FRACTURE PROPAGATION IN VISCOELASTIC MATERIALS: A MIXED MICRO-MACROSCOPIC APPROACH

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Abstract. This paper presents a theoretical approach to fracture propagation in viscoelastic media, which combines a micromechanical reasoning and macroscopic thermodynamics arguments. Unlike cracks, fractures can be viewed as interfaces that are able to transfer efforts. Their specific behavior under shear and normal stresses is a fundamental component of the deformation and fracture in brittle materials such as geomaterials. Based on the implementation of the Mori-Tanaka linear homogenization scheme and correspondence principle, the first part of the paper is dedicated to assess the exact homogenized behavior of fractured viscoelastic materials. An approximate model for effective viscoelastic properties is also formulated in the framework of Burger model. Based on macroscopic thermodynamics principles, the free energy at macroscopic scale is then formulated, allowing for the analysis of damage propagation. It is shown that the thermodynamic force associated with damage propagation can be computed from the derivative of macroscopic free energy density with respect to fracture density parameter. Expression for the propagation criterion is therefore formulated based on the closed form expression previously obtained for the homogenized viscoleastic relaxation tensor.

Keywords: Fracture, Viscoelasticity, Micromechanics, Damage propagation

1 INTRODUCTION

It is well-known that material engineering, such as rocks or geomaterials, frequently exhibit at different scales discontinuities of various sizes and orientations. The term discontinuity (crack or fracture) refers to a zone of small thickness along which the mechanical and physical properties of the matrix material are degraded. From a macroscopic viewpoint, discontinuities represent a fundamental component of the overall deformation, strength and fluid transport properties (e.g. Barton et al., 1985; Maghous et al., 2008). Discontinuities present in a medium are geometrically modeled as interfaces whose mechanical behavior is expressed by means of relationships linking the stress vector to displacement jump.

While the instantaneous behavior of fractured media has been widely investigated in the literature, few works have been dedicated to analyze the time-dependent behavior of such media. Recently, Nguyen et al. (2010) and Nguyen and Dormieux (2014) addressed the problem of crack propagation within a viscoelastic material by means of an approximate micromechanics-based approach. The latter relies upon the assumptions that the cracked medium can be modeled by an approximate homogenized Burger model and that the residual part of macroscopic free energy density can be disregarded. The present contribution is viewed as an extension of the model developed in Nguyen et al. (2013) to formulate a more comprehensive model that addresses either crack (discontinuity without stress transfer) or fracture (discontinuity without stress transfer) propagation in the context of viscoelasticity. The analysis of damage propagation in the viscoelastic fractured medium, which is regarded as a homogenized material, is based on energy dissipation concepts combining macroscopic thermodynamics arguments and a micromechanics reasoning, Unlike the Nguyen´s approach, the present modeling considers both recoverable and residual parts of the macroscopic free energy density.

2 HOMOGENIZED VISCOELASTIC BEHAVIOR OF FRACTURED MATERIALS

Prior to the analysis of damage propagation, the overall properties of the viscoelastic medium should first be formulated from those of its constituents, namely the intact matrix and the fractures. Indeed, the approach developed for the damage propagation requires to be able compute the macroscopic free energy, which explicitly depends on the homogenized viscoelastic relaxation tensor.

In the context of non-ageing linear viscoelasticity, the homogenized viscoelastic behavior can be determined combining elastic homogenization and the correspondence principle (Le et al., 2008). The correspondence principle expresses the equivalence between the viscoelastic problem in time domain and the associated elastic problem in the Laplace domain (Bland, 1960; Salençon, 2009). It is recalled that the Carson-Laplace transform $p \rightarrow \mathrm{u}^*(p)$ of a time-dependent function $t \rightarrow \mathrm{u}(t)$ is defined by:

$$\mathrm{u}^*(p) = \int_{-\infty}^{\infty} \dot{u}(t) e^{-pt} dt \tag{1}$$

The starting point is the elastic homogenization developed in Maghous et al. (2014) for elastic fracture media. Assuming isotropy at micro-scale, the homogenized elastic tensor $\mathbb{C}^{\text{hom}}$ has been determined from the elastic properties of the matrix and those of the fractures. Assuming elastic isotropy for the individual constituents of the fractured medium, together with a spatially isotropic distribution of fracture, it results that the homogenized material is described by an isotropic viscoelastic behavior. Denoting by $\mathbb{R}_{hom}$ the homogenized relaxation tensor, it follows from the correspondence principle that:

$$\mathbb{R}^*_{hom} = \mathbb{C}^*_{hom} = 3\,k^*_{hom}\,\mathbb{J} + 2\,\mu^*_{hom}\,\mathbb{K} \tag{2}$$

where the fourth-order tensors $\mathbb{J}$ and $\mathbb{K}$ are defined as $\mathbb{J} = \frac{1}{3}\underline{\underline{1}} \otimes \underline{\underline{1}}$ and $\mathbb{K} = \mathbb{I} - \mathbb{J}$. Parameters k^*_{hom} and μ^*_{hom} are the Carson-Laplace transform of the homogenized bulk and shear moduli, respectively :

$$k^*_{hom} = 3\,k^*_s\,\frac{\beta^*_1}{\beta^*_2}$$
$$\mu^*_{hom} = \mu^*_s\,\frac{45\beta^*_1\beta^*_3}{\beta^*_4} \tag{3}$$

where k^*_s and μ^*_s are respectively the bulk and shear moduli of matrix material in the operational space. We also k^*_j and μ^*_j that represent respectively the bulk and shear moduli of the fractures. Coefficients β^*_j dependent on the rheological model adopted for the viscoelastic behavior of matrix and fracture material, and on the crack density parameter $\epsilon = \mathcal{N}r^3$ ($\mathcal{N}$ is the crack density expressed by volume unity and r is mean radius of fractures):

$$\beta^*_1 = 9\,k^*_s\,\pi\,\mu^*_s + 9\,k^*_s\,r\,k^*_j + 12\,\mu^*_s\,r\,k^*_j + 12\,k^*_s\,r\,\mu^*_j + 16\,\mu^*_s\,r\,\mu^*_j + 3\,\pi\,\mu^{*2}_s$$
$$\beta^*_2 = 12\,k^{*2}_s\,\epsilon\,\pi + 16\,k^*_s\,\epsilon\,\pi\,\mu^*_s + 3\,\pi\,\mu^{*2}_s + 9\,k^*_s\,\pi\,\mu^*_s + 12\,r\,\mu^*_s\,k^*_j + 9\,k^*_s\,r\,k^*_j + 16\,r\,\mu^*_s\,\mu^*_j + 12\,k^*_s\,r\,\mu^*_j$$
$$\beta^*_3 = 9\,k^*_s\,\pi\,\mu^*_s + 12\,k^*_s\,r\,\mu^*_j + 16\,\mu^*_s\,r\,\mu^*_j + 6\,\pi\,\mu^{*2}_s \tag{4}$$
$$\beta^*_4 = \frac{16}{3}\pi\,\epsilon\,\mu^*_s\left(3k^*_s + 4\mu^*_s\right)\left(2\beta^*_1 + \beta^*_3\right) + 5\beta^*_1\,\beta^*_3$$

It is observed that k^*_s and μ^*_s are respectively the bulk and shear moduli of matrix material in the operational space. Their values are dependent on the rheological model utilized adopted for the viscoelastic behavior of matrix material. If the Kelvin-Voigt rheological model (represented in the Fig. 1) is adopted for instance, moduli k^*_s and μ^*_s take the fallowing form:

$$k^*_s = \frac{k^e_{m,s}\left(p\,k^v_{K,s} + k^e_{K,s}\right)}{p\,k^v_{K,s} + k^e_{K,s} + k^e_{m,s}}$$
$$\mu^*_s = \frac{\mu^e_{m,s}\left(p\,\mu^v_{K,s} + \mu^e_{K,s}\right)}{p\,\mu^v_{K,s} + \mu^e_{K,s} + \mu^e_{m,s}} \tag{5}$$

and similar expressions for k_j^* and μ_j^* associated with the behavior of fractures.
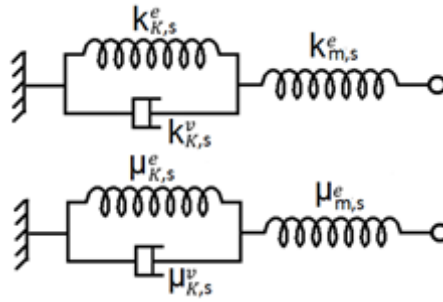

Figure 1: Rheological model of Kelvin-Voigt.

2.1   Inverse of Carson-Laplace transform

Aguiar (2016) presented a procedure to obtain the inverse Carson-Laplace transform that is valid for a wide range of rheological models. The approach has been developed in order to cover a large number of individual rheological models adopted for matrix material and fracture material, including Spring elastic model, Maxwell model, three-element standard model, Burger model or Generalized Maxwell model. For sake of simplicity, rheological models which do not exhibit instantaneous elasticity, such as the two-element Kelvin model, shall not be considered in the present analysis. It may be perceived from the analysis of Eq. (3) and Eq. (4) that the expressions of the bulk and shear moduli can always be written as a ratio of two polynomial functions of variable p. Referring either to k_{hom}^* or μ_{hom}^* by the generic relaxation function R_{hom}^*, it follows that:

$$\text{R}_{hom}^* = \frac{A(p)}{B(p)} \tag{6}$$

where polynomials $A(p)$ and $B(p)$ can be generally expressed as

$$A(p) = \sum_{k=0}^{n} a_k\, p^k$$

$$B(p) = \sum_{k=0}^{n} b_k\, p^k = \prod_{k=1}^{n}(p - R_k) \quad ; \quad b_n = 1 \tag{7}$$

where a_k and b_k are real coefficients depending on the rheological models of the matrix and fractures, and scalar R_k is the kth roots of $B(p)$. It should be noted that for usual rheological models, the polynomial $B(p)$ admits only simple roots and b_0 is non null (i.e., $b_0 \neq 0$). After mathematical developments aiming to compute the inverse of the Carson-Laplace transform of R_{hom}^*, it has been shown in Aguiar (2016) that the expression of the relaxation function R_{hom} in the time domain expresses as:

$$R_{hom}(t) = \left[\frac{a_0}{b_0} + \sum_{k=1}^{n} D_k e^{R_k t} \right] Y(t) \qquad (8)$$

where

$$D_k = \left[\frac{C(p)}{\frac{\partial B(p)}{\partial p}} \right]_{p=R_k} = \left[\frac{C(p)(p-R_k)}{B(p)} \right]_{p=R_k} \quad ; \quad C(p) = \frac{1}{b_0} \sum_{k=1}^{n} \left[(b_0 a_k - a_0 b_k) p^{k-1} \right] \qquad (9)$$

The above procedure can be developed separately for the moduli k_{hom}^* and μ_{hom}^*, provided that the polynomials $A(p)$ and $B(p)$ that are associated with each one of these moduli are considered.

