Cumulative equations for continuous time Chicago hyetograph method

Equações cumulativas sequenciais do hietograma do método de Chicago

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ABSTRACT

The Chicago method is a classical method based on IDF curves for obtaining design hyetographs which present rainfall rates as continuous functions of time, one valid for times before peak and another for after peak. The intensity peak of rainfall is arbitrarily positioned at time zero, the function before peak counting time reversely and the function after peak with time axis normally. The time duration of the hyetograph before peak is determined by multiplying the total duration by a displacement factor with value between zero and one. This technical note presents both equations for before and after peak with the same time axis (time zero at the beginning of hyetograph) to facilitate applications of the Chicago method. The equations presented show cumulated rainfall depths resulting from exact integrals over chronological time of the original Chicago method's rainfall intensity equations. That is, at every instant of time from the beginning of the hyetograph one has the exact value of cumulated rainfall. Thus, by decumulation, the sequential rainfall hyetograph depths are obtained. There is an advantage in using these equations which are easily introduced in spreadsheets.

Keywords: Chicago method; Design hyetograph; IDF.

RESUMO

O método de Chicago é um dos métodos clássicos em hidrologia para definição de hietogramas de projeto com base nas curvas de intensidade-duração-frequência (curvas IDF). O hietograma do método de Chicago apresenta as intensidades por funções contínuas no tempo, uma para antes e outra para depois do pico. O pico é posicionado arbitrariamente em um instante zero e as equações usam tempos contados de forma reversa (antes do pico) ou normal (depois do pico). A duração para antes do pico é determinada por um fator de pico entre zero e um multiplicado pela duração total do hietograma. Para facilitar o emprego do método de Chicago a presente nota técnica apresenta equações com tempo zero na origem do hietograma e valores obtidos de forma cronológica até a sua duração total. As equações aqui apresentadas são de precipitações cumulativas que representam a integral exata no tempo dos volumes das intensidades do Método de Chicago, ou seja, a cada instante de tempo a partir de zero (início da chuva) tem-se o valor exato da lâmina acumulada precipitada. Por desacumulação podem-se obter as lâminas precipitadas sequenciais do hietograma. Há uma vantagem de manuseio dessas equações que as tornam facilmente programáveis em planilhas eletrônicas.

Palavras-chave: Método de Chicago; Hietograma de projeto; IDF.

INTRODUCTION

The well-known "Chicago Method" was proposed by Keifer and Chu (1957) to calculate design rainfalls of urban storm water infrastructures. The method establishes two analytical equations for rainfall intensity over time, one valid before the peak rate and another valid after peak, both deduced from an IDF (intensity-duration-frequency) analytical expression, preserving the same volumes of all rainfall intensities.

The Chicago method is much utilized worldwide, but it should be emphasized that it calculates an artificial rainfall hyetograph as a design hyetograph, which the time distribution derives from the analytical expression of IDF without any relationship with real hyetographs patterns. Even if this method considers a parameter to locate the peak rate in time, estimated from real hyetographs, this conclusion does not change. Despite this, Alfieri, Laio and Claps (2008), after testing five design hyetographs by simulation, concluded that the Chicago method results are consistent and stable in flow calculations with several unit graphs applied to different watershed and climate parameters. They noted a systematic overestimation of flows, which could be corrected by a factor of 0.94.

This technical note does not aim to discuss different approaches to calculate design hyetographs. The focus is on the classical Chicago method, so the objective of this paper is to present the chronological and continuous cumulative equations of rainfall depths, to facilitate the calculation of design hyetographs by simple handheld calculators or electronic spreadsheets without requiring the use of complex software packages. The equations presented here do not need any integration because they provide cumulative rainfall depths directly, in a form not normally described in hydrology text books. While a software manual (SMITH, 2004) contains cumulative rainfall depth equations, as opposed to those that are presented in this technical note, they do not calculate chronological cumulative rainfall depths from the beginning to the end of rainfall duration.

CHICAGO METHOD

The method starts with an analytical and time differentiable IDF equation. Therefore, it is possible to employ the Sherman equation (GARCIA-BARTUAL; SCHNEIDER, 2001) presented below:

$$i_t = \frac{a}{(t+b)^n} \qquad (1)$$

Where i_t is the rainfall intensity, t is the duration, a, b and n being the parameters that represent local conditions and the return period, the latter considered in the parameter "a" by an expression such as this:

$$a = k\,T^m \qquad (2)$$

Where T is the return period in years and k a constant.
When n=1 and b > 0 the IDF equation is normally called the Talbot equation. When b=0 and n > 0 the equation is known as Montana equation.

