Partial characterization of graphs having a single large Laplacian eigenvalue

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Abstract

The parameter $\sigma(G)$ of a graph G stands for the number of Laplacian eigenvalues greater than or equal to the average degree of G. In this work, we address the problem of characterizing those graphs G having $\sigma(G) = 1$. Our conjecture is that these graphs are stars plus a (possible empty) set of isolated vertices. We establish a link between $\sigma(G)$ and the number of anticomponents of G. As a by-product, we present some results which support the conjecture, by restricting our analysis to cographs, forests, and split graphs.

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1 Introduction

Let G be a graph on n vertices and m edges and let $d_1 \geqslant \cdots \geqslant d_n$ be its degree sequence. We denote by $A(G)$ its adjacency matrix and by $D(G)$ the diagonal matrix having d_i in the diagonal entry (i,i), for every $1 \leqslant i \leqslant n$, and 0 otherwise. The *Laplacian matrix* of G is the positive semidefinite matrix $L(G) = D(G) - A(G)$. The eigenvalues of $L(G)$ are called *Laplacian eigenvalues* of G; the spectrum of $L(G)$ is the *Laplacian spectrum* of G and will be denoted by $Lspec(G)$. Since it is easily seen that 0 is a Laplacian eigenvalue and it is well-known that Laplacian eigenvalues are less than or equal to n it turns out

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that $Lspec(G) \subset [0, n]$. From now on, if $Lspec(G) = \{\mu_1, \mu_2, \ldots, \mu_n\}$, we will assume that $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_n$, where $\mu_n = 0$.

Understanding the distribution of Laplacian eigenvalues of graphs is a problem that is both relevant and difficult. It is relevant due to the many applications related to Laplacian matrices (see, for example, [9, 10]). It seems to be difficult because little is known about how the Laplacian eigenvalues are distributed in the interval $[0, n]$.

Our main motivation is understanding the structure of graphs that have few large Laplacian eigenvalues. In particular, we would like to characterize graphs that have a single large Laplacian eigenvalue. What do we mean by a large Laplacian eigenvalue? A reasonable measure is to compare this eigenvalue with the average of all eigenvalues. Since the average of Laplacian eigenvalues equals the average degree $\overline{d}(G) = \frac{2m}{n}$ of G, we say that a Laplacian eigenvalue is *large* if it is greater than or equal to the average degree.

Inspired by this idea, the paper [3] introduces the spectral parameter $\sigma(G)$ which counts the number of Laplacian eigenvalues greater than or equal to $\overline{d}(G)$. Equivalently, $\sigma(G)$ is the largest index i for which $\mu_i \geqslant \frac{2m}{n}$. Since the greatest Laplacian eigenvalue μ_1 is at least $\frac{2m}{n}$ then it follows that $\sigma(G) \geqslant 1$.

There is evidence that $\sigma(G)$ plays an important role in defining structural properties of a graph G. For example, it is related to the clique number ω of G (the number of vertices of the largest induced complete subgraph of G) and it also gives insight about the Laplacian energy of a graph [11, 3]. Pirzada and Ganie [11] conjectured that $\sigma(G) \geqslant \omega - 1$. Later, this conjectured was disproved in [3] by showing a counterexample within the class of threshold graphs. Moreover, several structural properties of a graph are related to σ (see, for example, [2, 3]).

In this paper we are concerned with furthering the study of $\sigma(G)$. In particular, we deal with a problem posed in [3] which asks for characterizing all graphs G having $\sigma(G) = 1$; *i.e.*, having only one large Laplacian eigenvalue. Our conjecture is that the only connected graph on n vertices having $\sigma = 1$ is the star $K_{1,n-1}$ and that the only disconnected graph on n vertices having $\sigma = 1$ is a star together with some isolated vertices. More precisely, we conjecture that graphs having $\sigma = 1$ are some stars plus a (possibly empty) set of isolated vertices. From now on, $K_{1,r} + sK_1$ denotes the star on $r + 1$ vertices plus s isolated vertices.

Conjecture 1. Let G be a graph. Then $\sigma(G) = 1$ if and only if G is isomorphic to K_1, $K_2 + sK_1$ for some $s \geqslant 0$, or $K_{1,r} + sK_1$ for some $r \geqslant 2$ and $0 \leqslant s < r - 1$.