2.2 Exact rheological model of homogenized viscoelastic-fractured materials

A classical model to represent non-ageing linear viscoelastic materials is the generalized Maxwell model showed in Fig. 2:
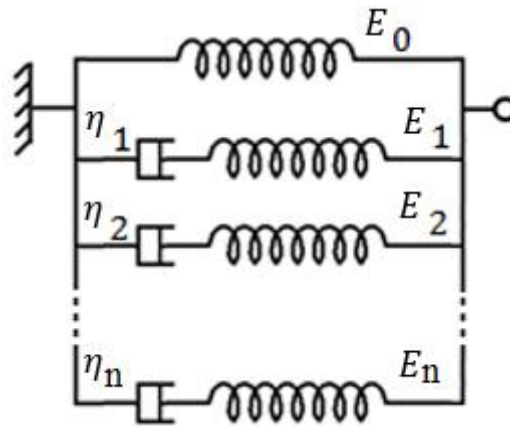

Figure 2: Generalized rheological model of Maxwell

The relaxation function R_{G-Max} associated with the generalized Maxwell model reads:

$$R_{G-max}(t) = \left[E_0 + \sum_{k=1}^{n} E_k e^{\left(-\frac{E_k}{\eta_k} t \right)} \right] Y(t) \qquad (10)$$

which is formally identical to the homogenized relaxation of the fractured medium expressed by Eq. (8) medium, with:

$$E_0 = \frac{a_0}{b_0} \quad ; \quad E_k = D_k \quad ; \quad -\frac{E_k}{\eta_k} = R_k \Rightarrow \eta_k = -\frac{D_k}{R_k} \qquad (11)$$

This correlation implies that the proposed model can represents exactly the homogenized material whenever the restrictions applied to the Inverse of Carson-Laplace transform are met. As observed previously, this model can represents a three dimensional solid through its moduli k_{hom} and μ_{hom}, adopting the generalized Maxwell model for them. The following expressions are found:

$$k_{hom}(t) = \left[k_0^e + \sum_{k=1}^{n} k_k^e\, e^{\left(-\frac{k_k^e}{k_k^v}t\right)} \right] Y(t) \quad ; \quad \mu_{hom}(t) = \left[\mu_0^e + \sum_{k=1}^{n} \mu_k^e\, e^{\left(-\frac{\mu_k^e}{\mu_k^v}t\right)} \right] Y(t) \tag{12}$$

Correlating with the Eq. (8) is obtained:

$$k_0^e = \frac{a_0^k}{b_0^k} \quad ; \quad k_k^e = D_k^k \quad ; \quad -\frac{k_k^e}{k_k^v} = R_k^k \Rightarrow k_k^v = -\frac{D_k^k}{R_k^k}$$

$$\mu_0^e = \frac{a_0^\mu}{b_0^\mu} \quad ; \quad \mu_k^e = D_k^\mu \quad ; \quad -\frac{\mu_k^e}{\mu_k^v} = R_k^\mu \Rightarrow \mu_k^v = -\frac{D_k^\mu}{R_k^\mu} \tag{13}$$

where the parameters $a_0^\alpha, b_0^\alpha, D_k^\alpha$ and R_k^α refer to expressions of the moduli k_{hom} and μ_{hom}.

2.3 Approximate rheological model for the homogenized viscoelastic behavior of fractured material

The exact model requires to be able computing the roots R_k of polynomial $B(p)$. This task reveals rather complex when the degree of the polynomial $B(p)$ is larger than four. In order to circumvent this difficulty, Aguiar (2016) proposed an approximate model that extends that proposed by Nguyen et al. (2011) for crack to deal with the situation of fractures . The approach is applicable to isotropic materials under isotropic or deviatoric loading. For sake of simplicity, the approximate model shall be only detailed for the bulk modulus. The results regarding the shear modulus are summarized in Appendix once that the procedure to the shear module is similar. Figure 3 shows the Burger model for the bulk modulus.
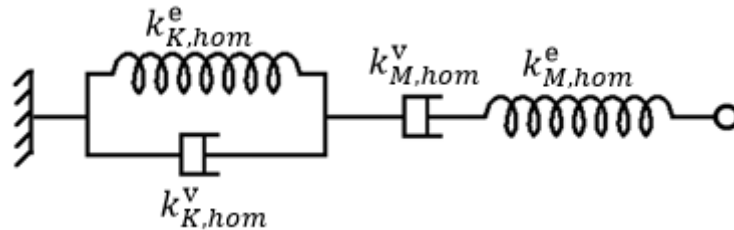

Figure 3: Rheological model of Burger applied to bulk module.

The moduli $k_{M,hom}^e$, $k_{K,hom}^e$, $k_{M,hom}^v$, and $k_{K,hom}^v$ are represented as:

$$\frac{1}{k_{M,hom}^e} = \frac{1+\epsilon Q_M^e}{k_{M,s}^e} \quad ; \quad \frac{1}{k_{K,hom}^e} = \frac{1+\epsilon Q_K^e}{k_{K,s}^e} \quad ; \quad \frac{1}{k_{M,hom}^v} = \frac{1+\epsilon Q_M^v}{k_{M,s}^v} \quad ; \quad \frac{1}{k_{K,hom}^v} = \frac{1+\epsilon Q_K^v}{k_{K,s}^v} \tag{14}$$

where the coefficients Q_M^e, Q_K^e, Q_M^v and Q_K^v are dependent on the rheological models adopted for the matrix and fractures materials. Their values are given in appendix A for the particular situation where a Burger model is considered to represent the viscoelastic behavior in bulk of the matrix, together with a Maxwell model to describe that of fractures. In this case, the Carson-Laplace transform of the homogenized bulk modulus reads:

$$\frac{1}{k_{hom}^*} = \frac{1}{k_{M,hom}^e} + \frac{1}{p\,k_{M,hom}^v} + \frac{1}{k_{K,hom}^e + p\,k_{K,hom}^v} \tag{15}$$

Similarly, the Carson-Laplace transform of the homogenized shear modulus expresses as:

$$\frac{1}{\mu_{hom}^*} = \frac{1}{\mu_{M,hom}^e} + \frac{1}{p\,\mu_{M,hom}^v} + \frac{1}{\mu_{K,hom}^e + p\,\mu_{K,hom}^v} \tag{16}$$

where

$$\frac{1}{\mu_{M,hom}^e} = \frac{1+\epsilon M_M^e}{\mu_{M,s}^e} \quad ; \quad \frac{1}{\mu_{K,hom}^e} = \frac{1+\epsilon M_K^e}{\mu_{K,s}^e} \quad ; \quad \frac{1}{\mu_{M,hom}^v} = \frac{1+\epsilon M_M^v}{\mu_{M,s}^v} \quad ; \quad \frac{1}{\mu_{K,hom}^v} = \frac{1+\epsilon M_K^v}{\mu_{K,s}^v} \tag{17}$$

The expressions of the above coefficients are given in appendix B. It is observed that the approximate Burger model is only formulated herein for the situation where the matrix material is also described by a Burger model.

3   FRACTURE PROPAGATION IN VISCOELASTIC MEDIA

Let us consider now a representative elementary volume of the fractured material (regarded as a homogenized medium) occupying a geometry domain $\Omega(\epsilon)$. We consider in the subsequent analysis that the family of embedded plane fractures with crack (fracture) density parameter ϵ is propagating under the loading defined by evolving macroscopic strain $\underline{\underline{\varepsilon}}$. Propagation of fractures is accounted for at macroscopic scale by considering evolving crack density $\dot{\epsilon} \neq 0$. In the context of elasticity, the dissipation associated with the fracture (damage) propagation takes the form:

$$\frac{\dot{D}}{|\Omega|} = -\frac{\partial W^{el}}{\partial \epsilon}\Big|_{\underline{\underline{\varepsilon}}} \left(\underline{\underline{\varepsilon}}, \epsilon\right) \dot{\epsilon} \tag{18}$$

Thus, the energy release rate $\mathcal{F} = -\partial W^{el}/\partial \epsilon$ appears to be the driving force of the propagation. Therefore, the criterion for propagation takes form of a threshold on $\mathcal{F}$. For elastic materials, the elastic energy W^{el} stored is represented by a quadratic function of $\underline{\underline{\varepsilon}}$:

$$W^{el} = \frac{1}{2} \underline{\underline{\varepsilon}} : \mathbb{C}_{hom}(\epsilon) : \underline{\underline{\varepsilon}} \tag{19}$$

where $\mathbb{C}_{hom}(\epsilon)$ is the elastic stiffness of the material, so the driving force of fracture propagation reads:

$$\mathcal{F} = \frac{1}{2}\underline{\underline{\varepsilon}} : \frac{\partial \mathbb{C}_{hom}(\epsilon)}{\partial \epsilon} : \underline{\underline{\varepsilon}} \tag{20}$$

