Universidade Federal do Rio Grande do Sul
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Programa de Pós Graduação em Matemática

Hypersurfaces of paralellisable Riemannian manifolds

Dissertação de mestrado

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Porto Alegre, 27 de março de 2017

Dissertação submetida por Eduardo Rosinato Longa[1], como requisito parcial para a obtenção do grau de Mestre em Matemática pelo Programa de Pós Graduação em Matemática do Instituto de Matemática da Universidade Federal do Rio Grande do Sul.

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Agradecimentos

Primeiramente, agradeço aos meus pais, sem os quais nada disso teria sido possível. Obrigado por sempre estar ao meu lado, tanto nos momentos difíceis quanto nos momentos de alegria.

Agradeço também ao Jaime pelos encontros produtivos e também pelos improdutivos. "Fazer pesquisa é assim", dizia ele. Aprendi a entender e enxergar os problemas de uma maneira mais ampla, sem se concentrar tanto nos detalhes, de início.

Também devo aos professores Álvaro, Miriam e Patrícia, por sempre estarem prontos para me acudir e por terem se disposto a assistir a alguns seminários.

Agradeço profundamente a todos os professores que passaram pela minha vida. Boa parte de vocês fez com que eu tivesse curiosidade e amor pelo conhecimento. Espero um dia inspirar outros assim como vocês me inspiraram.

Agradeço ainda ao Ícaro Gonçalves, com o qual discuti questões interessantes sobre folheações, culminando em um artigo cujos resultados aqui apresento.

Um muito obrigado a todos os meus amigos, dentre os quais não posso deixar de citar o Bohrer, a Luísa, a Rafaela e o Horta. A este último, também agradeço por me ajudar a revisar este trabalho.

Enfim, a todos que de alguma forma fizeram parte desta jornada.

Resumo

Introduzimos uma aplicação de Gauss para hipersuperfícies de variedades Riemannianas paralelizáveis e definimos uma curvatura associada. Após, provamos um teorema de Gauss-Bonnet. Como exemplo, estudamos cuidadosamente o caso no qual o espaço ambiente é uma esfera Euclidiana menos um ponto e obtemos um teorema de rigidez topológica. Ele é utilizado para dar uma prova alternativa para um teorema de Qiaoling Wang and Changyu Xia, o qual afirma que se uma hipersuperfície orientável imersa na esfera está contida em um hemisfério aberto e tem curvatura de Gauss-Kronecker não-nula então ela é difeomorfa a uma esfera. Depois, obtemos alguns invariantes topológicos para hipersuperfícies de variedades translacionais que dependem da geometria da variedade e do espaço ambiente. Finalmente, encontramos obstruções para a existência de certas folheações de codimensão um.

Abstract

We introduce a Gauss map for hypersurfaces of paralellisable Riemannian manifolds and define an associated curvature. Next, we prove a Gauss-Bonnet theorem. As an example, we carefully study the case where the ambient space is an Euclidean sphere minus a point and obtain a topological rigidity theorem. We use it to provide an alternative proof for a theorem of Qiaoling Wang and Changyu Xia, which asserts that if an orientable immersed hypersurface of the sphere is contained in an open hemisphere and has nowhere zero Gauss-Kronecker curvature, then it is diffeomorphic to a sphere. Later, we obtain some topological invariants for hypersurfaces of translational manifolds that depend on the geometry of the manifold and the ambient space. Finally, we find obstructions to the existence of certain codimension-one foliations.

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Introduction

In differential geometry, the study of submanifolds has always been a central topic. A particularly interesting question is to describe how a manifold curves inside another one. To tackle this and other issues, numerous definitions of curvatures were introduced throughout history. Various branches of geometry have then stemmed, including minimal and constant mean curvature surfaces, their higher dimensional analogues, curvature flows and numerous problems involving geodesics, to cite a few.

Two primary aspects that have to be considered in any of these areas are the ambient space — the manifold where the submanifold lies in — and the relationship between the dimension of the ambient space and that of the submanifold. Usual ambient spaces that appear in the literature are space forms, those of constant curvature, bounded curvature and simply connected ones. As for dimensions, there has been extensive work dealing with hypersurfaces — submanifolds of codimension one — and surfaces.

An important tool which has been used to study hypersurfaces M^n of the Euclidean space $\mathbb{R}^{n+1}$ is the Gauss map. Assuming M is orientable, there exists a smooth choice of a unit normal vector $\eta(p)$ for every point p of M. This yields a map η from M to the unit sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ if we consider the tangent spaces of M as being linear subspaces of $\mathbb{R}^{n+1}$. The study of its properties dates back to the 18th century, with the works of (not surprisingly) Gauss and Euler. The variation of the normal vector along different directions gives a clue on how the manifold bends in that direction. For example, if M is a right cylinder on $\mathbb{R}^3$, then the Gauss map is constant along any fixed generating line, indicating that it does not curve in these directions.

There have been several constructions of this map in other contexts, according to the geometry involved. For example, in [9] and [5], Epstein and Bryant define a Gauss map G for hypersurfaces of the hyperbolic space $\mathbb{H}^{n+1}$.

Their construction is as follows. Consider $\mathbb{H}^{n+1}$ as the ball $\mathbb{B}^{n+1}$ with the hyperbolic metric. Given an orientable hypersurface M^n with a unit normal vector field η and a point $p \in M$, let $G(p)$ be the intersection of the geodesic of $\mathbb{H}^n$ issuing from p in the direction of $\eta(p)$ with the sphere $\mathbb{S}^n = \partial \mathbb{B}^{n+1}$. This yields a map $G : M \to \mathbb{S}^n$.

In [22], Ripoll extended the definition for an orientable hypersurface M^n of a Lie group G^{n+1} with a left-invariant metric. If η is a normal vector field for M, then $\gamma(p)$, the Gauss map at p, is the translation of $\eta(p)$ to the Lie algebra $\mathfrak{g} \cong T_e G$ via the derivative $DL_{p^{-1}}(p)$ of the left translation $L_{p^{-1}}$. Thus, it is a map $\gamma : M^n \to \mathbb{S}^n$, where now $\mathbb{S}^n$ is the unit sphere of $\mathfrak{g}$.

In this setting, Meeks et al. [18] classified, using the stereographic projection of this Gauss map, the immersed constant mean curvature spheres in a compact, simply connected homogeneous three-manifold. For each $H \in \mathbb{R}$ there exists an immersed oriented sphere of cmc H and it is unique up to ambient isometry.

Later, in [2], Bittencourt and Ripoll defined a Gauss map for an orientable hypersurface M^n of a homogeneous Riemannian manifold $(G/H)^{n+1}$ with an invariant metric. Here G^{n+k+1} is a Lie group with a bi-invariant metric and H^k is a closed Lie subgroup of G. Let $\pi : G \to G/H$ be the natural projection. If $\eta : M \to T(G/H)$ is a normal vector field for M, then the Gauss map at a point $p \in M$ first lifts $\eta(p)$ to the orthogonal complement of $T_x(\pi^{-1}(p))$ in $T_x G$, where $\pi(x) = p$, and then right translates this vector to the Lie algebra $\mathfrak{g}$ of G via the derivative $DR_{x^{-1}}(x)$. This composition gives a map from M to the unit sphere $\mathbb{S}^{n+k}$ of $\mathfrak{g}$. Subsequently, Ramos and Ripoll in [21] performed the same construction for a Lie group G with a bi-invariant pseudo Riemannian metric instead. For other applications, see [8], [10].