The Chicago method development starts from the volume equation associated with the intensity of IDF for each t, that is, the product of IDF intensity by its duration t, resulting in:

$$V_t = \frac{a\,t}{(t+b)^n} \qquad (3)$$

The method considers an average rainfall intensity function curve $\bar{i}_t$, different from i_t, which integral over time match the volume Equation 3 above:

$$\int_0^t \bar{i}_t\,dt = V_t = \frac{a\,t}{(t+b)^n} \qquad (4)$$

Doing a derivation dV_t/dt, results in:

$$\bar{i}_t = \frac{a\left[t(1-n)+b\right]}{(t+b)^{1+n}} \qquad (5)$$

This equation has a maximum intensity for t = 0, as for the IDF equation. However, except for t=0, where the intensities are equal, it is clear that:

$$\bar{i}_t < i_t \qquad (6)$$

Figure 1 shows these intensity functions, where a rectangle area indicates the volume V_t.

The innovative idea by Keifer and Chu (1957) was to use a modified Equation 5 to explain hyetograph intensities after the peak, and a similar one, reversed, for intensities before peak. In other words the peak is fixed at t = 0, a modified Equation 5 on a normal t axis is then used for after peak intensities. Another modified Equation 5 with an opposed t axis provides before peak intensities. Thus, to maintain the same volume of Equation 5 both modified equations, representing a unique hyetograph, incorporate a peak factor γ, ranging from 0 to 1, known as storm advancement coefficient. In fact, the γ parameter locates the
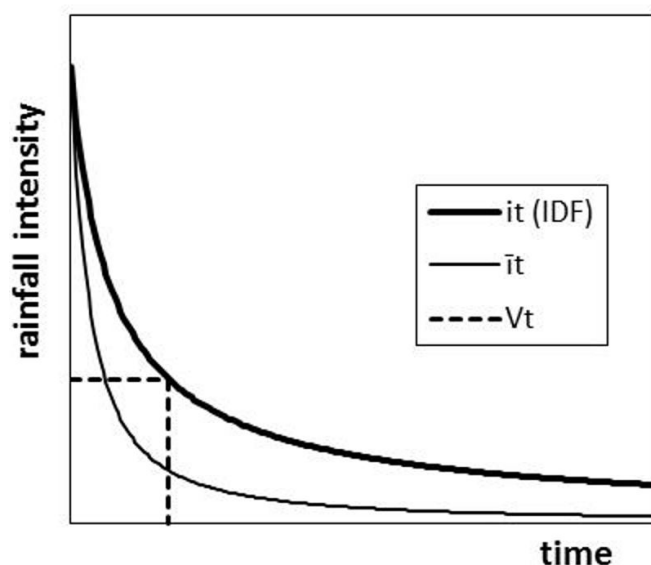

Figure 1. Functions $\bar{i}_t$ (mean intensity) and i_t (IDF).

hyetograph peak intensity over its duration. Before the peak the modified Equation 5 replaces the t variable of the opposite axis by t_1/γ. After the peak the modified Equation 5 replaces the t variable of a normal axis by $t_2/(1-\gamma)$.

As a result of this the well-known rainfall intensity equations of the Chicago method are obtained, given by:

Before peak

$$\overline{i}_{t_1} = \frac{a\left[\dfrac{t_1}{\gamma}(1-n)+b\right]}{\left(\dfrac{t_1}{\gamma}+b\right)^{1+n}} \tag{7}$$

After peak

$$\overline{i}_{t_2} = \frac{a\left[\dfrac{t_2}{1-\gamma}(1-n)+b\right]}{\left(\dfrac{t_2}{1-\gamma}+b\right)^{1+n}} \tag{8}$$

Figure 2 shows both opposite axes (t_1 and t_2) each of them with their $\overline{i}$ function. The example considered the Porto Alegre (Brazil) IDF at INMET (National Meteorology Institute) station (BEMFICA; GOLDENFUM; SILVEIRA, 2000), with a return period of T = 50 years and $\gamma = 0.3$.