In the viscoelastic case, the elastic energy stored in the system W is obtained as the sum of the elastic energy stored in each spring of the system. This definition demands on the determination of the viscoelastic model utilized to represent the material. Nguyen (2010) demonstrates that the elastic energy stored in the viscoelastic system can be viewed as a function of the current values of macroscopic deformation $\underline{\underline{\varepsilon}}$, crack density parameter ϵ and of viscous strain field $\left\{\underline{\underline{\varepsilon}}^{v}\right\}$. Hence,

$$\frac{\dot{D}}{|\Omega_0|} = \underline{\underline{\Sigma}} : \dot{\underline{\underline{\varepsilon}}} - \frac{\partial W}{\partial \underline{\underline{\varepsilon}}}\bigg|_{\epsilon,\left\{\underline{\underline{\varepsilon}}^{v}\right\}} : \dot{\underline{\underline{\varepsilon}}} - \frac{\partial W}{\partial \left\{\underline{\underline{\varepsilon}}^{v}\right\}}\bigg|_{\underline{\underline{\varepsilon}},\,\epsilon} : \left\{\dot{\underline{\underline{\varepsilon}}}^{v}\right\} - \frac{\partial W}{\partial \epsilon}\bigg|_{\underline{\underline{\varepsilon}},\left\{\underline{\underline{\varepsilon}}^{v}\right\}} \dot{\epsilon} \tag{21}$$

Considering first a reversible evolution in which $\left\{\dot{\underline{\underline{\varepsilon}}}^{v}\right\} = 0$ and $\dot{\epsilon} = 0$, one obtains:

$$\underline{\underline{\sigma}} = \frac{\partial W}{\partial \underline{\underline{\varepsilon}}}\bigg|_{\epsilon,\left\{\underline{\underline{\varepsilon}}^{v}\right\}} = \mathbb{R}^{0}_{hom}(\epsilon) : \left(\underline{\underline{\varepsilon}} - \underline{\underline{\varepsilon}}^{v}\left(\epsilon,\left\{\underline{\underline{\varepsilon}}^{v}\right\}\right)\right) \tag{22}$$

where $\underline{\underline{\sigma}}$ is the macroscopic stress, $\underline{\underline{\varepsilon}}^{v}$ is the macroscopic viscous deformation and $\mathbb{R}^{0}_{hom}(\epsilon)$ is the instantaneous homogenized relaxation tensor at $t = 0$. In the Eq. (21), the term involving the viscous strain rate $\left\{\dot{\underline{\underline{\varepsilon}}}^{v}\right\}$ represents the viscous dissipation $\dot{D}^{vis}$. Hence, the propagation term can be written as:

$$\frac{\dot{D} - \dot{D}^{vis}}{|\Omega_0|} = -\frac{\partial W}{\partial \epsilon}\bigg|_{\underline{\underline{\varepsilon}},\left\{\underline{\underline{\varepsilon}}^{v}\right\}} \dot{\epsilon} \tag{23}$$

The energy release rate takes so the following form:

$$\mathcal{F} = -\frac{\partial W}{\partial \epsilon}\bigg|_{\underline{\underline{\varepsilon}},\left\{\underline{\underline{\varepsilon}}^{v}\right\}} \left(\underline{\underline{\varepsilon}}, \epsilon, \left\{\underline{\underline{\varepsilon}}^{v}\right\}\right) \tag{24}$$

Again, the energy release rate can be used to formulate a propagation criterion, the difference with respect to elasticity is lying in the dependency on the viscous strain field. To deal with this issue, Nguyen et al. (2010) suggested reasoning similar to the energy analysis performed in microelastoplasticity, where the elastic energy can be put in the form:

$$W = \frac{1}{2}\left(\underline{\underline{\varepsilon}} - \underline{\underline{\varepsilon}}^{v}\right) : \mathbb{R}^{0}_{hom} : \left(\underline{\underline{\varepsilon}} - \underline{\underline{\varepsilon}}^{v}\right) + W_{res}\left(\epsilon,\left\{\underline{\underline{\varepsilon}}^{v}\right\}\right) \tag{25}$$

The first term represents the energy that is recoverable in an instantaneous unloading, while the second term can be interpreted as the residual elastic energy that is stored in the viscous system immediately after this unloading. The Propagation criterion stems from the assumption that the non viscous dissipation is proportional to the propagation rate $\dot{\epsilon}$:

$$\frac{\dot{D} - \dot{D}^{vis}}{|\Omega_0|} = -\mathcal{F}_c \dot{\epsilon} \tag{26}$$

where $\mathcal{F}_c$ is termed the critical energy. Comparing this equation with the Eq. (23) and (24), the propagation criterion is written on $\mathcal{F}$ by:

$$\mathcal{F} < \mathcal{F}_c \Rightarrow \dot{\epsilon} = 0 \quad ; \quad \mathcal{F} = \mathcal{F}_c \Rightarrow \dot{\epsilon} \geq 0 \tag{27}$$

However, $\mathcal{F}_c$ is not a material property because it depends on ϵ. Introducing the critical energy $\mathcal{F}_t$ associated with the propagation of a single fracture, it follows that (see Aguiar, 2016):

$$\mathcal{F}_c(\epsilon) = \frac{2\pi}{3} \mathcal{F}_t \left(\frac{\mathcal{N}}{\epsilon}\right)^{\frac{1}{3}} \tag{28}$$

It is observed that, unlike $\mathcal{F}_c$ which depends on ϵ, the critical energy $\mathcal{F}_t$ is a material property characterizing the energy per unit surface area that is dissipated due to the creation of a new interface. The equality $\mathcal{F}\left(\underline{\underline{\varepsilon}}, \epsilon, \left\{\underline{\underline{\varepsilon}}^v\right\}\right) = \mathcal{F}_c$ allows to derive the fracture propagation induced by the loading history $\underline{\varepsilon}(t)$, Even if it is assumed that $\mathcal{F}_t$ is a material constant, it is emphasized that the viscous strain field explicitly appears in this equation. Therefore, the loading rate is expected to affect the damage propagation.

Two approaches will be presented in the following analysis to deal with the problem of damage propagation in the homogenized viscoelastic medium.

3.1   A first simplified approach to damage propagation

Basically, the approach developed in this section consists in neglecting the residual elastic energy in the expression of the driving force of the propagation. Combining Eq. (24) and (25) yields:

$$\mathcal{F} = -\frac{1}{2}\left(\underline{\underline{\varepsilon}} - \underline{\underline{\varepsilon}}^v\right) : \frac{\partial \mathbb{R}^0_{hom}}{\partial \epsilon} : \left(\underline{\underline{\varepsilon}} - \underline{\underline{\varepsilon}}^v\right) + \frac{1}{2}\left(\underline{\underline{\varepsilon}} - \underline{\underline{\varepsilon}}^v\right) : \mathbb{R}^0_{hom} : \frac{\partial \underline{\underline{\varepsilon}}^v}{\partial \epsilon} - \frac{\partial W_{res}\left(\epsilon, \left\{\underline{\underline{\varepsilon}}^v\right\}\right)}{\partial \epsilon}\Bigg|_{\underline{\varepsilon}, \left\{\underline{\varepsilon}^v\right\}} \tag{29}$$

The main difficulty that raises from the above expression lies in the fact that the driving force $\mathcal{F}$ is strongly coupled to the viscoelastic strain field through W_{res} and $\dfrac{\partial \underline{\underline{\varepsilon}}^v}{\partial \epsilon}$, which are not a priori known. The driving force must generally be determined numerically. However, Nguyen et al. (2010) showed that the derivatives of the residual energy and of the macroscopic viscous strain can be neglected in some situations. Adopting such a simplification, the energy release rate then reduces to:

$$\mathcal{F} = -\frac{1}{2}\left(\underline{\underline{\varepsilon}} - \underline{\underline{\varepsilon}}^v\right) : \frac{\partial \mathbb{R}_{hom}^0}{\partial \epsilon} : \left(\underline{\underline{\varepsilon}} - \underline{\underline{\varepsilon}}^v\right) \tag{30}$$

Computation of $\mathcal{F}$ requires to know the expression of $\underline{\varepsilon}^v$, which is a rather complex task for a general viscoelastic behavior. The idea is to rewrite the above expression in terms of macroscopic stresses:

$$\mathcal{F} = \frac{1}{2}\underline{\underline{\sigma}} : \frac{\partial \mathbb{J}_{hom}^0}{\partial \epsilon} : \underline{\underline{\sigma}} \tag{31}$$

where $\mathbb{J}_{hom}^0$ is the instantaneous creep tensor. It follows from the assumption of isotropy that :

$$\mathbb{J}_{hom}^0 = \frac{1}{3k_{hom}^0}\mathbb{J} + \frac{1}{2\mu_{hom}^0}\mathbb{K} \tag{32}$$

Substituting into equation (31) leads to

$$\mathcal{F} = \frac{1}{2}\frac{\partial(1/k_{hom}^0)}{\partial \epsilon}\sigma_v^{\;2} + \frac{1}{4}\frac{\partial(1/\mu_{hom}^0)}{\partial \epsilon}\sigma_d^{\;2} \tag{33}$$

Where $\sigma_v = \frac{1}{3}\mathrm{tr}\underline{\underline{\sigma}}$ and $\sigma_d = (\underline{\underline{\sigma}} - \sigma_v\underline{\underline{1}}):(\underline{\underline{\sigma}} - \sigma_v\underline{\underline{1}})$. In the case of uniaxial tensile stress $\sigma_{11}\,\underline{e}_1 \otimes \underline{e}_1$, the energy release rate reduces to:

$$\mathcal{F} = \left(\frac{1}{18}\frac{\partial(1/k_{hom}^0)}{\partial \epsilon} + \frac{1}{6}\frac{\partial(1/\mu_{hom}^0)}{\partial \epsilon}\right)\sigma_{11}^{\;2} \tag{34}$$

3.2 An alternative approach to damage propagation

The main idea of this approach consists in preserving the complete expression (29) of the energy release rate, and to adopt the approximate homogenized viscoelastic behavior formulated in section 2.3, to compute the terms related to viscous contribution. As previously mentioned, the elastic energy stored into a viscoelastic system is obtained by the sum of the elastic energy stored in each spring of the system. Considering that the material is homogeneous, this definition writes the following equation to the elastic energy:

$$W = \frac{1}{2}\sum_{k=0}^{n}\underline{\underline{\varepsilon}}_k^e : \mathbb{C}_k^e : \underline{\underline{\varepsilon}}_k^e \tag{35}$$

Where $\underline{\underline{\varepsilon}}_k^e$ is the elastic deformation acting in the kth spring of the system, $\mathbb{C}_k^e$ is the elastic stiffness of the kth spring of the system and n is the total number of sprigs in the system.