In this work we introduce a Gauss map for hypersurfaces lying in parallelisable Riemannian manifolds — manifolds whose tangent bundle is trivial. The detailed definitions and properties comprise the first section of Chapter 2. In the second section of this chapter we thoroughly investigate the case in which the parallelisable manifold is the sphere with a point deleted and, as a consequence, prove a topological rigidity theorem. This constitutes a joint work with Jaime Ripoll, see [17].

In Chapter 3 we change our focus to foliations. The concept and some examples are introduced in the first section. In the sequel, we define some topological invariants for an immersed hypersurface of a parallelisable Riemannian manifold. Finally, we prove some results concerning the existence of totally geodesic foliations using the material developed in Chapter 2. This was a joint work with Ícaro Gonçalves, see [12].

Chapter 1 was included to familiarise the reader with some basic notation and vocabulary and to provide some tools that are used to prove Theorem

2.18 and Theorem 3.15. Strictly speaking, just a couple of pages could have been written about this, but all the beauty that permeates the subject would have been lost.

Finally, a note to the reader: all manifolds, maps, vector fields, differential forms, etc. in this work are smooth (of class C^∞) unless otherwise stated.

Transversality and Applications

Given a map $f : M \to N$ between manifolds and a point $q \in N$, what can we say about the solution set $\{p \in M : f(p) = q\}$? Is it a manifold? More generally, given a submanifold $Z \subseteq N$, what type of condition can we impose on f for $f^{-1}(Z)$ to be a submanifold of M? As we will see shortly, the key concept is transversality, a term that gained prominence in the 1950s, with the brilliant work of René Thom (see [29]). In the sequel, we present some applications of this concept, including an alternative definition for the Euler characteristic of a compact manifold.

For background material concerning the subject of this chapter, the main reference is the masterpiece "Differential Topology", by V. Guillermin and A. Pollack [13].

1.1 Basic notions and facts

Let us first recall some basic facts and definitions that will be used throughout in this chapter.

A fundamental result is the Inverse Function theorem, which we cite here for completeness. The reader can check [27].

Theorem (Inverse Function Theorem). *Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable in an open set containing a, and $\det Df(a) \neq 0$. Then there is an open set V containing a and an open set W containing $f(a)$ such that $f : V \to W$ has a continuous inverse $f^{-1} : W \to V$ which is differentiable and for all $y \in W$ satisfies*

$$D(f^{-1})(y) = [Df(f^{-1}(y))]^{-1}.$$

We use the following terms throughout this work.

Definition 1.1. *Let $f : M^m \to N^n$ be a map between manifolds and $p \in M$. If*

(i) *$Df(p)$ is injective (hence $m \leq n$), then f is is said to be an immersion at p;*

(ii) *$Df(p)$ is surjective (hence $m \geq n$), then f is said to be a submersion at p;*

(iii) *$Df(p)$ is bijective (hence $m = n$), then f is a local diffeomorphism around p (due to the Inverse Function Theorem).*

If (i), (ii) or (iii) holds for every point $p \in M$, then f is called an immersion, a submersion or a local diffeomorphism, accordingly. If f is an immersion, we say M is immersed in N. If, furthermore, f is injective and a homeomorphism between M and $f(M)$, with the topology induced by N, then we say M is embedded in N. The codimension of M in N is the number $\mathrm{codim}\, M = n - m$. If the codimension is 1, M is called a (immersed) hypersurface of N.

Example 1.2. Let $f : \mathbb{R}^n \to \mathbb{R}^{n+k}$ be the usual inclusion:

$$f(x_1, \ldots, x_n) = (x_1, \ldots, x_n, 0, \ldots, 0).$$

Clearly f is an immersion, since $Df(x) = f$ for every $x \in \mathbb{R}^{n+k}$ and f itself is injective.

Example 1.3. Let $f : \mathbb{R}^{n+k} \to \mathbb{R}^n$ be the projection onto the first coordinates:

$$f(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+k}) = (x_1, \ldots, x_n, 0, \ldots, 0).$$

One has that f is a submersion, since $Df(x) = f$ for every $x \in \mathbb{R}^{n+k}$ and f is surjective.

As the two next theorems show, all immersions and submersions look like the same in a certain coordinate system. The proofs for these results can be found in either [13] or [27].

Theorem (Local Immersion Theorem). *Let $f : M^m \to N^n$ be a map between manifolds and suppose f is an immersion at $p \in M$. Then there exist parametrisations $\varphi : U \to M$ and $\psi : V \to N$ around p and $f(p)$ such that $\varphi(0) = p$, $\psi(0) = f(p)$ and*

$$(\psi^{-1} \circ f \circ \varphi)(x_1, \dots, x_m) = (x_1, \dots, x_m, 0, \dots, 0), \quad (x_1, \dots, x_n) \in U.$$

In other words, f is locally given by the inclusion of Example 1.2.

Theorem (Local Submersion Theorem). *Let $f : M^m \to N^n$ be a map between manifolds and suppose f is a submersion at $p \in M$. Then there exist parametrisations $\varphi : U \to M$ and $\psi : V \to N$ around p and $f(p)$ such that $\varphi(0) = p$, $\psi(0) = f(p)$ and*

$$(\psi^{-1} \circ f \circ \varphi)(x_1, \dots, x_m) = (x_1, \dots, x_n), \quad (x_1, \dots, x_m) \in U.$$

Said differently, f is locally given by the inclusion of Example 1.3.

Before answering the first question of this chapter, let's introduce the following nomenclature.

Definition 1.4. *Let $f : M \to N$ be a map between manifolds. We say that $p \in M$ is a regular point for f if $Df(p)$ is surjective. Otherwise, p is called a critical point or singular point. A point $q \in N$ is called a regular value for f if every point in the inverse image $f^{-1}(q)$ is a regular point for f. This includes the case where q does not lie in the image of f. A point of N that is not a regular value is called a critical value.*

Example 1.5. Let $\{r_1, r_2, \dots\}$ be an enumeration of the rational numbers. For each i, let $f_i : \mathbb{R} \to \mathbb{R}$ be a bump function supported in the interval $[i, i+1]$ and with maximum value equal to r_i (Figure 1.1). Then every rational number is a critical value for the map $f = \sum_i f_i$.
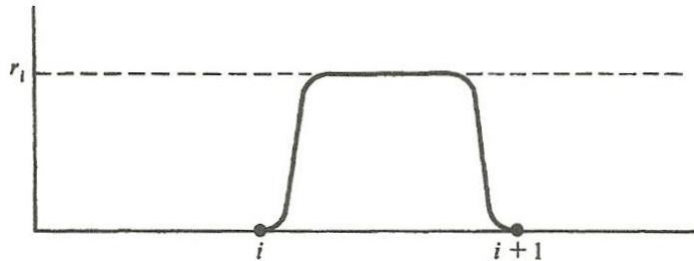

Figure 1.1: The function f_i

One might be worried that some exotic functions may have too many critical values, but a famous theorem, proved by A. Sard in 1942 shows that the opposite occurs, in some sense.

Theorem (Sard, [24]). *Let $f : M \to N$ be a map. Then, the set of critical values has measure zero in N. In other words, almost every point of N is a regular value for f.*

Some clarification is needed here. A set $X \subset \mathbb{R}^n$ has measure zero if for any $\varepsilon > 0$ it can be covered by a sequence of cubes in $\mathbb{R}^n$ having total n-dimensional volume less than ε. A set $S \subset N^n$ has measure zero if for every parametrisation $\varphi : U \subseteq \mathbb{R}^n \to N$ of N, the set $\varphi^{-1}(S)$ has measure zero in $\mathbb{R}^n$.

