\dagger a_2+a_3 a_2^\dagger\right)+\frac{1}{2}\mu\gamma\left(a_1^\dagger a_3+a_1 a_3^\dagger\right)\\&+U\gamma(N_1+N_3-N_2)\left(a_1^\dagger a_3+a_1 a_3^\dagger\right)+U\gamma^2Q^2\end{aligned}\tag{A.0.5}$$

Appendix B

Integrability and the Quantum Inverse Scattering Method

As seen in Section 2.2, the integrability of the system described by (2.2.7) is ensured by the fact that it's obtainable through the Quantum Inverse Scattering Method (QISM) [5]. In this appendix, we will describe the method in more detail, basing ourselves heavily on [10, 13, 46–48].

In order to use the QISM and the Algebraic Bethe Ansatz (ABA), we need two main ingredients: The Lax Operators, and the R-matrix.

The Lax Operators consists of representations of our local algebras [13]. For the general, $n+m$-wells problem, we will take two Lax Operators,

$$\tilde{L}_{A,n}(u) = \begin{pmatrix} u+\eta\tilde{N}_{A,n} & A_n \\ A_n^\dagger & \eta^{-1} \end{pmatrix} \tag{B.0.1}$$

$$\tilde{L}_{B,m}(u) = \begin{pmatrix} u+\eta\tilde{N}_{B,m} & B_m \\ B_m^\dagger & \eta^{-1} \end{pmatrix}. \tag{B.0.2}$$

The Lax Operator $\tilde{L}_{A,n}(u)$ is the representation of the algebra for the partition (or class) A, a subsystem with n wells, while $\tilde{L}_{B,m}(u)$ is the representation for the class B, with m wells.

The monodromy[1] matrix, also known as a transport operator [13, 50], and given by

$$\tilde{T}(u) = \tilde{L}_{A,n}(u+\omega)\tilde{L}_{B,m}(u-\omega) = \begin{pmatrix} \tilde{T}_{1,1}(u) & \tilde{T}_{1,2}(u) \\ \tilde{T}_{2,1}(u) & \tilde{T}_{2,2}(u) \end{pmatrix}, \tag{B.0.3}$$

is in our case

$$\begin{aligned}
\tilde{T}_{1,1}(u) &= (u+\omega+\eta\tilde{N}_{A,n})(u-\omega+\eta\tilde{N}_{B,m}) + A_n B_m^\dagger \\
\tilde{T}_{1,2}(u) &= (u+\omega+\eta\tilde{N}_{A,n})B_m + \eta^{-1}A_n \\
\tilde{T}_{2,1}(u) &= A_n^\dagger(u-\omega+\eta\tilde{N}_{B,m}) + \eta^{-1}B_m^\dagger \\
\tilde{T}_{2,2}(u) &= A_n^\dagger B_m + \eta^{-2},
\end{aligned}$$

[1] Monodromy means literally "running around", and is the study of how objects from mathematical analysis, algebraic topology, algebraic geometry and differential geometry behave as they "run round" a point, often a singularity. [49]

and it is the representation of the "global" algebra of our system, built by taking into consideration the individual algebras along our boundary condition. This is why our system is bi-partite, divided in two classes: our "global" algebra is composed by the local algebras of class A and class B. The center of our algebra is given by the diagonal elements of the monodromy matrix, so the transfer matrix $\tilde{\tau}(u) = \mathrm{Tr}\{\tilde{T}(u)\}$, given by

$$\begin{aligned}\tilde{\tau}(u) &= \tilde{T}_{1,1}(u) + \tilde{T}_{2,2}(u) \\ &= (u^2 - \omega^2 + \eta^{-2}) + (u+\omega)\eta\tilde{N}_{B,m} + (u-\omega)\eta\tilde{N}_{A,n} + \eta^2\tilde{N}_{A,n}\tilde{N}_{B,m} + A_n B_m^\dagger + A_n^\dagger B_m,\end{aligned}$$

contains the information about the conserved quantities (i.e, symmetries) induced by our algebra over our vector space.