The first step consists in identify the rheological model that represents the behavior of the material. Based on the model represented previously, the Burger model was adopted to represent the material in this application. Accordingly, Eq. (35) may be rewritten as:

$$W = \frac{1}{2}\left(\underline{\underline{\epsilon}}_M^e : \mathbb{C}_{M,hom}^e : \underline{\underline{\epsilon}}_M^e + \underline{\underline{\epsilon}}_K^e : \mathbb{C}_{K,hom}^e : \underline{\underline{\epsilon}}_K^e\right) \tag{36}$$

where $\underline{\underline{\epsilon}}_M^e, \underline{\underline{\epsilon}}_K^e, \mathbb{C}_{M,hom}^e$ and $\mathbb{C}_{K,hom}^e$ are presented in the Fig. 4.
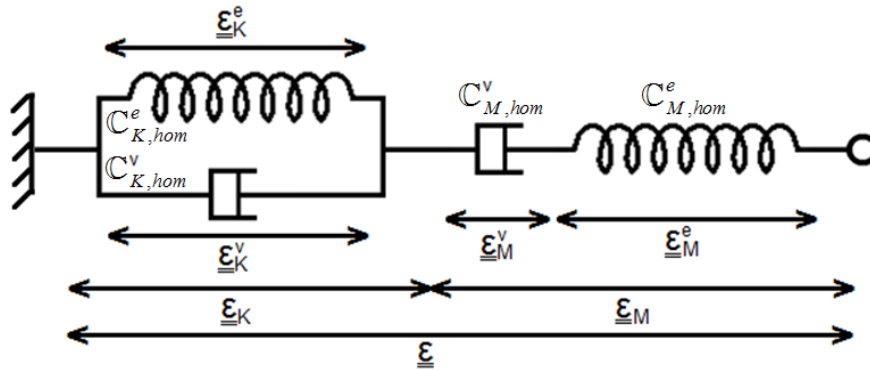

Figure 4: Burger's rheological model

For sake of simplicity, the following procedure will be described in the one-dimensional setting. Thus, the Eq. (36) takes the form:

$$W = \frac{1}{2}\left(\varepsilon_M^{e\,2} C_{M,hom}^e + \varepsilon_K^{e\,2} C_{K,hom}^e\right) \tag{37}$$

Referring to the Burger model, the unique element with instantaneous behavior is $\mathbb{C}_{M,hom}^e$. It is therefore possible to separate an instantaneous behavior from the total elastic energy:

$$W_{inst} = \frac{1}{2}\varepsilon_M^{e\,2} C_{M,hom}^e \tag{38}$$

In this context, the instantaneous relaxation function takes the value of $C_{M,hom}^e$ at $t = 0$. It follows from comparison with Eq. (25) that:

$$W_{res} = \frac{1}{2}\varepsilon_K^{e\,2} C_{K,hom}^e \tag{39}$$

It follows that the instantaneous and the residual energy are clearly defined. However, it is necessary to determine a relation between the total strain ε and the strains ε_M^e and ε_K^e associated with the springs of the Burger model. In the particular case of Burger rheological mode, this relation is derived from the following equations:

$$\varepsilon = \varepsilon_K + \varepsilon_M \quad ; \quad \varepsilon_M = \varepsilon_M^v + \varepsilon_M^e \quad ; \quad \varepsilon_K = \varepsilon_K^v = \varepsilon_K^e$$

$$\sigma = \sigma_K = \sigma_M \quad ; \quad \sigma_M = \sigma_M^v = \sigma_M^e \quad ; \quad \sigma_K = \sigma_K^v + \sigma_K^e \tag{40}$$

$$\sigma = R_{\text{hom}} \otimes \varepsilon = R_{\text{hom}}^0 \varepsilon^e \quad ; \quad \sigma_\alpha^e = C_{\alpha,\text{hom}}^e \varepsilon_\alpha^e \quad ; \quad \sigma_\alpha^v = C_{\alpha,\text{hom}}^v \dot{\varepsilon}_\alpha^v$$

The total strain as well as the strains in the springs are then computed as:

$$\varepsilon_M^e = \frac{R_{\text{hom}}}{C_{M,\text{hom}}^e} \otimes \varepsilon$$

$$\dot{\varepsilon}_K^e = \frac{R_{\text{hom}}}{C_{K,\text{hom}}^v} \otimes \varepsilon - \frac{C_{K,\text{hom}}^e}{C_{K,\text{hom}}^v} \varepsilon_K^e \tag{41}$$

where $\otimes$ denotes the boltzmanian product defined

$$A(\mathrm{t}_0,\mathrm{t}) \otimes B(t) = A(t_0, t = t_0)B(t) + \int_{t_0}^t \frac{dA(t_0, \mathrm{t}-\tau)}{d\tau} \mathrm{E}(\tau) \, \mathrm{d}\tau \tag{42}$$

Although the approach could be developed in the general case of prescribed macroscopic strain history, it will be now illustrated in the particular case $\varepsilon(t) = \varepsilon\, Y(t)$, where ε is a constant and $Y(t)$ is the Heaviside distribution. Eq. (41) reduces therefore to:

$$\varepsilon_M^e = \frac{R_{\text{hom}}}{C_{M,\text{hom}}^e} \varepsilon$$

$$\dot{\varepsilon}_K^e = \frac{R_{\text{hom}}}{C_{K,\text{hom}}^v} \varepsilon - \frac{C_{K,\text{hom}}^e}{C_{K,\text{hom}}^v} \varepsilon_K^e \tag{43}$$

Solving the above ordinary differential equation with respect to ε_K^e, it comes:

$$\varepsilon_M^e(t) = \frac{R_{\text{hom}}(t)}{C_{M,\text{hom}}^e} \varepsilon$$

$$\varepsilon_K^e(t) = \frac{\varepsilon}{C_{K,hom}^v} e^{-\frac{C_{K,hom}^e}{C_{K,hom}^v}t} \int_o^t R_{hom}(\tau) e^{\frac{C_{K,hom}^e}{C_{K,hom}^v}\tau} \, d\tau \tag{44}$$

Once the strains in the springs are calculated from the total strain, it is possible to express the elastic energy as a function of the total strain solely:
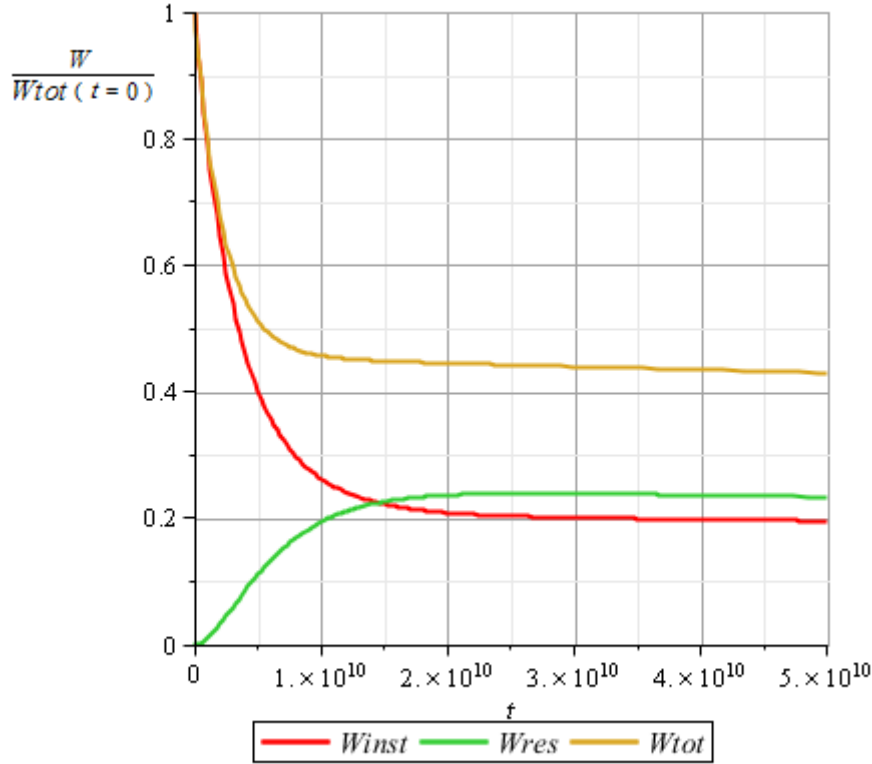
$$W = \frac{1}{2} \frac{R_{\text{hom}}^2(t)}{C_{M,\text{hom}}^e} \varepsilon^2 + \frac{1}{2} \frac{C_{K,hom}^e}{C_{K,hom}^{v\,2}} e^{-2\frac{C_{K,hom}^e}{C_{K,hom}^v}t} \left(\int_o^t R_{hom}(\tau) e^{\frac{C_{K,hom}^e}{C_{K,hom}^v}\tau} \, d\tau \right)^2 \varepsilon^2 \tag{45}$$

where

$$W_{ins} = \frac{1}{2}\frac{R_{\text{hom}}^{2}(t)}{C_{M,\text{hom}}^{e}}\varepsilon^{2}$$

$$W_{res} = \frac{1}{2}\frac{C_{K,\text{hom}}^{e}}{C_{K,\text{hom}}^{v\,2}}e^{-2\frac{C_{K,\text{hom}}^{e}}{C_{K,\text{hom}}^{v}}t}\left(\int_{o}^{t}R_{\text{hom}}(\tau)e^{\frac{C_{K,\text{hom}}^{e}}{C_{K,\text{hom}}^{v}}\tau}\,d\tau\right)^{2}\varepsilon^{2}$$

$$(46)$$

Figure 5 represents the instantaneous and residual parts of the total free energy.