It is easy to see that no open set has zero measure. As a consequence, we have the following useful corollary.

Corollary 1.6. *Let $f : M \to N$ be a map. Then, the set of regular values for f is dense in N.*

We now face the initial question of the chapter.

Theorem 1.7. *Let $f : M^m \to N^n$ be a map and let $q \in N$ be a regular value for f. Then $f^{-1}(q)$ is a submanifold of M of codimension n.*

Proof. Let $p \in f^{-1}(q)$. By the Local Submersion Theorem, there exist parametrisations $\varphi : U \to M$ and $\psi : V \to N$ around p and q such that $\varphi(0) = p$, $\psi(0) = q$ and

$$(\psi^{-1} \circ f \circ \varphi)(x_1, \ldots, x_n, x_{n+1}, \ldots, x_m) = (x_1, \ldots, x_n), \quad (x_1, \ldots, x_m) \in U.$$

Let $\tilde{U}$ be the set of points $(x_1, \ldots, x_{m-n})$ such that $(0, \ldots, 0, x_1, \ldots x_{m-n})$ lie in U. Define $\xi : \tilde{U} \to M$ by the rule

$$\xi(x_1, \ldots x_{m-n}) = \varphi(0, \ldots, 0, x_1, \ldots, x_{m-n}).$$

Then $\xi(\tilde{U}) = U \cap f^{-1}(q)$ and ξ is a parametrisation of $f^{-1}(q)$ around p. Hence, $f^{-1}(q)$ is a submanifold of M of dimension equal to $m - n$. $\qquad\square$

Example 1.8. Let V be a finite dimensional real vector space equipped with an inner product $\langle \cdot, \cdot \rangle$. Consider the standard differentiable structure on V. Explicitly, let $T : \mathbb{R}^n \to V$ be an isometry and endow V with the maximal differentiable atlas that contains T as an element. Then, a map $f : V \to \mathbb{R}$

is differentiable if and only if $f \circ T : \mathbb{R}^n \to \mathbb{R}$ is differentiable. In particular, every linear map is differentiable.

Concretely, let $f : V \to \mathbb{R}$ be defined by $f(x) = \langle x, x \rangle$. Then, for $x \in V$ and $v \in T_x V \cong V$, we have

$$Df(x) \cdot v = \frac{d}{dt} f(x + tv)|_{t=0} = \frac{d}{dt} \left[\langle x, x \rangle + 2t \langle x, v \rangle + t^2 \langle v, v \rangle \right] |_{t=0} = 2 \langle x, v \rangle.$$

Thus, every positive number is a regular value for f, and so each sphere $\mathbb{S}_r(0) = \{ x \in V : \langle x, x \rangle = r^2 \}$ of radius r is a hypersurface of V, by the previous theorem.

1.2   Transversality

We now tackle the question of whether $f^{-1}(Z)$ is a submanifold of M for a given map $f : M^m \to N^n$ and a submanifold Z^k. This is a local matter, that is, $f^{-1}(Z)$ is a manifold if and only if every point $p \in f^{-1}(Z)$ has a neighbourhood U in M such that $f^{-1}(Z) \cap U$ is a submanifold of U. This allows us to reduce the study of the relation $f(p) \in Z$ to the simpler case in which Z is a single point, as we next explain. If $q = f(p)$, we may write Z in a neighbourhood V of q by the zero set of a submersion $g : V \to \mathbb{R}^{n-k}$, by the Local Submersion Theorem. Then, $f^{-1}(Z)$ is given by the zero set of $g \circ f : f^{-1}(V) \to \mathbb{R}^{n-k}$. Now we may apply Theorem 1.7 to guarantee $(g \circ f)^{-1}(0)$ is a manifold around p. For that we need 0 to be a regular value for $g \circ f$.

Since $D(g \circ f)(p) = Dg(q) \circ Df(p)$, the map $D(g \circ f)(p)$ is surjective if and only if $Dg(q)$ maps the image of $Df(p)$ onto $\mathbb{R}^{n-k}$. But $Dg(q) : T_q N \to \mathbb{R}^{n-k}$ is a surjective linear map whose kernel is the subspace $T_q Z$. Thus, $Dg(q)$ carries a subspace of $T_q N$ onto $\mathbb{R}^{n-k}$ precisely if that subspace and $T_q Z$ together span $T_q N$. We conclude that $g \circ f$ is a submersion at p if and only if

$$Df(p)(T_p M) + T_q Z = T_q N.$$

This is the condition we were looking for.

Definition 1.9. *A map $f : M \to N$ is transversal to a submanifold Z of N if*

$$Df(p)(T_p M) + T_{f(p)} Z = T_{f(p)} N$$

for every point $p \in f^{-1}(Z)$. This is symbolised by $f \pitchfork Z$.

The previous argument then proves:

Theorem 1.10. *If the map $f : M \to N$ is transversal to a submanifold Z of N, then the preimage $f^{-1}(Z)$ is a submanifold of M. Moreover, the codimension of $f^{-1}(Z)$ in M equals the codimension of Z in N.*

Remark 1.11. If $f^{-1}(Z) = \emptyset$, then f is automatically transversal to Z, by vacuity.

Note that the case Z is a single point q of N, a map $f : M \to N$ is transversal to Z exactly when q is a regular value for f. This way, Theorem 1.10 generalises Theorem 1.7.

The most important and readily visualised special case concerns the transversality of the inclusion map $i : M \to N$ of some submanifold $M \subset N$ with another submanifold $Z \subset N$ (Figure 1.2).

Figure 1.2: Surfaces in $\mathbb{R}^3$
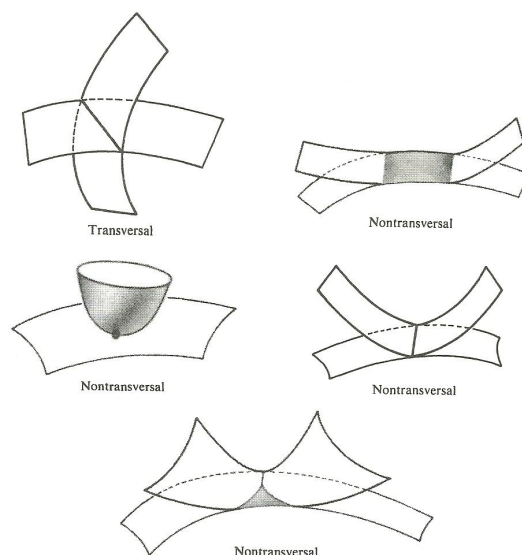
To say $p \in M$ belongs to the preimage $i^{-1}(Z)$ simply means that p belongs to the intersection $M \cap Z$. So, i is transversal to Z if and only if

$$T_pM + T_pZ = T_pN$$

for every $p \in M \cap Z$. Notice that this equation is symmetric in M and Z. When it holds, we say M and Z are transversal, and write $M \pitchfork Z$. We have the following corollary.