Before building our Hamiltonian from the transfer matrix, as it was done in Eq. (2.2.6), we must ensure that our conserved operators commute for different values of the spectral parameter, that is,

$$[\tilde{\tau}(u), \tilde{\tau}(v)] = 0.$$

This is the crucial step to build an integrable system, and the idea of the QISM is to induce the commutativity required between the conserved quantities by enveloping our system in a bigger algebra [48]. The structure constants of this bigger algebra are the $R(u)$ operators, which in our case will be

$$R(u) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(u) & c(u) & 0 \\ 0 & c(u) & b(u) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{B.0.4}$$

where $b(u) = u/(u+\eta)$ and $c(u) = \eta/(u+\eta)$, and that must satisfy the consistency relation known as the Yang-Baxter Equation,

$$R_{12}(u-v)R_{13}(u), R_{23}(v) = R_{23}(v)R_{13}(u), R_{12}(u-v). \tag{B.0.5}$$

In order to understand the meaning of the $R(u)$ matrices, it is useful to illustrate Eq. (B.0.5) in diagram form, given in Fig. (B.1).

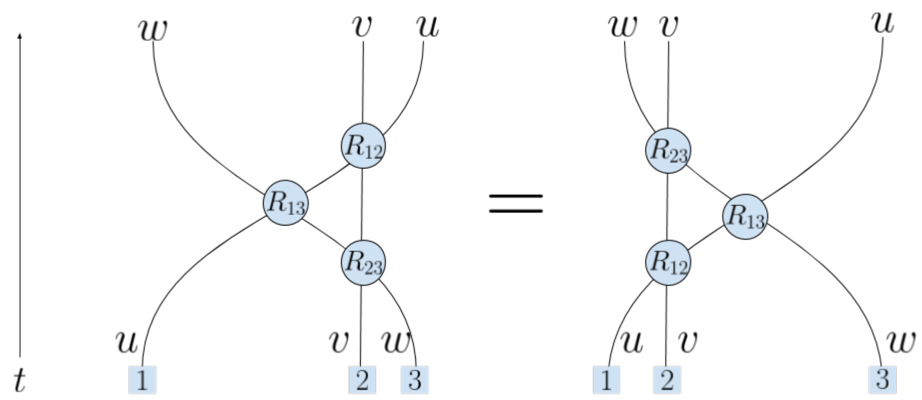

Figure B.1: Graphical representation of the Yang-Baxter Equation (B.0.5).

If we take the spectral parameter as rapidities of different particles, the consistency relation on Eq. (B.0.5) can be physically interpreted as the condition that the rapidities are conserved in a multiparticle collision [46]. The algebra induced by the Yang-Baxter Equation, then, is what guarantees that our interactions are non-diffractive, which in turn implies the many-body interactions are two-body reducible that, as discussed earlier, characterize an integrable system [5].

In order to "envelop" our system in this bigger algebra, which we will call Yang-Baxter Algebra [51] (YBA), our smaller algebras must also follow the consistency conditions given in the Yang-Baxter Equation. That means that our Lax operators must satisfy

$$R_{12}(u-v)\tilde{L}_1(u)\tilde{L}_2(v) = \tilde{L}_2(v)\tilde{L}_1(u)R_{12}(u-v) \,, \tag{B.0.6}$$

together with the $R(u)$ matrix[2]. One of the properties of the Yang-Baxter Algebra is called co-multiplication, which means that if the Lax Operators obey the YBA, then the monodromy matrix $\tilde{T}(u) = \tilde{L}_{A,n}(u+\omega)\tilde{L}_{B,m}(u-\omega)$, built as a multiplication of Lax Operators, also obeys the same algebra. Then, we have the so-called RTT relations,

$$R_{12}(u-v)\tilde{T}_1(u)\tilde{T}_2(v) = \tilde{T}_2(v)\tilde{T}_1(u)R_{12}(u-v) \,, \tag{B.0.7}$$

which, when satisfied, ensure that our global algebra $\tilde{T}(u) = \tilde{L}_{A,n}(u+\omega)\tilde{L}_{B,m}(u-\omega)$ describes a system that interacts with itself in a two-body reducible manner.