Figure 5: Elastic energy stored in the springs of a Burger model

$(\varepsilon = 10^{-3}, \epsilon = 0,1, k_{M,s}^{e} = 11,4.10^{9}\,Pa, k_{K,s}^{e} = 9,7.10^{9}\,Pa, \mu_{M,s}^{e} = 8,51.10^{9}\,Pa,$

$\mu_{K,s}^{e} = 7,28.10^{9}\,Pa, k_{M,s}^{v} = 9000.10^{18}\,Pa.s, k_{K,s}^{v} = 95.10^{18}\,Pa.s, \mu_{M,s}^{v} = 6750.10^{18}\,Pa.s,$

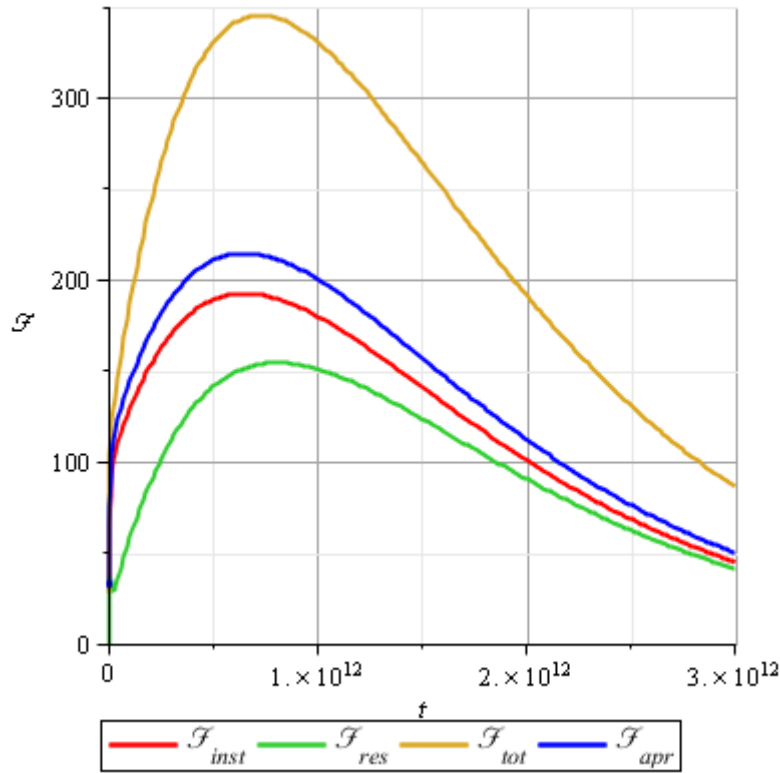$\mu_{K,s}^{v} = 71,3.10^{18}\,Pa.s, k_{M,j}^{e} = 12.10^{12}\,Pa/m. k_{M,j}^{v} = 120.10^{21}\,Pa.s/m, \mu_{M,j}^{e} = 10.10^{12}\,Pa/m,$

$\mu_{M,j}^{v} = 90.10^{21}\,Pa.s/m)$

Equation (29) of the energy release rate $\mathcal{F}$ takes the following form:

$$\mathcal{F} = -\frac{1}{2}\varepsilon^{2}\frac{\partial}{\partial\epsilon}\left(\frac{R_{\text{hom}}^{2}(t)}{C_{M,\text{hom}}^{e}} + \frac{C_{K,\text{hom}}^{e}}{C_{K,\text{hom}}^{v\,2}}e^{-2\frac{C_{K,\text{hom}}^{e}}{C_{K,\text{hom}}^{v}}t}\left(\int_{o}^{t}R_{\text{hom}}(\tau)e^{\frac{C_{K,\text{hom}}^{e}}{C_{K,\text{hom}}^{v}}\tau}\,d\tau\right)^{2}\right)$$

$$(47)$$

It is recalled that this equation was obtained using the approximate Burger model. The above expression of $\mathcal{F}$ is computed analytically. Figure 6 represents the evolution in time of

the energy release rate $\mathcal{F}$ for a fixed model data and a fixed value of crack density parameter ϵ:

Figure 6: Temporal evolution of energy release rate keeping $\epsilon = 0,1$ constant.

$$(\varepsilon = 10^{-3}, \epsilon = 0,1, k_{M,s}^e = 11,4.10^9 \, Pa, k_{K,s}^e = 9,7.10^9 \, Pa, \mu_{M,s}^e = 8,51.10^9 \, Pa,$$

$$\mu_{K,s}^e = 7,28.10^9 \, Pa, k_{M,s}^v = 9000.10^{18} \, Pa.s, k_{K,s}^v = 95.10^{18} \, Pa.s, \mu_{M,s}^v = 6750.10^{18} \, Pa.s,$$

$$\mu_{K,s}^v = 71,3.10^{18} \, Pa.s, k_{M,j}^e = 12.10^{12} \, Pa/m. \, k_{M,j}^v = 120.10^{21} \, Pa.s/m, \mu_{M,j}^e = 10.10^{12} \, Pa/m,$$

$$\mu_{M,j}^v = 90.10^{21} \, Pa.s/m)$$

Figure 6 indicates that the energy release rate evolves in time even when the crack (fracture) density parameter keeps a constant value. If it is assumed that the critical energy $\mathcal{F}_c$ is constant in time, it is therefore possible to define the time when the damage propagation starts from condition $\mathcal{F} = \mathcal{F}_c$. Curves in red, green and yellow refer respectively the instantaneous, residual and elastic (total) contributions to energy release rate corresponding to the alternative model, while the blue curve corresponds to the energy release rate computed in the simplified model disregarding the contribution of residual energy terms.

4 CONCLUSIONS

Taking advantage of the correspondence principle and an Eshelby-based elastic homogenization (Mori-Tanaka) scheme, the exact homogenized viscoelastic behavior of fractured materials was presented. The specific inverse Carson-Laplace transform developed in this paper allows for the analytical derivation of the homogenized relaxation tensor of

fractured medium. It can easily be applied for a large class of rheological models used to describe the individual viscoelastic behavior of matrix material or fracture material. An approximate model that would be more suitable for practical implementation in structural analyses has been formulated in the Burger framework.

Based on energy dissipation concepts, it has been shown that the derivative of the elastic energy with respect to the fracture density parameter represents the thermodynamic force responsible for the damage propagation. Similarly to damage analysis in elasticity, a propagation criterion has been formulated. The main difference with respect to elasticity lies in the fact that the expression of energy release rate depends on the viscous strain field, which are a priori unknown. A main consequence is that the loading rate is expected to affect the damage propagation.

Two distinct approaches have been developed to evaluate the expression of elastic energy considering separately the instantaneous and residual parts. The first approach disregards the viscous contribution to the expression of energy release rate, while the second approach makes use of the approximate Burger model to derive analytically the expressions of derivatives of the residual energy and of the macroscopic viscous strain.

REFERENCES

Aguiar, C. B., 2016. *Abordagem micromecânica da propagação de fraturas em meios elásticos e viscoelásticos*. Master's thesis, Federal University of Rio Grande do Sul/Porto Alegre.

Barton, N., Bandis, S., Bakhtar, K., 1985. Strength, deformation and conductivity coupling of rock joints. *International Journal of Rock Mechanics and Mining Sciences*, vol. 22, pp. 121–140.

Bland, D. R., 1960. *The theory of linear viscoelasticity*. Pergamon Press.

Budiansky, B., O'Connell, R. J., 1976. Elastic moduli of a cracked solid. *International Jounal of Solids Structures*, Pergamon Press, vol. 12, pp. 81–97.

Le, Q. V., Meftah, F., He, Q. –C., Le Pape, Y., 2008. Creep and relaxation functions of a heterogeneous viscoelastic porous medium using the Mori-Tanaka homogenization scheme and a discrete microscopic retardation spectrum. *Mechanics of Time-Dependent Materials*, vol. 11, n. 3, pp. 309–331.

Maghous, S., Bernaud, D., Fréarad, J., Garnier D., 2008. Elastoplastic behavior of jointed rock masses as homogenized media and finite element analysis. *International Journal of Rock Mechanics and Mining Sciences*, v. 45, n. 8, pp. 1273–1286.

Maghous, S., Lorenci, G., Bittencourt E., 2014. Effective poroelastic behavior of a jointed rock. *Mechanics Research Communications*, vol. 509, pp. 54–69

Nguyen, S. T., 2010, *Propagation de fissures et endommagement par microfissures dans un milieu viscoélastique linéaire non vieellissant*. PhD thesis, University of Paris-Est/ Champs-sur-Marne.

Nguyen, S. T., Dormieux, L., Le Pape, Y., Sanhuja, J., 2010, Crack propagation in viscoelastic structures: Theoretical and numerical analyses. *Computational Materials Sciences*, vol. 50, pp. 83–91.