Corollary 1.12. *The intersection of two transversal submanifolds M and Z of N is again a submanifold. Moreover,*

$$\mathrm{codim}(M \cap Z) = \mathrm{codim}\, M + \mathrm{codim}\, Z.$$

Consider now the situation where the manifolds are allowed to have boundary. We would like a result similar to Theorem 1.10 for these manifolds. Unfortunately, the transversality of f alone does not guarantee that $f^{-1}(Z)$ is a submanifold with boundary of M if M is a manifold with boundary. For example, let $f : \overline{\mathbb{H}}^2 = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \geq 0\} \to \mathbb{R}$ be the map $(x_1, x_2) \mapsto x_2$ and let $Z = \{0\}$. Then f is transversal to Z but $f^{-1}(Z) = \partial\mathbb{H}^2$ is not a submanifold of $\mathbb{H}^2$. The right condition is an additional assumption along the boundary.

Given a map f from a manifold with boundary M onto a boundaryless manifold N, let $\partial f : \partial M \to N$ denote the restriction of f to the boundary of M. The proof of next theorem is similar to that of Theorem 1.10.

Theorem 1.13. *Let f be as above and suppose that both f and ∂f are transversal to the boundaryless submanifold Z of N. Then the preimage $f^{-1}(Z)$ is a submanifold with boundary*

$$\partial \left[f^{-1}(Z) \right] = f^{-1}(Z) \cap \partial M,$$

and the codimension of $f^{-1}(Z)$ in M equals the codimension of Z in N.

Still in the spirit of generalising classical theorems for this broader class of manifolds, we have

Theorem 1.14 (Sard, bis)**.** *Let f be a map from a manifold with boundary M onto a boundaryless manifold N. Then almost every point of N is a regular value for both f and ∂f.*

Proof. If a point $p \in \partial M$ is regular for ∂f, then it is regular for f, since the derivative of ∂f at p is just the restriction of the derivative of f at p to $T_p(\partial M)$. Thus a point $q \in N$ is a critical value for f and ∂f only when q is a critical value for $f|_{\mathrm{int}\, M}$ or ∂f. But since $\mathrm{int}\, M$ and ∂M are boundaryless manifolds, both sets of critical values have measure zero, by the previous Sard theorem. Thus the complement of the set of common regular values for f and ∂f, being the union of sets of measure zero, itself has measure zero, as required. $\qquad\square$

We shall now see that transversality is a generic property. This means that any map $f : M \to N$, no matter how bizarre its behaviour with respect to a given submanifold Z of N is, may be deformed by an arbitrary small amount into a map that is transversal to Z (Figure 1.3).
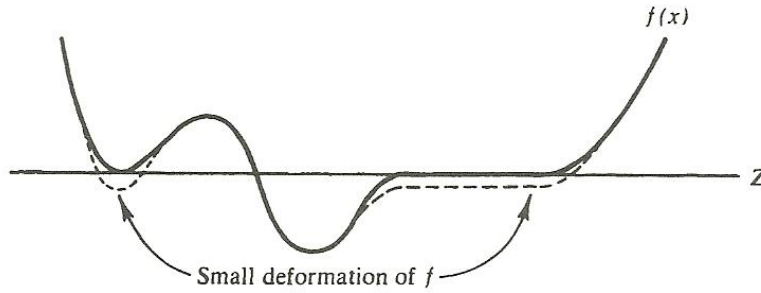

Figure 1.3: Transversality is a generic property

In order to prove this fact, we need the following theorem, which has importance on its own right. It deals with families of mappings. Given a family of maps $f_s : M \to N$, indexed by a parameter s that varies over a manifold S, consider the map $F : M \times S \to N$ defined by $F(p, s) = f_s(p)$. We require that the family vary smoothly by assuming F is smooth. We then have:

Theorem 1.15 (The Transversality Theorem). *Suppose that $F : M \times S \to N$ is a smooth map of manifolds, where only M has boundary, and let Z be a boundaryless submanifold of N. If both F and ∂F are transversal to Z, then for almost every $s \in S$, both f_s and ∂f_s are transversal to Z.*

Proof. Let $W = F^{-1}(Z)$, which, by Theorem 1.13 is a submanifold of $M \times S$ with boundary $\partial W = W \cap (\partial M \times S)$. Consider $\pi : M \times S \to S$ be the projection onto the second factor. We shall show that whenever $s \in S$ is a regular value for $\pi|_W$, then $f_s \pitchfork Z$, and whenever s is a regular value for $\partial \pi|_W$, then $\partial f_s \pitchfork Z$. Since almost every $s \in S$ is a regular value for both maps, by Sard's theorem, the result follows.

In order to show that $f_s \pitchfork Z$ if s is a regular value for $\pi|W$, let $f_s(p) = z \in Z$. Because $F \pitchfork Z$, we know that

$$DF(p, s)(T_p M \times T_s S) + T_z Z = T_z N,$$

that is, given $a \in T_z N$, there is a vector $(w, e) \in T_p M \times T_s S$ such that $DF(p, s) \cdot (w, e) - a \in T_z Z$. We want to exhibit a vector $v \in T_p M$ such

that $Df_s(p) \cdot v - a \in T_z Z$. Since $D\pi(p, s) : T_p M \times T_s S \to T_s S$ is just the natural projection map and s is a regular value for $\pi|_W$, there exists a vector of the form $(u, e) \in T_{(p,s)} W$. But $F(W) \subseteq Z$, so that $DF(p, s) \cdot (u, e) \in T_z Z$. Consequently, the vector $v = w - u$ is a solution. Indeed

$$Df_s(p) \cdot v - a = DF(p, s) \cdot (v, 0) - a = DF(p, s) \cdot [(w, e) - (u, e)] - a$$
$$= [DF(p, s) \cdot (w, e) - a] - DF(p, s) \cdot (u, e) \in T_z Z.$$

The same argument shows that $\partial f_s \pitchfork Z$ when s is a regular value for $\partial\pi|_W$. $\qquad\square$

The Transversality Theorem implies that transversal maps are generic, at least in the case of maps $f : M \to \mathbb{R}^n$. To see why this is true, let S be an open ball of $\mathbb{R}^n$ and define $F : M \times S \to \mathbb{R}^n$ by $F(p, s) = f(p) + s$. For any fixed $p \in M$, $F_p = F(p, \cdot)$ is a translation of the ball S, so a submersion. Thus, F itself is a submersion and therefore transversal to any submanifold Z of $\mathbb{R}^n$. According to the Transversality Theorem, for almost every $s \in S$, the map $f_s = f + s$ is transversal to Z. Hence, f may be deformed into a transversal map just by adding a small quantity s.

When the target manifold of f is a boundaryless manifold N, we cannot add points. However, we may embed N as a submanifold in some Euclidean space $\mathbb{R}^n$ and perform the previous construction. The problem is that the map f_s do not map M into N anymore, but into a neighbourhood of N. To solve this issue, we need to use the celebrated ε-Neighbourhood Theorem:

Theorem 1.16 (ε-Neighbourhood Theorem). *Let N be a boundaryless compact submanifold of $\mathbb{R}^n$. Then, there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, the following properties hold:*

(i) the set

$$N_\varepsilon = \{y \in \mathbb{R}^n : d(y, N) < \varepsilon\}$$

is an n-dimensional submanifold of $\mathbb{R}^n$;

(ii) the map $\pi : N_\varepsilon \to N$, given by setting $\pi(y)$ as the point in N that minimises the distance to y, is well-defined and is a submersion.