The main difficulty of the QISM is then finding a solution $R(u)$ to the Yang-Baxter Equation, and also Lax Operators that satisfy the Yang-Baxter Algebra simultaneosuly with $R(u)$. Having solved that, all there is left for us is, fortunately, some simple, but somewhat extensive algebraic manipulation, which comprises most of the work done in this dissertation.

[2] The indexes in Eq. (3.2.4) refer to the vector spaces in which our operators are acting. In the remaining spaces, they act as the identity.

Appendix C

Commutation Relations

In this Appendix, we will develop a few of the commutation relations used throughout the text. It's important to keep in mind the definitions of the operators involved and their basic commutation relations, so we will write them below before starting.

- $a_i^\dagger,\ a_i$ are the bosonic creation and destruction operators for the i$-th$ site, such that together with $n_i = a_i^\dagger a_i$, obey

$$[a_i, a_j^\dagger] = \delta_{ij}, \qquad [a_i^\dagger, a_j] = -\delta_{ij}, \qquad [n_i, a_j^\dagger] = \delta_{ij} a_j^\dagger, \qquad [n_i, a_j] = -\delta_{ij} a_j.$$

- $N_{A,n} = \sum_{i=1}^n n_i$ is the sum of the n number operators from class A,
- $A_n^\dagger,\ A_n$ are generalized bosonic creation and destruction operators, creating/destroying a state in superposition over every site of class A. When $\sum_{i=1}^n \alpha_i^2 = 1$, they follow the bosonic algebra

$$A_n^\dagger = \sum_{i=1}^n \alpha_i a_i^\dagger, \qquad A_n = \sum_{i=1}^n \alpha_i a_i.$$

$$[A_n, A_n^\dagger] = \sum_{i=1}^n \alpha_i^2 = 1, \qquad [A_n^\dagger, A_n] = -\sum_{i=1}^n \alpha_i^2 = -1,$$

- $N'_{A,n} = A_n^\dagger A_n$ is the number operator associated with the generalized creation and destruction operators over A. It also follows the bosonic algebra for the number operator when $\sum_{i=1}^n \alpha_i^2 = 1$,

$$[N'_{A,n}, A_n^\dagger] = A_n^\dagger, \qquad [N'_{A,n}, A_n] = -A_n.$$

- The Gamma operators, $\Gamma_i^\dagger,\ \Gamma_i$, are $i = 1, \cdots, n-1$ creation/destruction operators for the states orthogonal to the ones created/destructed by $A_n^\dagger,\ A_n$. They follow the bosonic algebra when $\sum_{i=1}^n \mu_{i_j}^2 = 1$.

$$\Gamma_j^\dagger = \sum_{i=1}^n \mu_{i_j} a_i^\dagger, \quad \Gamma_j = \sum_{i=1}^n \mu_{i_j} a_i \quad \text{such that} \quad \sum_{i=1}^n \mu_{i_j} \alpha_i = 0$$

$$[\Gamma_j, \Gamma_j^\dagger] = \sum_{i=1}^n \mu_{i_j}^2 = 1, \quad [\Gamma_j^\dagger, \Gamma_j] = -\sum_{i=1}^n \mu_{i_j}^2 = -1,$$

- $Q_i = \Gamma_i^\dagger \Gamma_i$ are $i = 1, \cdots, n-1$ number operators associated to the $\Gamma_i^\dagger$, Γ_i states. They too follow the bosonic algebra of the number operators when $\sum_{i=1}^n \mu_{i_j}^2 = 1$

$$[Q_i, \Gamma_i^\dagger] = \Gamma_i^\dagger, \quad [Q_i, \Gamma_i] = -\Gamma_i$$