CUMULATIVE EQUATIONS

The contribution of this technical note is a sequential integration of Equations 7 and 8 over an interval (0,t), where t = 0 identifies the beginning of whole hyetograph. As a result two cumulative rainfall equations (providing rainfall cumulative depths from the beginning of the hyetograph) are obtained. The first is valid for times before t_p, the time peak. The second is valid for times after the peak.
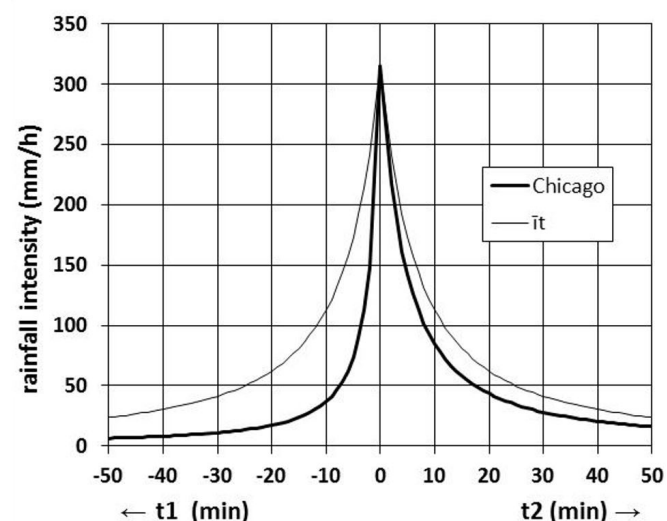Before peak

Figure 2. Mirrored $\overline{i}_t$ functions and those of the Chicago hyetograph with γ=0.3 considering the Porto Alegre IDF with T=50 years.

$$P_t = \gamma P_{TOT} - \frac{a\left(t_p - t\right)}{\left(b + \dfrac{t_p - t}{\gamma}\right)^n} \tag{9}$$

After peak

$$P_t = \gamma P_{TOT} + \frac{a\left(t - t_p\right)}{\left(b + \dfrac{t - t_p}{1-\gamma}\right)^n} \tag{10}$$

Where P_t is the cumulative rainfall at time t from the beginning of the hyetograph.

P_{TOT}, the total rainfall depth of the hyetograph is given by:

$$P_{TOT} = \frac{a\,t_{TOT}}{\left(t_{TOT} + b\right)^n} \tag{11}$$

Where t_{TOT} is the total duration of the hyetograph.

It should be mentioned that Equations 9 and 10 are exact integrals over time of intensities given by the Chicago method, that is, Equations 9 and 10 do not introduce any simplification and are valid for an IDF like Equation 1.

Example 1

The example 5.10 of Bertoni and Tucci (1993) was first taken to demonstrate the performance of Equations 9 and 10.

Without specifying location these authors utilized the following IDF equation:

$$i_t = \frac{1100\,T^{0.15}}{\left(t + 30\right)^{0.75}} \tag{12}$$

Where i_t is the rainfall intensity in mm/h, T is the return period in years and t the time in minutes.

The example of Bertoni and Tucci (1993) presents an application of the Chicago method for a hyetograph with total duration of 90 minutes, time step of 10 minutes, return period of 10 years and $\gamma = 0.35$.

The first step is to calculate the parameter "a". This parameter is simply the IDF numerator divided by 60 to obtain intensities in mm/min, consistent with time t in minutes.

$$a = \frac{1100 \times 10^{0.15}}{60} = 25.90 \tag{13}$$

Parameters "b" and "c" were taken directly from IDF:

$$b = 30\,min \tag{14}$$

$$c = 0.75 \tag{15}$$

With t_{TOT} = 90 min, Equation 11 provides:

$$P_{TOT} = \frac{25.90 \times 90}{\left(90 + 30\right)^{0.75}} = 64.29\,mm \tag{16}$$

Thus,

$$\gamma P_{TOT} = 0.35 \times 64.29 = 22.50 \; mm \qquad (17)$$

The peak time is given by:

$$t_p = \gamma t_{TOT} = 0.35 \times 90 = 31.5 \; min \qquad (18)$$

With these parameter values the resulting cumulative rainfall equations are:

Before peak (t <= 31.5 min)

$$P_t = 22.50 - \frac{25.90\,(31.5 - t)}{\left(30 + \dfrac{31.5 - t}{0.35}\right)^{0.75}} \qquad (19)$$

After peak (t >= 31.5 min)

$$P_t = 22.50 + \frac{25.90\,(t - 31.5)}{\left(30 + \dfrac{t - 31.5}{0.65}\right)^{0.75}} \qquad (20)$$

The results of Equations 19 and 20 are presented in Table 1 (P$_t$), with the corresponding non cumulated values which are the hyetograph rainfall depths. The last column of Table 1 shows the hyetograph values by Bertoni and Tucci (1993).

Except for rounding errors, the results of Equations 19 and 20 are virtually the same shown in Bertoni and Tucci (1993) denoting the correction of the proposed Equations 9 and 10.