Moreover, when N is not compact, the same conclusions hold by replacing the constant ε_0 by a positive smooth map $\varepsilon_0 : N \to (0, \infty)$ and defining

$$N_\varepsilon = \{y \in \mathbb{R}^n : \|y - q\| < \varepsilon_0(q) \text{ for some } q \in N\}.$$

Corollary 1.17. *Let $f : M \to N$ be a map, N being boundaryless. Then there is an open ball S in some Euclidean space and a map $F : M \times S \to N$ such that $F(p, 0) = f(p)$, and for any fixed $p \in M$, the map $s \mapsto F(p, s)$ is a submersion from S to N.*

Proof. Let N be embedded inside $\mathbb{R}^n$ and let S be the unit ball of this Euclidean space. Define $F : M \times S \to N$ by

$$F(p, s) = \pi \left[f(p) + \varepsilon(f(p))s \right],$$

where $\pi : N_\varepsilon \to N$ is the projection map from the Tubular Neighbourhood Theorem. It is clear that $F(p, 0) = f(p)$ for any $p \in M$. For fixed $p \in M$,

$$s \mapsto f(p) + \varepsilon(f(p))s$$

is certainly a submersion from S to N_ε. As the composition of submersions is another, $s \mapsto F(p, s)$ is a submersion. $\qquad\square$

As a result of this corollary, we obtain that transversality is a generic property in the general case of a map $f : M \to N$. We shall need another form of this result. Recall that two maps $f, g : M \to N$ are homotopic if one can be continuously deformed into the other. More explicitly, there must exist a continuous map $H : M \times [0, 1] \to N$ such that $H(p, 0) = f(p)$ and $H(p, 1) = g(p)$ for every point $p \in M$. If H is smooth, we say that f and g are smoothly homotopic.

Theorem 1.18 (Transversality Homotopy Theorem)**.** *For any map $f : M \to N$ and any boundaryless submanifold of the boundaryless manifold N, there exists a map $g : M \to N$ smoothly homotopic to f such that $g \pitchfork Z$ and $\partial g \pitchfork Z$.*

Proof. Let $F : M \times S \to N$ be the map of Corollary 1.17. By the Transversality Theorem, $f_s \pitchfork Z$ and $\partial f_s \pitchfork Z$ for almost $s \in S$. But each f_s is smoothly homotopic to f via $(p, t) \mapsto F(p, ts)$. $\qquad\square$

Definition 1.19. *Let $f : M \to N$ be a map, where N is boundaryless, and let C be a subset of M. We say f is transversal to a boundaryless submanifold Z of N on C if the transversality condition*

$$Df(p)(T_p M) + T_{f(p)}Z = T_{f(p)}N$$

holds for every point $p \in C \cap f^{-1}(Z)$.

Theorem 1.20 (Extension Theorem). *Supose Z^k is a closed submanifold of N^n, both boundaryless, and C is a closed subset of M^m. Let $f : M \to N$ be a map with $f \pitchfork Z$ on C and $\partial f \pitchfork Z$ on $C \cap \partial M$. Then there exists a map $g : M \to N$ smoothly homotopic to f, such that $g \pitchfork Z$, $\partial g \pitchfork Z$ and $g = f$ on a neighbourhood of C.*

Lemma 1.21. *Let C be a closed subset of the manifold M and let U be an open set containing C. Then there exists a smooth map $\beta : M \to [0, 1]$ that is identically equal to one outside U and equal to zero on a neighbourhood of C.*

Proof. Let V be a closed set of M that is contained in U and contains C in its interior: for example, endow M with a Riemannian metric, consider the Urysohn function $r : M \to [0, 1]$ defined by

$$r(p) = \frac{d(p, C)}{d(p, C) + d(p, M \setminus U)},$$

and set $V = \{p \in M : r(p) \leq \frac{1}{2}\}$. Consider a partition of unity $\{\varphi, \psi\}$ subordinate to the open cover $\{U, M \setminus V\}$, with $\operatorname{supp} \varphi \subset U$ and $\operatorname{supp} \psi \subset M \setminus V$. Now just set $\beta = \psi$. $\qquad\square$

Proof of Theorem 1.20. We first show that $f \pitchfork Z$ on a neighbourhood of C. If $p \in C$ but $p \notin f^{-1}(Z)$, then since Z is closed, $M \setminus f^{-1}(Z)$ is a neighbourhood of p on which $f \pitchfork Z$ clearly. If $p \in C \cap f^{-1}(Z)$, then there is a neighbourhood W of $f(p)$ in N and a submersion $\varphi : W \to \mathbb{R}^{n-k}$ such that $f \pitchfork Z$ at a point $p' \in f^{-1}(Z \cap W)$ if and only if $\varphi \circ f$ is regular at p' (see discussion before Definition 1.9). But since being regular is an open condition, $\varphi \circ f$ is regular on an open neighbourhood of p. Thus $f \pitchfork Z$ on a neighbourhood of every point of C, and so $f \pitchfork Z$ on an open neighbourhood U of C.

Let now $\beta : M \to [0, 1]$ be the function given by Lemma 1.21 for the open set U and set $\tau = \beta^2$. Let $F : M \times S \to N$ be the function used in Corollary 1.17 and define $G : M \times S \to N$ by $G(p, s) = F(p, \tau(p)s)$.

Claim. $G \pitchfork Z$.

Proof of the Claim. Let $(p, s) \in G^{-1}(Z)$ and suppose initially that $\tau(p) \neq 0$. Then the map $r \in S \mapsto G(x, r)$ is a submersion as it is the composition of the diffeomorphism $r \mapsto \tau(p)r$ with the submersion $r \in S \mapsto F(p, r)$. This way, G is regular at (p, s) and certainly $G \pitchfork Z$ at (p, s). When $\tau(p) = 0$, we calculate $DG(p, s) : T_p M \times \mathbb{R}^{n'} \to T_{G(p,s)} N$. For clarity, define $\mu : M \times S \to M \times S$ by $\mu(p, s) = (p, \tau(p)s)$. Then

$$D\mu(p,s) \cdot (v,w) = (v, \tau(p)w + (D\tau(p) \cdot v)s)$$

for $(v,w) \in T_pM \times \mathbb{R}^{n'}$. Since $G = F \circ \mu$, we have

$$\begin{aligned} DG(p,s) \cdot (v,w) &= DF(p, \tau(p)s) \cdot (v, \tau(p)w + (D\tau(p) \cdot v)s) \\ &= DF(p,0) \cdot (v,0) = Df(p) \cdot v \end{aligned}$$

because $\tau(p) = 0$ and F equals f when restricted to $M \times \{0\}$. But if $\tau(p) = 0$, then $p \in U$ and $f \pitchfork Z$ at p, so that

$$DG(p,s)(T_pM \times \mathbb{R}^{n'}) + T_{G(p,s)}Z = Df(p)(T_pM) + T_{f(p)}Z = T_{f(p)}N,$$

that is, G is transversal to Z at (p,s). $\square$

An analogous argument shows that $\partial G \pitchfork Z$. By the Transversality Theorem there exists an $s \in S$ such that the map $g : M \to N$ given by $g(p) = G(p,s)$ satisfies $g \pitchfork Z$ and $\partial g \pitchfork Z$. As before, g is smoothly homotopic to f via $(p,t) \mapsto G(p,ts)$. Finally, if p belongs to the neighbourhood V, where $\tau = 0$, then $g(p) = G(p,s) = F(p,0) = f(p)$, as we wanted. $\square$

The next corollaries will be of great value. For the first, just notice that ∂M is always closed in M. The second follows immediately from the first.