In this work, we show that this conjecture is true if it holds for graphs which are simultaneously connected and co-connected (Conjecture 11) and prove that Conjecture 1 is true for cographs, forests, and split graphs. The main tool for proving our results is an interesting link we have found between σ and the number of anticomponents of G (see Section 2). The interesting feature of this result is that it relates a spectral parameter with a classical structural parameter. Studying structural properties of the anticomponents of G may shed light on the distribution of Laplacian eigenvalues and, reciprocally, the distribution of Laplacian eigenvalues should give insight about the structure of the graph.

This article is organized as follows. In Section 2 we state definitions and previous results concerning Laplacian eigenvalues. In Section 3, we present some new results which establish the connection between σ and the number of nonempty anticomponents of G. In Section 4, we present some evidence on the validity of Conjecture 1 by proving that the conjecture is true when G is a cograph, a forest, or a split graph.

2 Definitions

In this article, all graphs are finite, undirected, and without multiple edges or loops. All definitions and concepts not introduced here can be found in [13]. We say that a graph is *empty* if it has no edges. A *trivial* graph is a graph with precisely one vertex; every trivial graph is isomorphic to the graph which we will denote by K_1. A graph is *nontrivial* if it has more than one vertex.

We use the standard notation $\Delta(G)$ to denote the maximum degree of a graph G.

Let G_1 and G_2 be two graphs such that $V(G_1) \cap V(G_2) = \emptyset$. The *disjoint union* of G_1 and G_2, denoted $G_1 + G_2$, is the graph whose vertex set is $V(G_1) \cup V(G_2)$, and its edge set is $E(G_1) \cup E(G_2)$. We write kG to represent the disjoint union $G + \cdots + G$ of k copies of a graph G. The *join* of G_1 and G_2, denoted $G_1 \vee G_2$, is the graph obtained from $G_1 + G_2$ by adding new edges from each vertex of G_1 to every vertex of G_2.

A vertex v of a graph G is a *twin* of another vertex w of G if they both have the same neighbors in $V(G) \setminus \{v, w\}$. We say that a graph G' is obtained from G *by adding a twin* v' to a vertex v of G if $V(G') = V(G) \cup \{v'\}$, v' is a twin of v in G', and $G' - v'$ is isomorphic to G.

By $G[S]$ we denote the subgraph of G induced by a subset $S \subseteq V(G)$.

We use $\overline{G}$ to denote the complement graph of a graph G. An *anticomponent* of a graph G is the subgraph of G induced by the vertex set of a connected component of $\overline{G}$. More precisely, an induced subgraph H of G is an anticomponent if $\overline{H}$ is a connected component of $\overline{G}$. Notice that if $G_1, G_2, \ldots, G_k$ are the anticomponents of G, then $G = G_1 \vee \cdots \vee G_k$. A graph G is *co-connected* if $\overline{G}$ is connected.

A *forest* is a graph with no cycles and a *tree* is a connected forest. The complete graph on n vertices is denoted by K_n. A *universal vertex* of a graph G is a vertex v adjacent to every vertex w different from v. A *star* is a graph isomorphic to K_1 or to a tree with a universal vertex. We use $K_{1,n-1}$ to denote the star on n vertices, where $K_{1,0}$ is isomorphic to K_1 and $K_{1,1}$ is isomorphic to K_2. The *chordless path* (respectively, *cycle*) on k vertices is denoted by P_k (respectively, C_k).

A *stable set* of a graph is a set of pairwise nonadjacent vertices. A *clique* of a graph is a set of pairwise adjacent vertices.

Throughout this article, given two graphs G and H, we write $G = H$ to point out that G and H belong to the same isomorphism class.

Brouwer and Haemers [1] provided a lower bound for the k-th Laplacian eigenvalue of a graph in terms of d_k, answering a conjecture raised by Guo [6].

Theorem 2 ([1])**.** *Let G be a graph on n vertices. If G is not isomorphic to $K_k+(n-k)K_1$, then $\mu_k(G) \geqslant d_k - k + 2$.*

Throughout this article we will use the lower bounds corresponding to the cases $k = 1$ [5] and $k = 2$ [7].

It is easy to prove that the Laplacian spectrum of the disjoint union $G_1 + G_2$ is the union of the Laplacian spectrums of G_1 and G_2. The next result allows to determine the Laplacian spectrum of the join $G_1 \vee G_2$, from those of G_1 and G_2.