Example 2

A second example for the evaluation of Equations 9 and 10 is given by Zahed Filho and Marcellini (1995) example 2.4.

The IDF equation utilized by these authors was provided by Paulo Sampaio Wilken for São Paulo (Brazil):

$$i_t = \frac{3462.7\,T^{0.172}}{(t + 22)^{1.025}} \qquad (21)$$

Where i$_t$ is the rainfall intensity in mm/h, T is the return period in years and t the time in minutes.

The example of Zahed Filho and Marcellini (1995) presents an application of the Chicago method for a hyetograph with a total duration of 90 minutes, time step of 5 minutes, return period of 10 years and γ = 0.39.

Table 1. Results of Example 1.

t (min)	P$_t$ (mm)	P$_{hiet}$ (1) (mm)	P$_{hiet}$ (2) (mm)
10	3.67	3.67	3.66
20	9.16	5.49	5.49
30	19.76	10.60	10.60
40	35.59	15.83	15.83
50	45.16	9.57	9.54
60	51.80	6.64	6.67
70	56.85	5.05	5.04
80	60.90	4.05	4.06
90	64.29	3.39	3.38

(1) Results of this technical note calculated by an electronic spreadsheet;
(2) Results of Bertoni and Tucci (1993).

As indicated previously the first step is to calculate parameter "a".

$$a = \frac{3462.7 \times 10^{0.172}}{60} = 85.76 \qquad (22)$$

Parameters "b" and "c" were taken directly from the IDF:

$$b = 22 \; min \qquad (23)$$

$$c = 1.025 \qquad (24)$$

With t$_{TOT}$ = 90 min, Equation 11 provides:

$$P_{TOT} = \frac{85.76 \times 90}{(90 + 22)^{1.025}} = 61.25 \; mm \qquad (25)$$

Thus,

$$\gamma P_{TOT} = 0.39 \times 61.25 = 23.89 \; mm \qquad (26)$$

The peak time is given by:

$$t_p = \gamma t_{TOT} = 0.39 \times 90 = 35.1 \; min \qquad (27)$$

With these parameter values the resulting cumulative rainfall equations are:

Before peak (t <= 35.1 min)

$$P_t = 23.89 - \frac{85.76\,(35.1 - t)}{\left(22 + \dfrac{35.1 - t}{0.39}\right)^{1.025}} \qquad (28)$$

After peak (t >= 35.1 min)

$$P_t = 23.89 + \frac{85.76\,(t - 35.1)}{\left(22 + \dfrac{t - 35.1}{0.61}\right)^{1.025}} \qquad (29)$$

Equations 28 and 29 were applied with a 5 min step providing cumulated values P$_t$ (Table 2). The corresponding non cumulated values which are the hyetograph rainfall depths were shown in the second column. The last column of Table 2 shows the hyetograph values by Zahed Filho and Marcellini (1995).

It can be observed that results of the proposed Equations 9 and 10 are very similar to those obtained by Zahed Filho and Marcellini (1995). The differences noted are due to the fact that these authors calculated rainfall depths by averaging rainfall intensities of the beginning and end of each time step, while this technical note equation provides exact integrals over the entire time interval curve. The total rainfall calculated by Zahed Filho and Marcellini (1995) has rounding errors because the sum of last column values is 64 mm, slightly different from the expected value (61.2 mm) of IDF with a total duration of 90 minutes.

Example 3

Example 3 is provided by AAS (SMITH, 2004), in which IDF, with a fixed 5 year return period, does not have a specified location:

$$i_t = \frac{1140}{(t+6)^{0.84}} \tag{30}$$

Where i_t is the rainfall intensity in mm/h and t is the time in minutes.

The AAS (SMITH, 2004) hyetograph example has a total duration of 120 minutes with 5 minute time intervals. The peak factor was $\gamma = 0.35$. The return period (5 years) was fixed in the IDF.

Table 2. Results of Example 2.

t (min)	P_t (mm)	P_{hiet} (1) (mm)	P_{hiet} (2) (mm)
5	0.68	0.68	0.69
10	1.59	0.90	0.92
15	2.83	1.24	1.27
20	4.64	1.81	1.85
25	7.47	2.83	2.96
30	12.47	5.01	5.36
35	23.52	11.05	12.53
40	36.73	13.20	13.73
45	44.15	7.42	7.57
50	48.88	4.73	4.79
55	52.14	3.26	3.29
60	54.52	2.38	2.39
65	56.33	1.81	1.81
70	57.74	1.41	1.42
75	58.88	1.13	1.14
80	59.81	0.93	0.93
85	60.58	0.77	0.78
90	61.23	0.65	0.65

(1) Results of this technical note calculated by an electronic spreadsheet; (2) Results of Zahed Filho and Marcellini (1995).