Corollary 1.22. *If, for $f : M \to N$, the map $\partial f : \partial M \to N$ is transversal to Z, then there exists a map $g : M \to N$ smoothly homotopic to f such that $\partial g = \partial f$ and $g \pitchfork Z$.*

Corollary 1.23. *Suppose that $f : \partial M \to N$ is a map transversal to a boundaryless submanifold Z of the boundaryless submanifold N. If f extends to any map $M \to N$, then it also extends to a map that is transversal to Z on all of M.*

1.3   Intersection number

Our objective is to define the intersection number of a map $f : M \to N$ and a submanifold Z of N. In order to do this, we need the notion of orientable manifolds.

Definition 1.24. *An orientation on a finite dimensional real vector space V is an equivalence class of ordered basis for V, two of these being equivalent if the matrix of the linear transformation that maps one into the other has positive determinant. A particular basis that belongs to the chosen orientation is called positive.*

Definition 1.25. *A linear map between oriented vector spaces preserves orientation if it maps positive basis into positive basis, and reverse orientation otherwise.*

For the next definition, let $\overline{\mathbb{H}}^m = \{x \in \mathbb{R}^m : x_m \geq 0\}$ denote the half-space $\{x_m \geq 0\}$.

Definition 1.26. *An orientation of a manifold with boundary M is a smooth choice of an orientation on each tangent space T_pM, meaning that every point $p \in M$ lies in the image of a parametrisation $\varphi : U \subseteq \overline{\mathbb{H}}^m \to M$ such that $D\varphi(x) : \mathbb{R}^m \to T_{\varphi(x)}M$ preserves orientation for each $x \in U$. If M admits an orientation, then it is said to be orientable. By an oriented manifold we mean a manifold together with a specified smooth orientation. If M is oriented, we write $-M$ to denote the same manifold with the opposite orientation.*

It is an easy exercise to prove that if M is connected and orientable, then the only two possible orientations for the underlying manifold are those of M and $-M$.

If M and N are oriented and one of them is boundaryless, we define a product orientation on $M \times N$ as follows. Given $p \in M$ and $q \in N$, select two positive ordered basis $\{v_1, \ldots, v_m\}$ and $\{w_1, \ldots, w_n\}$ of T_pM and T_qN. We specify the orientation of $M \times N$ by declaring the ordered basis $\{(v_1, 0), \ldots, (v_m, 0), (0, w_1), \ldots, (0, w_n)\}$ of $T_pM \times T_qN$ to be positive.

An orientation on M naturally induces an orientation on ∂M, called the boundary orientation. At every point $p \in \partial M$, let $\nu(p) \in T_pM$ be the outward unit normal. We specify the orientation of ∂M by declaring the ordered basis $\{v_1, \ldots, v_{m-1}\}$ of $T_p(\partial M)$ to be positive whenever $\{\nu(p), v_1, \ldots, v_{m-1}\}$ is a positive basis of T_pM.

Example 1.27. The orientation of a zero dimensional vector space is just a choice of sign: $+$ or $-$. Consider the unit interval $M = [0, 1]$ with its induced standard orientation from $\mathbb{R}$. At $p = 1$ the outward unit normal is $1 \in \mathbb{R} \cong T_1\mathbb{R}$, which is positively oriented, and at $p = 0$, the unit normal is $-1 \in \mathbb{R} \cong T_0\mathbb{R}$, which is negatively oriented. Thus, the orientation of $T_1(\partial M)$ is $+$ and that of $T_0(\partial M)$ is $-$.

Example 1.28. Let M be a boundaryless oriented manifold and consider the space $[0,1] \times M$. For each $t \in [0,1]$ the slice $M_t = \{t\} \times M$ is naturally diffeomorphic to M, so let us orient M_t in order to make this diffeomorphism $p \mapsto (t,p)$ orientation preserving. We have $\partial([0,1] \times M) = M_0 \cup M_1$ as a set, but a simple analysis show that, as an oriented manifold,

$$\partial([0,1] \times M) = M_1 - M_0.$$

Let now M be a compact oriented one-dimensional manifold with boundary. Since the boundary points of M are connected by diffeomorphic copies of the unit interval (see [13]), we have:

Remark 1.29. The orientation signs at the boundary points of any compact one-dimensional manifold with boundary cancel out in pairs and thus add up to zero.

We are now ready for the main definition of this section. This is the setting to be assumed : M^m, N^n, Z^k are boundaryless oriented manifolds, M is compact, Z is a closed submanifold of N, and $m + k = n$.

Definition 1.30. *Suppose $f : M \to N$ is transversal to Z. Since $f^{-1}(Z)$ is a zero-dimensional submanifold of M (see Theorem 1.10), the compactness of M implies that it consists of a finite number of points $\{p_1, \ldots, p_r\}$. We attribute signs to each point p_i as follows. Let $\{v_1, \ldots, v_m\}$ and $\{w_1, \ldots, w_k\}$ be positive ordered basis of $T_{p_i}M$ and $T_{f(p_i)}Z$. Then p_i is positive if the ordered basis $\{Df(p_i) \cdot v_1, \ldots, Df(p_i) \cdot v_m, w_1, \ldots, w_k\}$ is a positive basis of $T_{f(p_i)}N$, and we write $\mathrm{sign}_{p_i}(f) = 1$; otherwise, p_i is negative and $\mathrm{sign}_{p_i}(f) = -1$. The intersection number of f and Z, denoted $I(f, Z)$, is the sum of the signs of the points p_i:*

$$I(f, Z) = \sum_{f(p_i) \in Z} \mathrm{sign}_{p_i}(f).$$

We also say that the orientation number of f at p_i is $\mathrm{sign}\, p_i(f)$, or still, that the intersection number between f and Z at p_i is $\mathrm{sign}_{p_i}(f)$.

Example 1.31. For a positive integer n, let $f : \mathbb{S}^1 \subset \mathbb{C} \to \mathbb{R}^2 \cong \mathbb{C}$ be the map $z \mapsto z^n$ and let $Z = \{(x_1, 0) : x_1 \in \mathbb{R}\}$ be the x-axis. Orient $\mathbb{S}^1$ anticlockwise, Z in the positive x direction and $\mathbb{R}^2$ in the standard way. We have

$$Df(p) \cdot v = nz^{n-1}v,$$

for $v \in T_z \mathbb{S}^1 = \{siz : s \in \mathbb{R}\}$ and we are using complex multiplication. Also,

$$f^{-1}(Z) = \{(cos\theta, \sin\theta) : \sin(n\theta) = 0\}$$
$$= \left\{ p_j = \left(\cos\frac{j\pi}{n}, \sin\frac{j\pi}{n} \right) : j \in \{0, \ldots, 2n-1\} \right\}.$$

This way, if $\{v_j = ip_j\}$ is a positive basis for $T_{p_j}\mathbb{S}^1$,

$$Df(p_j) \cdot v_j = np_j^{n-1}v_j = inp_j^n = ine^{ij\pi} = in(-1)^j.$$

Thus, letting $\{1\}$ be a positive basis for $T_1 Z$ and $T_{-1} Z$, the ordered basis $\{in(-1)^j, 1\}$ of $\mathbb{R}^2$ is positive when j is odd and negative when j is even. So,

$$I(f, Z) = \sum_{p_j} \text{sign}_{p_j}(f) = \sum_{j=0}^{2n-1}(-1)^{j+1} = 0.$$

Example 1.32. Let $f : \mathbb{S}^1 \to \mathbb{S}^1 \times \mathbb{S}^1$ be the map $z \mapsto (z, 1)$, which parametrises a meridian on the torus and let $Z = \{1\} \times \mathbb{S}^1$ be a parallel on the torus (see Figure 1.4). Orient $\mathbb{S}^1$ anticlockwise and give $\mathbb{S}^1 \times \mathbb{S}^1$ the product orientation. It is clear that $f^{-1}(Z) = 1$. Choose $\{i\}$ and $\{(0, i)\}$ as positive basis for $T_1\mathbb{S}^1$ and $T_{(1,1)}Z$. Then, since $Df(p)$ is the inclusion $v \mapsto (v, 0)$, the ordered basis $\{(i, 0), (0, i)\}$ of $T_{(1,1)}(\mathbb{S}^1 \times \mathbb{S}^1)$ is positive. Thus, $I(f, Z) = 1$.
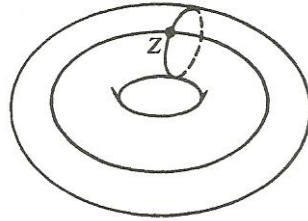

Figure 1.4: Two circles on the torus

One remarkable fact is that the intersection number is a homotopy invariant, as the next theorem shows.