Having that settled, we now present a few additional commutators that are important for our calculations

$$
\begin{aligned}
[\Gamma_i,\, a_j] &= \left[\sum_{l=1}^n \mu_{l_i} a_l,\, a_j\right] = \sum_{l=1}^n \mu_{l_i}\,[a_l,\, a_j] = 0 && \text{(C.0.1)}\\
\left[\Gamma_i^\dagger,\, a_j^\dagger\right] &= 0 \quad \text{(analogous to C.0.1)} && \text{(C.0.2)}\\
\left[\Gamma_i^\dagger,\, a_j\right] &= \left[\sum_{l=1}^n \mu_{l_i} a_l^\dagger,\, a_j\right] = \sum_{l=1}^n \mu_{l_i}\left[a_l^\dagger,\, a_j\right] = -\sum_{l=1}^n \mu_{l_i}\delta_{lj} = -\mu_{j_i} && \text{(C.0.3)}\\
\left[\Gamma_i^\dagger,\, a_j\right] &= \mu_{j_i} \quad \text{(analogous to C.0.3)} && \text{(C.0.4)}\\
[n_i,\, \Gamma_j] &= \left[n_i,\, \sum_{l=1}^n \mu_{l_j} a_l\right] = \sum_{l=1}^n \mu_{l_j}\,[n_i,\, a_l] = -\sum_{l=1}^n \mu_{l_j}\delta_{il} a_l = -\mu_{i_j} a_i && \text{(C.0.5)}\\
\left[n_i,\, \Gamma_j^\dagger\right] &= \mu_{i_j} a_i^\dagger \quad \text{(analogous to C.0.5)} && \text{(C.0.6)}\\
[N_{A,n},\, \Gamma_j] &= \left[\sum_{i=1}^n n_{a_i},\, \sum_{l=1}^n \mu_{l_j} a_l\right] = \sum_{i=1}^n \sum_{l=1}^n \mu_{l_j}\,[n_{a_i},\, a_l]\\
&= -\sum_{i=1}^n \sum_{l=1}^n \mu_{l_j}\delta_{il} a_l = -\sum_{i=1}^n \mu_{i_j} a_i = -\Gamma_j && \text{(C.0.7)}\\
\left[N_{A,n},\, \Gamma_j^\dagger\right] &= \sum_{i=1}^n \mu_{i_j} a_i^\dagger = \Gamma_j^\dagger \quad \text{(analogous to C.0.7)} && \text{(C.0.8)}\\
\left[A_n^\dagger,\, \Gamma_j\right] &= \left[\sum_{i=1}^n \alpha_i a_i^\dagger,\, \sum_{l=1}^n \mu_{l_j} a_l\right] = \sum_{i=1}^n \sum_{l=1}^n \alpha_i \mu_{l_j}\left[a_i^\dagger,\, a_l\right]\\
&= -\sum_{i=1}^n \sum_{l=1}^n \mu_{l_j}\alpha_i\delta_{il} = -\sum_{i=1}^n \mu_{i_j}\alpha_i = 0 \text{ (by definition)} && \text{(C.0.9)}\\
[A_n,\, \Gamma_j] &= \left[A_n^\dagger,\, \Gamma_j^\dagger\right] = \left[A_n,\, \Gamma_j^\dagger\right] = 0 \quad \text{(analogous to C.0.9)} && \text{(C.0.10)}\\
[N'_{A,n},\, \Gamma_j] &= \left[A_n^\dagger A_n,\, \Gamma_j\right] = \left[A_n^\dagger,\, \Gamma_j\right] A_n = 0 && \text{(C.0.11)}\\
\left[N'_{A,n},\, \Gamma_j^\dagger\right] &= 0 \quad \text{(see C.0.11)} && \text{(C.0.12)}\\
[Q_i, A_n] &= [\Gamma_i^\dagger \Gamma_i, A_n] = [\Gamma_i^\dagger, A_n]\Gamma_i = 0 && \text{(C.0.13)}\\
\left[Q_i, A_n^\dagger\right] &= 0 \quad \text{(see C.0.13)} && \text{(C.0.14)}\\
[Q_i,\, N'_{A,n}] &= \Gamma^\dagger \Gamma A_n^\dagger A_n - A_n^\dagger A_n \Gamma^\dagger \Gamma = 0 && \text{(C.0.15)}
\end{aligned}
$$