Nguyen, S. T., Dormieux, L., Le Pape, Y., Sanhuja, J., 2011, A burger model for the effective behavior of a microcracked viscoelastic solid. *International Journal of Damage Mechanics*, vol. 20, n. 8, pp. 1116–1129

Nguyen, S. T., Dormieux, L., 2014, Propagation of micro-cracks in viscoelastic materials: Analytical and numerical methods. *International Journal of Damage Mechanics*, vol. 24, n. 4, pp. 562–581

Nguyen, S. T., Jeannin, L., Dormieux, L., Renard, F., 2013, Fracturing of viscoelastic geomaterials and application to sedimentary layered rocks. *Mechanics Research Communications*, vol. 49, pp. 50–56

Salençon, J., 2009. *Viscoélasticité pour le calcul des structures*. Les Éditions de l'École Polytechnique – Les Presses des Ponts et Chaussées, Paris.

APPENDIX A: APPROXIMATE HOMOGENIZED BURGER MODEL: BULK MODULUS

Expression of the relaxation function in shear associated with the approximate model is provided herein. The principle of determination is quite to that followed for the bulk relaxation modulus.

$$
\frac{1}{k^e_{M,Bur}} = \frac{1 + \epsilon\, Q^e_M}{k^e_{M,s}} \quad ; \quad \frac{1}{k^e_{K,Bur}} = \frac{1 + \epsilon\, Q^e_K}{k^e_{K,s}}
$$

$$
\frac{1}{k^v_{M,Bur}} = \frac{1 + \epsilon\, Q^v_M}{k^v_{M,s}} \quad ; \quad \frac{1}{k^v_{K,Bur}} = \frac{1 + \epsilon\, Q^v_K}{k^v_{K,s}}
$$

(A.1)

with

$$
Q^e_M = Q^\infty_0 \quad ; \quad Q^e_K = Q^0_0 + \frac{k^e_{K,s}}{k^v_{M,s}} Q^0_1 - \frac{k^e_{K,s}}{k^e_{M,s}}\left(Q^\infty_0 - Q^0_0\right)
$$

$$
Q^v_M = Q^0_0 \quad ; \quad Q^v_K = Q^\infty_0 + \frac{k^v_{K,s}}{k^e_{M,s}} Q^\infty_{-1} - \frac{k^v_{K,s}}{k^v_{M,s}}\left(Q^0_0 - Q^\infty_0\right)
$$

(A.2)

And

$$
Q^0_0 = \frac{4\pi\, k^v_{M,s}\left(3k^v_{M,s} + 4\mu^v_{M,s}\right)}{3\pi\mu^v_{M,s}\left(3k^v_{M,s} + \mu^v_{M,s}\right) + a\left(3\mu^v_{M,s}k^v_{M,j} + 3k^v_{M,s}\mu^v_{M,j} + 4\mu^v_{M,s}\mu^v_{M,j}\right)}
$$

$$
Q^0_1 = \frac{4\pi\, k^v_{M,s}\left(\dfrac{P_1}{3}\mu^e_{K,s}\mu^e_{M,s}\mu^e_{M,j}\left(3k^v_{M,s} + 4\mu^v_{M,s}\right)P_2 + P_3\right)}{k^e_{M,s}k^e_{K,s}\mu^e_{M,s}\mu^e_{K,s}k^e_{M,j}\mu^e_{M,j}\left(P_1\right)^2}
$$

$$
Q^\infty_0 = \frac{4\pi\, k^e_{M,s}\left(3k^e_{M,s} + 4\mu^e_{M,s}\right)}{3\pi\mu^e_{M,s}\left(3k^e_{M,s} + \mu^e_{M,s}\right) + a\left(3k^e_{M,s}k^e_{M,j} + 12\mu^e_{M,s}k^e_{M,j} + 12k^e_{M,s}\mu^e_{M,j} + 16\mu^e_{M,s}\mu^e_{M,j}\right)}
$$

$$
Q^\infty_{-1} = \frac{4\pi\, k^e_{M,s}\left(\dfrac{1}{3}\mu^v_{K,s}\mu^v_{M,s}\mu^v_{M,j}\left(3k^e_{M,s} + 4\mu^e_{M,s}\right)S_1 S_2 + S_3\right)}{k^v_{M,s}k^v_{K,s}\mu^v_{M,s}\mu^v_{K,s}k^v_{M,j}\mu^v_{M,j}\left(S_1\right)^2}
$$

(A.3)

where

$$
P_1 = \left(3\pi\mu^v_{M,s}\left(3k^v_{M,s} + \mu^v_{M,s}\right) + a\left(3k^v_{M,s} + 4\mu^v_{M,s}\right)\left(3k^v_{M,j} + 4\mu^v_{M,j}\right)\right)
$$

$$
P_2 = \left(3k^e_{K,s}k^v_{M,s}k^e_{M,j} - 3k^e_{K,s}k^e_{M,s}k^v_{M,j} + 4k^e_{K,s}k^e_{M,s}\mu^v_{M,j} + 3k^e_{K,s}k^v_{M,s}k^e_{M,j}\right)
$$

$$
S_1 = \left(3\pi\mu^e_{M,s}\left(3k^e_{M,s} + \mu^e_{M,s}\right) + a\left(3k^e_{M,s} + 4\mu^e_{M,s}\right)\left(3k^e_{M,j} + 4\mu^e_{M,j}\right)\right)
$$

$$
S_2 = \left(3k^v_{K,s}k^e_{M,s}k^v_{M,j} + 3k^v_{M,s}k^e_{M,s}k^v_{M,j} + 4k^v_{K,s}k^e_{M,s}\mu^e_{M,j} - 3k^v_{K,s}k^v_{M,s}k^e_{M,j}\right)
$$

(A.4)

$$P_3 = -3\pi k_{K,s}^e k_{M,j}^v k_{M,s}^e \mu_{K,s}^e \mu_{M,j}^e \mu_{M,s}^e \mu_{M,s}^v \left(3k_{M,s}^v + 4\mu_{M,s}^v\right)\left(3k_{M,s}^v + \mu_{M,s}^v\right)$$

$$+ \frac{4}{3}a k_{K,s}^e k_{M,s}^e \mu_{K,s}^e {\mu_{M,j}^v}^2 \mu_{M,s}^e \left(3k_{M,s}^v + 4\mu_{M,s}^v\right)\left(3k_{M,j}^e + 4\mu_{M,j}^e\right)$$

$$+ 36\pi k_{K,s}^e k_{M,s}^e {k_{M,s}^v}^2 \mu_{K,s}^e \mu_{M,j}^e \mu_{M,j}^v \mu_{M,s}^e \mu_{M,s}^v + 60\pi k_{K,s}^e k_{M,s}^e k_{M,s}^v \mu_{K,s}^e \mu_{M,j}^e \mu_{M,j}^v \mu_{M,s}^e {\mu_{M,s}^v}^2$$

$$+ 16\pi k_{K,s}^e k_{M,s}^e \mu_{K,s}^e \mu_{M,j}^e \mu_{M,j}^v {\mu_{M,s}^v}^3 + 18\pi k_{K,s}^e k_{M,j}^e k_{M,s}^e \mu_{M,j}^e \left(\mu_{K,s}^e + \mu_{M,s}^e\right) k_{M,s}^v {\mu_{M,s}^v}^3$$

$$+ 27\pi k_{M,j}^e \mu_{M,j}^e {k_{M,s}^v}^2 {\mu_{M,s}^v}^2 \left(k_{K,s}^e k_{M,s}^e \mu_{K,s}^e + k_{K,s}^e k_{M,s}^e \mu_{M,s}^e + k_{K,s}^e \mu_{K,s}^e \mu_{M,s}^e + k_{M,s}^e \mu_{K,s}^e \mu_{M,s}^e\right)$$

$$+ 12\pi k_{K,s}^e k_{M,j}^e k_{M,s}^e \mu_{M,j}^e \left(\mu_{K,s}^e + \mu_{M,s}^e\right){\mu_{M,s}^v}^4$$

$$S_3 = -\pi k_{K,s}^v k_{M,s}^e \mu_{K,s}^v \mu_{M,j}^v \mu_{M,s}^v \mu_{M,s}^e \left(3k_{M,s}^e + 4\mu_{M,s}^e\right)\left(3k_{M,s}^e + \mu_{M,s}^e\right)\left(3k_{M,j}^e - 4\mu_{M,j}^e\right)$$

$$+ \frac{4}{3}a k_{K,s}^v k_{M,s}^v \mu_{K,s}^v {\mu_{M,j}^e}^2 \mu_{M,s}^v \left(3k_{M,s}^e + 4\mu_{M,s}^e\right)\left(3k_{M,j}^v + 4\mu_{M,j}^v\right)$$

$$- 6\pi k_{M,j}^v \mu_{M,j}^v {\mu_{M,s}^e}^2 k_{K,s}^v k_{M,s}^v \mu_{M,s}^e \left(\mu_{M,s}^v + \mu_{K,s}^v\right)\left(3k_{M,s}^e + 2\mu_{M,s}^e\right)$$

$$- 27\pi k_{M,j}^v \mu_{M,j}^v {\mu_{M,s}^e}^2 {k_{M,s}^e}^2 \left(\mu_{K,s}^v \mu_{M,s}^v \left(k_{K,s}^v + k_{M,s}^v\right) + k_{M,s}^v k_{K,s}^v \left(\mu_{K,s}^v + \mu_{M,s}^v\right)\right)$$

APPENDIX B: APPROXIMATE HOMOGENIZED BURGER MODEL: SHEAR MODULUS

Expression of the relaxation function in shear associated with the approximate model is provided herein. The principle of determination is quite to that followed for the bulk relaxation modulus.