Theorem 3 ([8, Theorem 2.20]). *Let G_1 and G_2 be two graphs with Laplacian spectrums $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_{n_1} = 0$ and $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_{n_2} = 0$, respectively. Then the Laplacian eigenvalues of $G_1 \vee G_2$ are $n_1 + n_2$; $n_2 + \mu_i$, for $1 \leqslant i \leqslant n_1 - 1$; $n_1 + \lambda_i$, for $1 \leqslant i \leqslant n_2 - 1$ and 0.*

3 Relating σ and the number of anticomponents

This section is devoted to establish a link between $\sigma(G)$ and the number of anticomponents of G.

In virtue of Theorem 3, the following result immediately holds.

Lemma 4. *If $G = G_1 \vee \cdots \vee G_k$, with $k \geqslant 1$, is a graph on n vertices, then n is a Laplacian eigenvalue of G with multiplicity at least $k - 1$.*

Lemma 5. *If G has k anticomponents, then $k \leqslant \sigma(G) + 1$.*

Proof. Let $G = G_1 \vee \cdots \vee G_k$ where $G_1, \ldots, G_k$ are the anticomponents of G. For any graph G with at least one vertex we have that $\sigma(G) \geqslant 1$ and thus the assertion follows when $k = 1$. We may assume that $k \geqslant 2$. Lemma 4 implies that n is a Laplacian eigenvalue of G with multiplicity at least $k - 1$ in G. Thus $\mu_{k-1}(G) = n$ which implies that $\sigma(G) \geqslant k - 1$. □

Remark 6. The upper bound given by Lemma 5 is sharp when $\sigma(G) > 1$. Indeed, for $s \geqslant 2$ consider the graph $G = 4K_2 \vee K_1 \vee \cdots \vee K_1$, where s is the number of K_1's. The average degree of G is $s + 7 - \frac{48}{s+8}$ and it has $s + 1$ anticomponents. Since its Laplacian eigenvalues are $s + 8$, $s + 2$, s, and 0 with multiplicities s, 4, 3, and 1, respectively, it follows that $\sigma(G) = s$.

We use $\ell(G)$ to denote the number of nonempty anticomponents of a graph G. Recall that a nontrivial graph has at least two vertices. The following result looks further into the case where equality holds in Lemma 5 showing that $\sigma(G)$ is an upper bound for $\ell(G)$.

Theorem 7. *Let G be a graph having $k = \sigma(G) + 1$ anticomponents. Then $\ell(G) \leqslant \sigma(G)$. Moreover, if $\sigma(G) = \ell(G)$, then the remaining anticomponent of G is empty but nontrivial.*

Proof. Write $G = G_1 \vee \cdots \vee G_k$ where $G_1, \ldots, G_k$ are the anticomponents of G. Since $\sigma(G) \geqslant 1$ then $k \geqslant 2$. We set the following notations for each $i \in \{1, \ldots, k\}$:

$$n_i = |V(G_i)|, \quad m_i = |E(G_i)|, \quad \mu_1^{(i)} = \mu_1(G_i).$$

Assume that $G_1, \dots, G_\ell$ are the nonempty anticomponents. Since $k \geqslant 2$ and we are assuming that $\sigma(G) = k - 1$ it turns out that $\mu_k(G) < \overline{d}(G)$. Therefore, for each $i \in \{1, \dots, k\}$ such that $n_i > 1$ we have that

$$n - n_i + \mu_1^{(i)} \leqslant \mu_k(G) < \frac{2m}{n} = \frac{2\sum_{j=1}^{k} m_j + 2\sum_{1\leqslant i<j\leqslant k} n_i n_j}{n},$$

the first inequality holds by Theorem 3. Equivalently,

$$\begin{aligned}\mu_1^{(i)} &< \frac{2\sum_{j=1}^{k} m_j - (n^2 - 2\sum_{1\leqslant i<j\leqslant k} n_i n_j)}{n} + n_i \\ &= \frac{2\sum_{j=1}^{k} m_j - \sum_{j=1}^{k} n_j^2 + n n_i}{n}.\end{aligned} \tag{1}$$

As a consequence of Theorem 2, we obtain the following lower bound for each $i \in \{1, \dots, \ell\}$:

$$\mu_1^{(i)} \geqslant \Delta(G_i) + 1 \geqslant \overline{d}(G_i) + 1 = \frac{2m_i}{n_i} + 1. \tag{2}$$