Appendix D

Matrix Representation of the General Three-well Hamiltonian

In this Appendix, we present a matrix representation for our Hamiltonian (3.2.10), with $n = 2,\ m = 1$, for $\alpha_1 = \alpha_2 = \frac{1}{\sqrt{2}}$, and two particles, $N = 2$.

$$\begin{pmatrix} 4U-2\mu & t & 0 & t & 0 & 0 \\ t & \mathscr{A} & t & \mathscr{B} & \frac{t}{\sqrt{2}} & 0 \\ 0 & t & \mathscr{C} & 0 & \mathscr{D} & \frac{U{\gamma_A}^2}{2} \\ t & \mathscr{B} & 0 & \mathscr{A} & \frac{t}{\sqrt{2}} & t \\ 0 & \frac{t}{\sqrt{2}} & \mathscr{D} & \frac{t}{\sqrt{2}} & \mathscr{E} & \mathscr{D} \\ 0 & 0 & \frac{U{\gamma_A}^2}{2} & t & \mathscr{D} & \mathscr{C} \end{pmatrix}$$

with

$$\mathscr{A} = \mu\left(\frac{\gamma_A}{2} + \chi_A - 1\right) + U\left(\frac{{\gamma_A}^2}{4} + \left(\frac{\gamma_A}{2} + \chi_A - 1\right)^2\right) \tag{D.0.1}$$

$$\mathscr{B} = \frac{\gamma_A \mu}{2} + U\gamma_A\left(\frac{\gamma_A}{2} + \chi_A - 1\right) \tag{D.0.2}$$

$$\mathscr{C} = \mu(\gamma_A + 2\chi_A) + U\left(\frac{{\gamma_A}^2}{2} + (\gamma_A + 2\chi_A)^2\right) \tag{D.0.3}$$

$$\mathscr{D} = \frac{\gamma_A \mu}{\sqrt{2}} + \sqrt{2}U\gamma_A(\gamma_A + 2\chi_A) \tag{D.0.4}$$

$$\mathscr{E} = \mu(\gamma_A + 2\chi_A) + U\left({\gamma_A}^2 + (\gamma_A + 2\chi_A)^2\right) \tag{D.0.5}$$

Appendix E

Matrix Representation of the General Four-well Hamiltonian

In this Appendix, we present a matrix representation for our Hamiltonian (3.2.10), with $n = 2,\ m = 2$, for $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 \frac{1}{\sqrt{2}}$, and two particles, $N = 2$.