Theorem 1.33. *Let $f, g : M \to N$ be two maps, transversal to the submanifold Z. If f and g are smoothly homotopic, then $I(f, Z) = I(g, Z)$.*

Proof. Let $H : [0,1] \times M \to N$ be a smooth homotopy between f and g, with $H(0, p) = f(p)$ and $H(1, p) = g(p)$. By Corollary 1.23, we can suppose $H \pitchfork Z$. By Theorem 1.13, $H^{-1}(Z)$ is a compact one-dimensional manifold with boundary

$$\partial(H^{-1}(Z)) = H^{-1}(Z) \cap (M_0 \cup M_1) = \{0\} \times f^{-1}(Z) \cup \{1\} \times g^{-1}(Z).$$

As an oriented manifold, however,

$$\partial(H^{-1}(Z)) = \{1\} \times g^{-1}(Z) - \{0\} \times f^{-1}(Z).$$

Remark 1.29 then implies that $I(g, Z) - I(f, Z) = 0$, as we wanted. $\square$

This theorem allows us to define the intersection number for an arbitrary smooth map $f : M \to N$ and a submanifold Z. Just select a homotopic map $g : M \to N$ that is transversal to Z and put $I(f, Z) = I(g, Z)$. The theorem guarantees that this does not depend on the choice of g.

Another very important concept is that of degree of a map, which we now present. It will be used in the future.

Definition 1.34. *Let $f : M^n \to N^n$ be a map between manifolds of the same dimension, where M is compact, N is connected and both are boundaryless. The degree $\deg_q(f)$ of f at a point $q \in N$ is defined as the intersection number $I(f, \{q\})$.*

Example 1.35. Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be the map of Example 1.31, given by $f(z) = z^n$. Every point z in the circle is a regular value for f, and $f^{-1}(z)$ consists of n points, all positive. Thus, $\deg_z(f) = n$ for all $z \in \mathbb{S}^1$.

As the next theorem shows, what happens in the preceding example can be generalised.

Theorem 1.36. *For a map $f : M \to N$ between boundaryless compact manifolds of the same dimension supposing N is connected, the degree $\deg_q(f)$ does not depend on the point $q \in N$.*

In the proof, we will make use of the following lemma.

Lemma 1.37. *If $q \in N$ is a regular value of $f : M \to N$ as in the theorem, there exists an open neighbourhood V of q in N such that $f^{-1}(V) = U_1 \cup \cdots \cup U_r$ is a disjoint union of open connected sets of M, each of which is diffeomorphic to V under f.*

Proof. The set $f^{-1}(q)$ consists of a finite number of points $p_1, \ldots, p_r$. For each i there exists an open set $\tilde{U}_i$ containing p_i which is diffeomorphic to an open neighbourhood V_i of q, by the Inverse Function Theorem. Since M is compact, the image of the closed set $C = M \setminus (\tilde{U}_1 \cup \cdots \cup \tilde{U}_r)$ is a closed set that does not contain q. Let $V \subset N$ an open set such that $q \in V \subset (V_1 \cap \cdots \cap V_r) \cap (N \setminus f(F))$. Defining $U_i = \tilde{U}_i \cap f^{-1}(V)$ finishes the proof. $\qquad\square$

Proof of Theorem 1.36. Given $q \in N$, alter f homotopically, if necessary, to make it transversal to $\{q\}$ (see Transversality Homotopy Theorem). Let V be the neighbourhood provided by Lemma 1.3. Since V and each set U_i are connected, all points in U_i have the same sign with respect to f, for fixed i. Thus, the function $q' \mapsto \deg_{q'}(f)$ is locally constant. Since N is connected, it is globally constant. $\qquad\square$

Note that, in order to calculate the degree of a map $f : M \to N$, we simply select a regular value $q \in N$ of f and count the number of preimages of q, except that a point $p \in f^{-1}(q)$ makes a contribution of $+1$ or -1 to the sum, depending on whether the isomorphism $Df(p) : T_p M \to T_q N$ preserves or reverses orientation.

1.4   Vector fields and the Euler characteristic

We will now use the ideas hitherto presented to define the index of a simple singularity of a vector field and, subsequently, define the Euler characteristic of a compact manifold. As usual, we start by reviewing some concepts. In this section, our manifolds are assumed to be oriented and boundaryless.

Definition 1.38. *The tangent bundle of a manifold M^m is the space $TM = \{(p, w) : p \in M, w \in T_p M\}$. If $\{\varphi_\alpha : U_\alpha \to M\}_\alpha$ is a differentiable atlas for M, then the collection of functions $\tilde{\varphi}_\alpha : U_\alpha \times \mathbb{R}^m \to TM$, defined by*

$$\tilde{\varphi}_\alpha(x, u_1, \ldots, u_m) = \left(\varphi(x), \sum_{i=1}^{m} u_i \frac{\partial \varphi}{\partial x_i}(x) \right), \quad x \in U_\alpha, (u_1, \ldots, u_m) \in \mathbb{R}^m,$$

is the differentiable atlas we endow TM for it to become a differentiable manifold of dimension $2m$.

Notice that TM contains a naturally embedded copy of M, namely, the zero section $M_0 = \{(p, 0) : p \in M\}$.

Definition 1.39. *A vector field on M is a map $v : M \to TM$ such that $\pi \circ v = id_M$, where $\pi : (p, w) \mapsto p$ is the natural projection from TM to M. The set of vector fields on M is denoted $\mathfrak{X}(M)$.*

It is customary to think of a vector field as a choice of vector $v(p) \in T_p M$ for each point $p \in M$. Whenever it is convenient, we shall adopt this practise.

Definition 1.40. *A point $p \in M$ is a singularity for a vector field v if $v(p) = (p, 0)$, and is called regular otherwise.*

An ODE theorem — the Tubular Flow Theorem (see [1]) — tells us that the behaviour of a vector field around a regular point is very simple and similar irrespective of the point. It is then natural to focus attention on the singular points. We start with the main definition.

Definition 1.41. *A vector field on a manifold M has simple singularities if it is transversal to the zero section M_0.*

If M is compact, then there are only finitely many simple singularities for a vector field on M.

The next proposition establishes a useful criterion for identifying these singularities.