Combining (1) and (2), we deduce that, for each $i \in \{1, \dots, \ell\}$,

$$2n_i \sum_{j=1}^{k} m_j - n_i \sum_{j=1}^{k} n_j^2 + n n_i^2 - 2n m_i - n n_i > 0. \tag{3}$$

Arguing towards a contradiction, suppose that $\ell(G) = k$. If we sum up the left-hand side of (3) for each $i \in \{1, \dots, k\}$, we obtain

$$2n \sum_{j=1}^{k} m_j - n \sum_{j=1}^{k} n_j^2 + n \sum_{i=1}^{k} n_i^2 - 2n \sum_{i=1}^{k} m_i - n^2 = -n^2$$

which is not a positive quantity. This contradiction proves that G has at most $k-1 = \sigma(G)$ nonempty anticomponents and our first assertion follows.

Assume now that $\ell(G) = k-1$. Suppose that G_k is trivial. Hence $n_k = 1$ and $m_k = 0$. Summing up to the left-hand side of (3) for each $i \in \{1, \dots, k-1\}$, we obtain that

$$-2\sum_{j=1}^{k-1} m_j + \sum_{j=1}^{k} n_j^2 - n^2 = -2\sum_{j=1}^{k-1} m_j - 2\sum_{1\leqslant i<j\leqslant k} n_i n_j$$

should be a positive number. This contradiction shows that G_k must be nontrivial. $\square$

Recall that a *bipartite graph* is a graph whose set of vertices can be partitioned into two (possibly empty) stable sets called *partite sets* of the bipartite graph. A *complete bipartite graph* is a bipartite graph isomorphic to $rK_1 \vee sK_1$ for two positive integers r and s. We denote by $K_{r,s}$ the complete bipartite graph isomorphic to $rK_1 \vee sK_1$. The upper bound $\sigma(G)$ on $\ell(G)$ for those graphs having exactly $\sigma(G)+1$ anticomponents is not tight when $\sigma(G) = 1$. Indeed, the following result shows that if a graph G has $\sigma(G) = 1$, then G has no nonempty anticomponents.

Corollary 8. *If G is a graph on n vertices with $\sigma(G) = 1$ and $\overline{G}$ is disconnected, then $G = K_{1,n-1}$.*

Proof. In virtue of Lemma 5, the number of anticomponents of G is at most 2. Since $\overline{G}$ is disconnected, we conclude that G has precisely two anticomponents G_1 and G_2 and thus $G = G_1 \vee G_2$.

Suppose, for a contradiction, that G_1 is a nonempty anticomponent of G. Because of Theorem 7, we conclude that G_2 is empty but nontrivial. Following the notation used in the proof of Theorem 7, we have that $m_2 = 0$. For $i = 1$, inequality (3) becomes

$$-2n_2m_1 - n_1n_2^2 + n_2n_1^2 - n_1^2 - n_1n_2 > 0. \tag{4}$$

Since G_2 is a nontrivial empty graph it follows that $\mu_1^{(2)} = 0$ and hence, for $i = 2$, inequality (1) becomes

$$2m_1 - n_1^2 + n_1n_2 > 0. \tag{5}$$

Summing up (4) and n_2 times (5) gives

$$-n_1^2 - n_1n_2 > 0.$$

This contradiction arose from supposing that G has some nonempty anticomponent. Hence, both anticomponents of G are empty; *i.e.*, G is a complete bipartite graph.

Since $G = K_{n_1,n_2}$, where $n_2 \geqslant n_1 \geqslant 1$ and $n = n_1 + n_2$, the average degree of G is equal to $\frac{2n_1n_2}{n}$. In virtue of Theorem 3, the Laplacian eigenvalues of K_{n_1,n_2} are n, n_2, n_1 and 0, each with multiplicity 1, $n_1 - 1$, $n_2 - 1$ and 1, respectively.

Arguing towards a contradiction, suppose that $n_1 \geqslant 2$. Hence $\mu_2(G) = n_2$. Since $2n_1 \leqslant n$, we deduce that $\overline{d}(G) = \frac{2n_1n_2}{n} \leqslant \mu_2(G)$, which contradicts the fact that $\sigma(G) = 1$. This contradiction proves that $n_1 = 1$ and therefore we conclude that $G = K_{1,n-1}$. $\square$

4 Graphs with $\sigma = 1$

In this section we provide some evidence in order to make plausible Conjecture 1. We first verify Conjecture 1 for graphs having disconnected complement; namely, we prove that the only graphs having $\sigma = 1$ and disconnected complement are the stars (including the trivial star K_1). Then, we prove that Conjecture 1 can be reduced to proving that the only connected and co-connected graph with $\sigma = 1$ is K_1. We then verify Conjecture 1 for cographs, forests, and split graphs.