$$
\begin{pmatrix}
\mathscr{A} & \mathscr{B} & \frac{U\gamma_B^2}{2} & \frac{t}{\sqrt{2}} & 0 & 0 & \frac{t}{\sqrt{2}} & 0 & 0 & 0 \\
\mathscr{B} & \mathscr{C} & \mathscr{B} & \frac{t}{2} & \frac{t}{2} & 0 & \frac{t}{2} & \frac{t}{2} & 0 & 0 \\
\frac{U\gamma_B^2}{2} & \mathscr{B} & \mathscr{A} & 0 & \frac{t}{\sqrt{2}} & 0 & 0 & \frac{t}{\sqrt{2}} & 0 & 0 \\
\frac{t}{\sqrt{2}} & \frac{t}{2} & 0 & \mathscr{D} & \mathscr{E} & \frac{t}{\sqrt{2}} & \mathscr{F} & -\frac{1}{2}U\gamma_A\gamma_B & \frac{t}{2} & 0 \\
0 & \frac{t}{2} & \frac{t}{\sqrt{2}} & \mathscr{E} & \mathscr{D} & \frac{t}{\sqrt{2}} & -\frac{1}{2}U\gamma_A\gamma_B & \mathscr{F} & \frac{t}{2} & 0 \\
0 & 0 & 0 & \frac{t}{\sqrt{2}} & \frac{t}{\sqrt{2}} & \mathscr{G} & 0 & 0 & \mathscr{I} & \frac{U\gamma_A^2}{2} \\
\frac{t}{\sqrt{2}} & \frac{t}{2} & 0 & \mathscr{F} & -\frac{1}{2}U\gamma_A\gamma_B & 0 & \mathscr{D} & \mathscr{E} & \frac{t}{2} & \frac{t}{\sqrt{2}} \\
0 & \frac{t}{2} & \frac{t}{\sqrt{2}} & -\frac{1}{2}U\gamma_A\gamma_B & \mathscr{F} & 0 & \mathscr{E} & \mathscr{D} & \frac{t}{2} & \frac{t}{\sqrt{2}} \\
0 & 0 & 0 & \frac{t}{2} & \frac{t}{2} & \mathscr{I} & \frac{t}{2} & \frac{t}{2} & \mathscr{H} & \mathscr{I} \\
0 & 0 & 0 & 0 & 0 & \frac{U\gamma_A^2}{2} & \frac{t}{\sqrt{2}} & \frac{t}{\sqrt{2}} & \mathscr{I} & \mathscr{G}
\end{pmatrix}
$$

with

$$\mathscr{A} = \mu\left(-\gamma_B - 2\chi_B\right) + U\left(\frac{\gamma_B^2}{2} + \left(-\gamma_B - 2\chi_B\right)^2\right) \tag{E.0.1}$$

$$\mathscr{B} = -\sqrt{2}U\left(-\gamma_B - 2\chi_B\right)\gamma_B - \frac{\mu\gamma_B}{\sqrt{2}} \tag{E.0.2}$$

$$\mathscr{C} = \mu\left(-\gamma_B - 2\chi_B\right) + U\left(\gamma_B^2 + \left(-\gamma_B - 2\chi_B\right)^2\right) \tag{E.0.3}$$

$$\mathscr{D} = U\left(\frac{\gamma_A^2}{4} + \frac{\gamma_B^2}{4} + \left(\frac{\gamma_A}{2} + \chi_A - \chi_B - \frac{\gamma_B}{2}\right)^2\right) + \mu\left(\frac{\gamma_A}{2} + \chi_A - \chi_B - \frac{\gamma_B}{2}\right) \tag{E.0.4}$$

$$\mathscr{E} = -U\left(\frac{\gamma_A}{2} + \chi_A - \chi_B - \frac{\gamma_B}{2}\right)\gamma_B - \frac{\mu\gamma_B}{2} \tag{E.0.5}$$

$$\mathscr{F} = \frac{\mu\gamma_A}{2} + U\left(\frac{\gamma_A}{2} + \chi_A - \chi_B - \frac{\gamma_B}{2}\right)\gamma_A \tag{E.0.6}$$

$$\mathscr{G} = \mu\left(\gamma_A + 2\chi_A\right) + U\left(\frac{\gamma_A^2}{2} + \left(\gamma_A + 2\chi_A\right)^2\right) \tag{E.0.7}$$

$$\mathscr{H} = \mu\left(\gamma_A + 2\chi_A\right) + U\left(\gamma_A^2 + \left(\gamma_A + 2\chi_A\right)^2\right) \tag{E.0.8}$$

$$\mathscr{I} = \frac{\mu\gamma_A}{\sqrt{2}} + \sqrt{2}U\gamma_A\left(\gamma_A + 2\chi_A\right) \tag{E.0.9}$$

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