Table 3. Results of Example 3.

t (min)	P_t (mm)	P_{hiet} (1) (mm)	P_{hiet} (2) (mm)
5	0.35	0.35	0.35
10	0.75	0.40	0.40
15	1.21	0.47	0.47
20	1.78	0.57	0.57
25	2.51	0.72	0.72
30	3.51	1.00	1.00
35	5.11	1.61	1.61
40	8.92	3.81	3.81
45	21.57	12.65	12.65
50	39.23	5.38	5.38
55	26.95	2.78	2.78
60	29.73	1.82	1.82
65	31.55	1.34	1.34
70	32.89	1.05	1.05
75	33.94	0.87	0.87
80	34.81	0.74	0.74
85	35.54	0.64	0.64
90	36.18	0.57	0.57
95	36.75	0.51	0.51
100	37.26	0.46	0.46
105	37.72	0.42	0.42
110	38.14	0.39	0.39
115	38.53	0.36	0.36
120	38.89	0.34	0.34

(1) Results of this technical note calculated by an electronic spreadsheet; (2) Results of AAS (SMITH, 2004).

As shown before, parameter "a" is the numerator of the IDF equation divided by 60.

$$a = \frac{1140}{60} = 19.00 \tag{31}$$

Parameters "b" and "c" are issued directly from IDF:

$$b = 6 \ min \tag{32}$$

$$c = 0.84 \tag{33}$$

$t_{TOT} = 120$ min being the total duration of the hyetograph, Equation 11 provides:

$$P_{TOT} = \frac{19.00 \times 120}{(120+6)^{0.84}} = 39.23 \ mm \tag{34}$$

Thus,

$$\gamma P_{TOT} = 0.35 \times 39.23 = 13.73 \ mm \tag{35}$$

The peak time is given by:

$$t_p = \gamma \, t_{TOT} = 0.35 \times 120 = 42.0 \ min \tag{36}$$

Replacing the above parameter values in Equations 9 and 10, results for this case in the cumulative equations for rainfall:

Before peak (t <= 42.0 min)

$$P_t = 13.73 - \frac{19.00(42.0-t)}{\left(6 + \dfrac{42.0-t}{0.35}\right)^{0.84}} \tag{37}$$

After peak (t >= 42.0 min)

$$P_t = 13.73 + \frac{19.00(t-42.0)}{\left(6 + \dfrac{t-42.0}{0.65}\right)^{0.84}} \tag{38}$$

Equations 37 and 38 when applied in 5 minute steps give the cumulative values of Table 3 with their corresponding non cumulated ones (the hyetograph). The results from AAS (SMITH, 2004) are also presented for comparison.

It can be seen in Table 3 that the hyetograph obtained by cumulative equations is identical to those of AAS (SMITH, 2004).

CONCLUSION

Cumulative rainfall equations of the Chicago hyetograph method are presented considering time from zero (beginning of hyetograph) to total duration (ending of hyetograph). Normally, this does not happen, because classical Chicago equations consider peak at time zero and regressive time counting for intensities before peak and progressive counting for after peak. This technical note thus presents a unique timeline from beginning to end of the hyetograph for rainfall calculation. An equation "before peak" gives the cumulated rainfall from beginning to time peak hyetograph. A second one, an "after peak" equation, is applied for times after time peak to ending and gives the cumulated rainfall from the beginning of the hyetograph. The absolute rainfall blocks are then obtained by subtracting the cumulated values between two time steps.

The equations provided by this technical note are advantageous for use in simple electronic spreadsheets, besides the fact they are exact integrals of original intensity curves by Keifer and Chu (1957). Three different examples are presented to demonstrate the proposed cumulative equations. The method is valid for any IDF with mathematical expression of Sherman, Talbot or Montana (GARCIA-BARTUAL; SCHNEIDER, 2001).

This technical note is not aimed at also validating the Chicago method. For novice users it must be highlighted that the Chicago method was originally developed to calculate stormwater hyetographs with any duration for design of urban drainage infrastructures but not limited to these conditions (BERTONI; TUCCI, 1993). The Chicago method considers the same return period for the entire hyetograph, and this can cause overestimation. Alfieri, Laio and Claps (2008) demonstrated this overestimation, by experiments with the simulation of convolution of Chicago hyetographs by general unitgraphs, and concluded that overestimations are small and can be corrected by using a multiplying factor of 0.94.

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