4.1 Reduction to co-connected graphs

We first obtain a result which proves the validity of Conjecture 1 for graphs having disconnected complement.

Corollary 9. *Let G be a graph on n vertices such that $\overline{G}$ is disconnected. Then $\sigma(G) = 1$ if and only if $G = K_{1,n-1}$.*

Proof. Assume first that $G = K_{1,n-1}$. Then $\overline{d}(G) = 2 - \frac{2}{n}$. If $n = 2$, the Laplacian eigenvalues of G are 2 and 0. If $n \geqslant 3$, the Laplacian eigenvalues of G are n, 1 and 0, each with multiplicity 1, $n-2$ and 1, respectively. In any case we have that $\sigma(G) = 1$.

The 'only if' part follows from Corollary 8. $\square$

As a consequence of Corollary 9, Conjecture 1 is equivalent to the validity of the following weaker conjecture.

Conjecture 10. Let G be a graph with connected complement. Then, $\sigma(G) = 1$ if and only if G is isomorphic to K_1, $K_2 + sK_1$ for some $s > 0$, or $K_{1,r} + sK_1$ for some $r \geqslant 2$ and $0 < s < r - 1$.

4.2 Reduction to connected and co-connected graphs

We next show that the validity of Conjectures 1 and 10 can be reduced to the validity of the following even weaker conjecture.

Conjecture 11. Let G be a connected graph with connected complement. Then, $\sigma(G) = 1$ if and only if G is isomorphic to K_1.

A graph class $\mathcal{G}$ is *closed by taking components* if every connected component of every graph in $\mathcal{G}$ also belongs to $\mathcal{G}$. In particular, the class of all graphs is closed by taking components. Below we prove that the reduction from Conjecture 1 to Conjecture 11 holds even when restricted to any graph class closed by taking components.

Theorem 12. *Let $\mathcal{G}$ be a graph class closed by taking components. If Conjecture 11 holds for $\mathcal{G}$, then Conjecture 1 also holds for $\mathcal{G}$.*

Proof. Let G be a graph in $\mathcal{G}$ with $\sigma(G) = 1$. Assume first that G is connected. If G is co-connected, by hypothesis, G is isomorphic to K_1. If G is not co-connected, then G is isomorphic to $K_{1,r}$ for some $r \geqslant 1$, by virtue of Corollary 9.

Assume now that G is disconnected and let $G = G_1 + G_2$, where each of G_1 and G_2 has at least one vertex. We can assume, without loss of generality, that G_1 is connected and $\mu_1(G_1) \geqslant \mu_1(G_2)$. If G_1 were empty, then G_2 would also be empty, contradicting $\sigma(G) = 1$. Hence we can assume, without loss of generality, that G_1 is nonempty. Let n_i and m_i denote the number of vertices and edges of G_i, respectively, for each $i \in \{1, 2\}$. Since $\sigma(G) = 1$,

$$\frac{2m_2}{n_2} \leqslant \mu_1(G_2) < \overline{d}(G) = \frac{2(m_1 + m_2)}{n_1 + n_2}.$$

This implies that

$$\frac{2m_2}{n_2} < \frac{2m_1 + 2m_2}{n_1 + n_2} < \frac{2m_1}{n_1}. \tag{6}$$

As a consequence of (6) we have that

$$\mu_2(G_1) < \frac{2m_1 + 2m_2}{n_1 + n_2} < \frac{2m_1}{n_1} = \overline{d}(G_1).$$

We conclude that $\sigma(G_1) = 1$. Since $\mathcal{G}$ is closed by taking components, $G_1 \in \mathcal{G}$. Thus, if G_1 were co-connected, then $G_1 = K_1$, contradicting the assumption that G_1 is nonempty. Hence G_1 is not co-connected and, by Corollary 9, we have that $G_1 = K_{1,r}$ for some $r \geqslant 1$.