$$\frac{1}{\mu_{M,hom}^e} = \frac{1 + \epsilon M_M^e}{\mu_{M,s}^e} \quad ; \quad \frac{1}{\mu_{K,hom}^e} = \frac{1 + \epsilon M_K^e}{\mu_{K,s}^e}$$

$$\frac{1}{\mu_{M,hom}^v} = \frac{1 + \epsilon M_M^v}{\mu_{M,s}^v} \quad ; \quad \frac{1}{\mu_{K,hom}^v} = \frac{1 + \epsilon M_K^v}{\mu_{K,s}^v} \tag{B.1}$$

with

$$M_M^e = M_0^\infty \quad ; \quad M_K^e = M_0^0 + \frac{\mu_{K,s}^e}{\mu_{M,s}^v}M_1^0 - \frac{\mu_{K,s}^e}{\mu_{M,s}^e}\left(M_0^\infty - M_0^0\right)$$

$$M_M^v = M_0^0 \quad ; \quad M_K^v = M_0^\infty + \frac{\mu_{K,s}^v}{\mu_{M,s}^e}M_{-1}^\infty - \frac{\mu_{K,s}^v}{\mu_{M,s}^v}\left(M_0^0 - M_0^\infty\right) \tag{B.2}$$

and

$$M_0^0 = \frac{16}{5} \frac{\pi \mu_{M,s}^v \left(3k_{M,s}^v + 4\mu_{M,s}^v\right) P_3}{P_1 P_2} \quad ; \quad M_1^0 = -\frac{16}{5} \frac{\pi \mu_{M,s}^v P_5}{\mu_{K,s}^e \mu_{M,s}^e k_{K,s}^e k_{M,s}^e k_{M,j}^e \mu_{M,j}^e P_4^2 P_2^2}$$

$$M_0^\infty = \frac{16}{5} \frac{\pi \mu_{M,s}^e \left(3k_{M,s}^e + 4\mu_{M,s}^e\right) Q_3}{Q_1 Q_2} \quad ; \quad M_{-1}^\infty = \frac{16}{5} \frac{\pi \mu_{M,s}^e Q_5}{\mu_{K,s}^v \mu_{M,s}^v k_{K,s}^v k_{M,s}^v k_{M,j}^v \mu_{M,j}^v Q_4^2 Q_2^2}$$

(B.3)

where

$$P_5 = U_3 P_2^3 + U_2 P_2^2 + U_1 P_2 + U_0 \quad ; \quad Q_5 = V_3 Q_2^3 + V_2 Q_2^2 + V_1 Q_2 + V_0$$

$$P_1 = 3\pi\mu_{M,s}^v\left(3k_{M,s}^v + \mu_{M,s}^v\right) + r\left(k_{M,j}^v k_{M,s}^v + 12k_{M,j}^v \mu_{M,s}^v + 12\mu_{M,s}^v \mu_{M,j}^v + 16\mu_{M,s}^v \mu_{M,j}^v\right)$$

$$P_2 = 3\pi\mu_{M,s}^v\left(3k_{M,s}^v + 2\mu_{M,s}^v\right) + 4r\mu_{M,j}^v\left(3k_{M,s}^v + 4\mu_{M,s}^v\right)$$

$$P_3 = \pi\mu_{M,s}^v\left(9k_{M,s}^v + 4\mu_{M,s}^v\right) + 2r\left(3k_{M,s}^v + 4\mu_{M,s}^v\right)\left(k_{M,j}^v + 2\mu_{M,j}^v\right)$$

$$P_4 = 3\pi\mu_{M,s}^v\left(3k_{M,s}^v + \mu_{M,s}^v\right) + r\left(3k_{M,s}^v + 4\mu_{M,s}^v\right)\left(k_{M,j}^v + 2\mu_{M,j}^v\right)$$

(B.4)

$$Q_1 = 3\pi\mu_{M,s}^e\left(3k_{M,s}^e + \mu_{M,s}^e\right) + r\left(k_{M,j}^e k_{M,s}^e + 12k_{M,j}^e \mu_{M,s}^e + 12\mu_{M,s}^e \mu_{M,j}^e + 16\mu_{M,s}^e \mu_{M,j}^e\right)$$

$$Q_2 = 3\pi\mu_{M,s}^e\left(3k_{M,s}^e + 2\mu_{M,s}^e\right) + 4r\mu_{M,j}^e\left(3k_{M,s}^e + 4\mu_{M,s}^e\right)$$

$$Q_3 = \pi\mu_{M,s}^e\left(9k_{M,s}^e + 4\mu_{M,s}^e\right) + 2r\left(3k_{M,s}^e + 4\mu_{M,s}^e\right)\left(k_{M,j}^e + 2\mu_{M,j}^e\right)$$

$$Q_4 = 3\pi\mu_{M,s}^e\left(3k_{M,s}^e + \mu_{M,s}^e\right) + r\left(3k_{M,s}^e + 4\mu_{M,s}^e\right)\left(k_{M,j}^e + 2\mu_{M,j}^e\right)$$

Finally, the value of U_i and V_i are:

$$U_3 = -k_{K,s}^e k_{M,s}^e k_{M,j}^e (3k_{M,s}^v + 4\mu_{M,s}^v)(\mu_{M,s}^e \mu_{K,s}^e (k_{M,j}^v + \mu_{M,j}^v) - \mu_{M,s}^v \mu_{M,j}^e (k_{M,s}^e + \mu_{K,s}^e))$$

$$U_2 = -\pi\mu_{M,s}^v(9k_{M,s}^v + 10\mu_{M,s}^v)U_3 - \frac{1}{2}k_{K,s}^e k_{M,j}^v k_{M,s}^e (3k_{M,s}^v + 4\mu_{M,s}^v)^2 \cdot$$

$$\cdot (k_{M,j}^v \mu_{K,s}^e \mu_{M,s}^e (3k_{M,j}^e + 2\mu_{M,j}^e) - 10\mu_{M,s}^v \mu_{M,j}^e k_{M,j}^e (\mu_{K,s}^e + \mu_{M,s}^e))$$

$$+ \pi k_{M,j}^e \mu_{M,s}^v (k_{K,s}^e k_{M,j}^v k_{M,s}^e \mu_{K,s}^e \mu_{M,s}^e (3k_{M,s}^v + 4\mu_{M,s}^v)(9k_{M,s}^v + 5\mu_{M,s}^v)$$

$$+ 21k_{M,s}^v \mu_{M,j}^e \mu_{M,s}^v (k_{K,s}^e k_{M,s}^e \mu_{M,s}^v (\mu_{M,s}^e + \mu_{K,s}^e) - k_{M,s}^v \mu_{K,s}^e \mu_{M,s}^e (k_{K,s}^e + k_{M,s}^e)))$$

$$U_1 = (36\pi^2 k_{M,s}^v (\mu_{M,s}^v)^3 + 30\pi^2 (\mu_{M,s}^v)^4)U_3$$

(B.5)

$$+ 6r^2 k_{K,s}^e k_{M,j}^e (k_{M,j}^v)^2 k_{M,s}^e \mu_{M,s}^v \mu_{M,j}^e (3k_{M,s}^v + 4\mu_{M,s}^v)^3 (\mu_{M,s}^e + \mu_{K,s}^e)$$

$$+ 9r\pi(3k_{M,s}^v + 4\mu_{M,s}^v)k_{M,j}^e k_{M,j}^v \mu_{M,s}^v (k_{K,s}^e k_{M,j}^v k_{M,s}^e \mu_{K,s}^e \mu_{M,s}^e (3k_{M,s}^v + 4\mu_{M,s}^v)(2\mu_{M,s}^v + 3k_{M,s}^v)$$

$$- 4\mu_{M,j}^e \mu_{M,s}^v ((\mu_{M,s}^e + \mu_{K,s}^e)(3k_{K,s}^e k_{M,s}^e k_{M,s}^{v2} + 5k_{K,s}^e k_{M,s}^e k_{M,s}^v \mu_{M,s}^v + 4k_{K,s}^e k_{M,s}^e \mu_{M,s}^{v2})$$

$$+ 2(k_{M,s}^v)^2 \mu_{K,s}^e \mu_{M,s}^e (k_{K,s}^e + k_{M,s}^e))) + 72\pi^2 (\mu_{M,s}^v)^4 (k_{M,s}^v)^2 \mu_{M,j}^e k_{M,j}^e \mu_{K,s}^e \mu_{M,s}^e (k_{K,s}^e + k_{M,s}^e)$$

$$- 3\pi^2 k_{K,s}^e k_{M,j}^e k_{M,s}^e (\mu_{M,s}^v)^2 (81k_{M,j}^v (k_{M,s}^v)^3 \mu_{K,s}^e \mu_{M,s}^e + 234k_{M,j}^v (k_{M,s}^v)^2 \mu_{K,s}^e \mu_{M,s}^e \mu_{M,s}^v$$

$$+ 210k_{M,j}^v k_{M,s}^v \mu_{K,s}^e \mu_{M,s}^e (\mu_{M,s}^v)^2 + 56k_{M,j}^v \mu_{K,s}^e \mu_{M,s}^e (\mu_{M,s}^v)^3 + 24k_{M,s}^v \mu_{K,s}^e \mu_{M,j}^e (\mu_{M,s}^v)^3$$

$$+ 24k_{M,s}^v \mu_{M,j}^e \mu_{M,s}^e (\mu_{M,s}^v)^3)$$

$$U_0 = (-54\pi^3 k_{M,s}^v (\mu_{M,s}^v)^5 - 36\pi^3 (\mu_{M,s}^v)^6) U_3$$

$$-18\pi r^2 k_{M,j}^e (k_{M,j}^v)^2 \mu_{M,j}^e (\mu_{M,s}^v)^2 (3k_{M,s}^v + 4\mu_{M,s}^v)^2 ((\mu_{M,s}^v + \mu_{K,s}^e)(9k_{K,s}^e k_{M,s}^e (k_{M,s}^v)^2$$