From (6) we deduce that

$$\mu_1(G_2) < \frac{2m_1 + 2m_2}{n_1 + n_2} < \frac{2m_1}{n_1} = \frac{2r}{r+1} < 2,$$

and hence, by virtue of Theorem 2, we conclude that G_2 must be empty. Then there exists an integer $s \geqslant 1$ such that $G_2 = sK_1$ and therefore it turns out that $G = K_{1,r} + sK_1$. The average degree of G is $\overline{d}(G) = \frac{2r}{r+1+s}$. If $r = 1$, then $\sigma(G) = 1$ because the second largest Laplacian eigenvalue of G is 0. If $r \geqslant 2$, then, as the second largest eigenvalue of G is 1 it follows that $\sigma(G) = 1$ if and only if $s < r - 1$. □

A *cograph* is a graph with no induced P_4. It is well-known that the only connected and co-connected cograph is K_1 [12]. Hence, Conjecture 11 holds trivially for cographs and, by Theorem 12, Conjecture 1 holds for cographs.

4.3 Characterizing forests and split graphs with $\sigma = 1$

In this section, we verify Conjecture 1 for forests and split graphs.

A graph class $\mathcal{G}$ is *monotone* if $G \in \mathcal{G}$ implies that every subgraph of G also belongs to $\mathcal{G}$. Notice that every monotone graph class is closed by taking components. It can be easily seen that the class of all forests is monotone and thus it is closed by taking components.

Theorem 13. *Conjecture 1 holds for forests.*

Proof. Notice that if T is a connected and co-connected forest, then T is either K_1 or a tree with diameter greater than two. By virtue of Theorem 12, it suffices to show that if T is a tree with diameter greater than two, then $\sigma(T) \geqslant 2$. Assume that T is a tree with diameter greater than two. Hence there exists two vertices v_1 and v_2 such that $d(v_1) \geqslant d(v_2) \geqslant 2 > 2 - \frac{2}{n} = \overline{d}(T)$. By Theorem 2, $\mu_2(T) \geqslant d_2(T) \geqslant 2 > \overline{d}(T)$. Therefore, $\sigma(T) \geqslant 2$. □

Let $\mathcal{H}$ be a set of graphs. We use the term *$\mathcal{H}$-free* for referring to the family of those graphs having no graph in $\mathcal{H}$ as induced subgraph. If $\mathcal{H}$ has just one element H, we write H-free for simplicity. A *split graph* [4] is a graph whose vertex set can be partitioned into a clique C and a stable set S, such a partition (C, S) of its vertices is called a *split partition.* It is well known that the class of split graphs coincides with the class of $\{2K_2, C_4, C_5\}$-free graphs.

Theorem 14. *Conjecture 1 holds for split graphs.*

Proof. Let (C, S) be a split partition of the graph on n vertices G such that $|C| = c$ and $|S| = n - c$. We label the vertices of G so that $C = \{v_1, \ldots, v_c\}$ and $S = \{v_{c+1}, \ldots, v_n\}$.

We can assume, without loss of generality, that C is a maximal clique of G under inclusion and $d_i \geqslant d_{i+1}$, for each $i \in \{1, \ldots, n-1\}$.

We claim that if G is a split graph with $\sigma(G) = 1$, then G is isomorphic to $K_{1,r-1} + (n-r)K_1$ for some r such that $2 \leqslant r \leqslant n$.

In order to prove our claim we assume that G is nonisomorphic to $K_{1,r-1} + (n-r)K_1$, for each $r \in \{2, \ldots, n\}$ and we will prove that $\sigma(G) \geqslant 2$. By virtue of Theorem 2, it suffices to prove that $d_2 \geqslant \overline{d}(G)$ or equivalently that

$$\sum_{i=3}^{n} (d_2 - d_i) \geqslant d_1 - d_2.$$

We will consider two cases.

1. *Assume that* $d_2 \geqslant c$. Since C is a maximal clique, $d_2 - d_i \geqslant 1$ for each $i \in \{c+1, \ldots, n\}$. Hence

$$\sum_{i=3}^{n} (d_2 - d_i) \geqslant \sum_{i=c+1}^{n} (d_2 - d_i) \geqslant n - c \geqslant d_1 - d_2.$$

2. *Assume that* $d_2 = c - 1$. Our assumption on G implies that $c > 2$. Moreover, we have that $d_i \leqslant 1$ for each $i \in \{c+1, \ldots, n\}$. Consequently, $d_2 - d_i \geqslant 1$ for each such i and the reasoning follows as above.

Thus we have proved our claim. In particular, the only connected and co-connected split graph with $\sigma = 1$ is K_1; i.e., Conjecture 11 holds for split graphs. Therefore, by virtue of Theorem 12, Conjecture 1 holds for split graphs. $\square$

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