$$+12 k_{K,s}^e k_{M,s}^e k_{M,s}^v \mu_{M,s}^v + 8k_{K,s}^e k_{M,s}^e \mu_{M_{,s}^2}^v) + 6(k_{M,s}^v)^2 \mu_{K,s}^e \mu_{M,s}^v (k_{K,s}^e + k_{M,s}^e))$$

$$-\frac{9}{2}\pi^2 (3k_{M,s}^v + 4\mu_{M,s}^v) k_{M,j}^e k_{M,j}^v (\mu_{M,s}^v)^2 (3k_{K,s}^e k_{M,j}^v k_{M,s}^e \mu_{K,s}^e \mu_{M,s}^v (3k_{M,s}^v + 4\mu_{M,s}^v)(2\mu_{M,s}^v + 3k_{M,s}^v)^2$$

$$-8(\mu_{M,s}^v)^2 \mu_{M,j}^e ((\mu_{M,s}^v + \mu_{K,s}^e)(9k_{K,s}^e k_{M,s}^e (k_{M,s}^v)^2 + 12 k_{K,s}^e k_{M,s}^e k_{M,s}^v \mu_{M,s}^v + 8k_{K,s}^e k_{M,s}^e (\mu_{M,s}^v)^2)$$

$$+6(k_{M,s}^v)^2 \mu_{K,s}^e \mu_{M,s}^v (k_{K,s}^e + k_{M,s}^e))) + 9\pi^3 k_{M,j}^e (\mu_{M,s}^v)^4 (-12(\mu_{M,s}^v)^2 \mu_{M,j}^e (k_{M,s}^v)^2 \mu_{K,s}^e \mu_{M,s}^v (k_{K,s}^e + k_{M,s}^e)$$

$$+ k_{K,s}^e k_{M,s}^e (12 k_{K,s}^e k_{M,s}^e k_{M,s}^v (\mu_{M,s}^v)^3 \mu_{M,j}^e (\mu_{M,s}^v + \mu_{K,s}^e)$$

$$+ k_{K,s}^e k_{M,j}^v k_{M,s}^e \mu_{K,s}^e \mu_{M,s}^v (2\mu_{M,s}^v + 3k_{M,s}^v)(3k_{M,s}^v + 4\mu_{M,s}^v)(4\mu_{M,s}^v + 9k_{M,s}^v)))$$

$$V_3 = k_{K,s}^v k_{M,j}^v k_{M,s}^v (3k_{M,s}^e + 4\mu_{M,s}^e)(\mu_{K,s}^v \mu_{M,s}^v (k_{M,j}^e + \mu_{M,j}^e) - \mu_{M,s}^e \mu_{M,j}^v (\mu_{K,s}^v + \mu_{M,s}^v))$$

$$V_2 = \frac{1}{2} r k_{K,s}^v k_{M,j}^e k_{M,s}^v (3k_{M,s}^e + 4\mu_{M,s}^e)^2 (k_{M,j}^v \mu_{K,s}^v \mu_{M,s}^v (2\mu_{M,j}^v + 3k_{M,j}^v)$$

$$-10 k_{M,j}^v \mu_{M,s}^e \mu_{M,j}^v (\mu_{K,s}^v + \mu_{M,s}^v))$$

$$-\pi k_{M,j}^v \mu_{M,s}^e (k_{K,s}^v (3k_{M,s}^e + 4\mu_{M,s}^e) k_{M,s}^v \mu_{K,s}^v \mu_{M,s}^v (18 k_{M,j}^e k_{M,s}^e + 15 k_{M,j}^v \mu_{M,s}^e$$

$$+9 k_{M,s}^e \mu_{M,j}^v + 10 \mu_{M,j}^v \mu_{M,s}^e) - \mu_{M,s}^e \mu_{M,j}^v ((\mu_{K,s}^v + \mu_{M,s}^v)(27 k_{K,s}^v (k_{M,s}^e)^2 k_{M,s}^v$$

$$+45 k_{K,s}^v k_{M,s}^e k_{M,s}^v \mu_{M,s}^e + 40 k_{K,s}^v k_{M,s}^v (\mu_{M,s}^e)^2) + 21 (k_{M,s}^e)^2 \mu_{K,s}^v \mu_{M,s}^v (k_{K,s}^v + k_{M,s}^v)))$$

$$V_1 = -6r^2 (k_{M,j}^e)^2 k_{M,j}^v k_{K,s}^v k_{M,s}^v \mu_{M,s}^e \mu_{M,j}^v (3k_{M,s}^e + 4\mu_{M,s}^e)^3 (\mu_{K,s}^v + \mu_{M,s}^v)$$

$$-9r\pi (3k_{M,s}^e + 4\mu_{M,s}^e) k_{M,j}^v k_{M,j}^v \mu_{M,s}^e (k_{K,s}^v k_{M,j}^e k_{M,s}^v \mu_{K,s}^v \mu_{M,s}^v (3k_{M,s}^e + 4\mu_{M,s}^e)(2\mu_{M,s}^v + 3k_{M,s}^v)$$

$$-4\mu_{M,s}^e \mu_{M,j}^v ((\mu_{K,s}^v + \mu_{M,s}^v)(3k_{K,s}^v (k_{M,s}^e)^2 k_{M,s}^v + 5k_{K,s}^v k_{M,s}^e k_{M,s}^v \mu_{M,s}^e + 4k_{K,s}^v k_{M,s}^v (\mu_{M,s}^e)^2)$$

$$+2(k_{M,s}^e)^2 \mu_{K,s}^v \mu_{M,s}^v (k_{K,s}^v + k_{M,s}^v)))$$

$$+3\pi^2 k_{M,j}^v (\mu_{M,s}^e)^2 (k_{K,s}^v (3k_{M,s}^e + 4\mu_{M,s}^e) k_{M,s}^v \mu_{K,s}^v \mu_{M,s}^v (27 k_{M,j}^e (k_{M,s}^e)^2 + 54 k_{M,j}^v k_{M,s}^e \mu_{M,s}^e$$

$$+24 k_{M,j}^e (\mu_{M,s}^e)^2 + 12 k_{M,s}^e \mu_{M,j}^e \mu_{M,s}^e + 10 \mu_{M,j}^e (\mu_{M,s}^e)^2)$$

$$-2(\mu_{M,s}^e)^2 \mu_{M,j}^v ((\mu_{K,s}^v + \mu_{M,s}^v)(18 k_{K,s}^v (k_{M,s}^e)^2 k_{M,s}^v + 27 k_{K,s}^v k_{M,s}^e k_{M,s}^v \mu_{M,s}^e$$

$$+20 k_{K,s}^v k_{M,s}^v (\mu_{M,s}^e)^2) + 12 (k_{M,s}^e)^2 \mu_{K,s}^v \mu_{M,s}^v (k_{K,s}^v + k_{M,s}^v)))$$

$$V_0 = 18 r^2 p k_{M,j}^e k_{M,j}^v \mu_{M,j}^v \mu_{M_{,s}^2}^e (3k_{M,s}^e + 4\mu_{M,s}^e)^2 ((\mu_{K,s}^v + \mu_{M,s}^v)(9k_{K,s}^v k_{M_{,s}^2}^e k_{M,s}^v + 12 k_{K,s}^v k_{M,s}^e k_{M,s}^v \mu_{M,s}^v$$

$$+8 k_{K,s}^v k_{M,s}^v \mu_{M_{,s}^2}^e) + 6 k_{M_{,s}^2}^e \mu_{K,s}^v \mu_{M,s}^v (k_{K,s}^v + k_{M,s}^v))$$

$$+(9/2) p^2 (3k_{M,s}^e + 4\mu_{M,s}^e) k_{M,j}^e k_{M,j}^v \mu_{M_{,s}^2}^e (3k_{K,s}^v k_{M,j}^e k_{M,s}^v \mu_{K,s}^v \mu_{M,s}^v (3k_{M,s}^e + 4\mu_{M,s}^e)(2\mu_{M,s}^e + 3k_{M,s}^e)^2$$

$$-8\mu_{M_{,s}^2}^e \mu_{M,j}^v ((\mu_{K,s}^v + \mu_{M,s}^v)(9k_{K,s}^v k_{M_{,s}^2}^e k_{M,s}^v + 12 k_{K,s}^v k_{M,s}^e k_{M,s}^v \mu_{M,s}^v + 8 k_{K,s}^v k_{M,s}^v \mu_{M_{,s}^2}^e)$$

$$+6 k_{M_{,s}^2}^e \mu_{K,s}^v \mu_{M,s}^v (k_{K,s}^v + k_{M,s}^v)))$$

$$-9 p^3 k_{M,j}^v \mu_{M_{,s}^4}^e (k_{K,s}^v \mu_{K,s}^v \mu_{M,s}^v k_{M,s}^v (3k_{M,s}^e + 4\mu_{M,s}^e)(2\mu_{M,s}^e + 3k_{M,s}^e)(9k_{M,j}^v k_{M,s}^e + 6k_{M,j}^e \mu_{M,s}^e + 2\mu_{M,j}^e \mu_{M,s}^e)$$

$$-2\mu_{M_{,s}^2}^e \mu_{M,j}^v ((\mu_{K,s}^v + \mu_{M,s}^v)(9k_{K,s}^v k_{M_{,s}^2}^e k_{M,s}^v + 12 k_{K,s}^v k_{M,s}^e k_{M,s}^v \mu_{M,s}^v + 8 k_{K,s}^v k_{M,s}^v \mu_{M_{,s}^2}^e)$$

$$+6 k_{M_{,s}^2}^e \mu_{K,s}^v \mu_{M,s}^v (k_{K,s}^v + k_{M,s}^v)))$$

(B.6)