Proposition 1.42. *Let v be a vector field on the manifold M^m. Suppose that $p \in M$ is an isolated singularity for v. Given a parametrisation $\varphi : U \to M$ covering a neighbourhood where p is the only singularity for v, write*

$$v(\varphi(x)) = -\sum_{i=1}^{m} a_i(x) \frac{\partial \varphi}{\partial x_i}(x), \quad x \in U,$$

for some functions $a_i : U \to \mathbb{R}$. Let $\varphi(x_0) = p$. Then p is a simple singularity for v if and only if $\det\left(\frac{\partial a_i}{\partial x_j}(x_0)\right) \neq 0$.

Proof. Let $\tilde{\varphi} : U \times \mathbb{R}^m$ be the parametrisation of TM given in Definition 1.38. Define $\theta : M \to TM$ as the zero vector field and set

$$\tilde{v} = \tilde{\varphi}^{-1} \circ v \circ \varphi : x \mapsto (x, -a_1(x), \dots, -a_m(x))$$
$$\tilde{\theta} = \tilde{\varphi}^{-1} \circ \theta \circ \varphi : x \mapsto (x, 0).$$

Notice that

$$\left\{ D\theta(p) \cdot \frac{\partial \varphi}{\partial x_1}(x_0), \ldots, D\theta(p) \cdot \frac{\partial \varphi}{\partial x_n}(x_0) \right\}$$

is a basis for $T_{(p,0)}M_0$. So, p is simple if and only if the set

$$\left\{ \frac{\partial(v \circ \varphi)}{\partial x_1}(x_0), \ldots, \frac{\partial(v \circ \varphi)}{\partial x_m}(x_0), \frac{\partial(\theta \circ \varphi)}{\partial x_1}(x_0), \ldots, \frac{\partial(\theta \circ \varphi)}{\partial x_m}(x_0) \right\}$$

is linearly independent on $T_{(p,0)}(TM)$, or, equivalently, if the set

$$\left\{ \frac{\partial \tilde{v}}{\partial x_1}(x_0), \ldots, \frac{\partial \tilde{v}}{\partial x_1}(x_0), \frac{\partial \tilde{\theta}}{\partial x_1}(x_0), \ldots, \frac{\partial \tilde{\theta}}{\partial x_m}(x_0) \right\}$$

is linearly independent on $\mathbb{R}^m \times \mathbb{R}^m$. Arranging these vectors as columns in a $2m \times 2m$ matrix, we obtain:

$$\begin{bmatrix} I_m & I_m \\ \left(-\frac{\partial a_i}{\partial x_j}(x_0) \right) & 0 \end{bmatrix},$$

where I_m is the identity matrix of order m. Thus, p is a simple singularity precisely when the determinant of the above matrix is nonzero, yielding the result. $\qquad\square$

Example 1.43. Let $\mathbb{S}^n$ be the unit sphere of $\mathbb{R}^{n+1}$ and let $p_0 = (0, \ldots, 1)$ be the north pole. Consider the vector field v on $\mathbb{S}^n$ defined by

$$v(p) = p_0 - \langle p, p_0 \rangle p, \quad p \in \mathbb{S}^n,$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^{n+1}$ (see Figure 1.5).
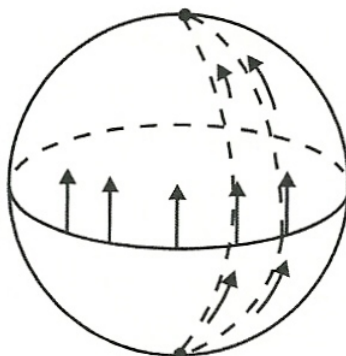
Figure 1.5: The vector field v.

The two singularities of v are p_0 and $-p_0$. We will show that they are simple. For this, let $\varphi : B(0,1) \to \mathbb{S}^n$ be the parametrisation

$$\varphi(x) = (x_1, \ldots, x_n, sq(x)), \quad x = (x_1, \ldots, x_n) \in B(0,1),$$

where $sq(x) = \sqrt{1 - \sum_{i=1}^n x_i^2}$ and $B(0,1)$ is the open unit ball of $\mathbb{R}^n$. Then, some straightforward computations yield

$$\frac{\partial \varphi}{\partial x_i}(x) = \left(0, \ldots, 1, \ldots, 0, \frac{-x_i}{sq(x)}\right).$$

and

$$v(\varphi(x)) = \left(-x_1 sq(x), \ldots, -x_n sq(x), \sum_{i=1}^n x_i^2\right).$$

So, if $a_i(x) = x_i \, sq(x)$, we have

$$v(\varphi(x)) = -\sum_{i=1}^m a_i(x) \frac{\partial \varphi}{\partial x_i}(x), \quad x \in B(0,1)$$

Another simple calculation shows that

$$\frac{\partial a_i}{\partial x_j}(x) = \delta_{ij} sq(x) - \frac{x_i x_j}{sq(x)}.$$

So, at $x = 0$ the matrix $\left(\frac{\partial a_i}{\partial x_j}(x) \right)$ is the identity, which is invertible. Thus, p_0 is a simple singularity. A similar argument works for $-p_0$, now using the parametrisation

$$\psi(x) = (x_1, \ldots, x_n, -sq(x)), \quad x = (x_1, \ldots, x_n) \in B(0, 1)$$

instead. It is possible to show that

$$v(\psi(x)) = -\sum_{i=1}^{m} b_i(x) \frac{\partial \psi}{\partial x_i}(x), \quad x \in B(0, 1),$$

for $b_i(x) = -a_i(x)$. Then, at $x = 0$, the matrix $\left(\frac{\partial b_i}{\partial x_j}(x) \right)$ is minus the identity and $-p_0$ is simple as well.

Definition 1.44. *Let v be a vector field with simple singularities on a compact manifold M. The index of v at a singularity $p \in M$, denoted by $I_p(v)$, is the intersection number between v and M_0 at $(p, 0)$.*

A careful examination of the proof of Proposition 1.42 shows that the index can be calculated as follows:

Proposition 1.45. *According to the notation of Proposition 1.42 and supposing the parametrisation φ preserves orientation, the index of v at the simple singularity p is given by*

$$I_p(v) = \begin{cases} +1, & \text{if } \det\left(\frac{\partial a_i}{\partial x_j}(x_0) \right) > 0 \\ -1 & \text{otherwise.} \end{cases}$$

Since the intersection number is an homotopy invariant (see Theorem 1.33) and any two vector fields v, w on a manifold M are homotopic via $(p, t) \mapsto tv(p) + (1 - t)w(p)$, the following definition makes sense.

Definition 1.46. *The Euler characteristic of a compact manifold M, denoted by $\chi(M)$, is obtained by selecting a vector field with simple singularities and summing the indices of the vector field at its singularities.*

Remark 1.47. Our definition of Euler characteristic is a differential invariant, but there is a more general definition which is invariant under homeomorphisms:

$$\chi(M) = \sum_{i=0}^{\dim M} \operatorname{rank} H^i(M),$$

where $H^i(M)$ is the i-th cohomology group of M. It is well defined for non-orientable manifolds as well (actually, it makes sense for any topological space), and coincides with our definition in the orientable case.

Example 1.48. The index of the vector field v of Example 1.43 at the points p_0 and $-p_0$ is $+1$ and $(-1)^n$, respectively. So,

$$\chi(\mathbb{S}^n) = \begin{cases} 0, & \text{if } n \text{ is odd} \\ 2, & \text{if } n \text{ is even.} \end{cases}$$

The next theorem tells us that one aspect of this example can be generalised.

Theorem 1.49. *Every odd-dimensional manifold has zero Euler characteristic.*

Proof. Let M^m be an odd dimensional (orientable) compact manifold and let v be a vector field on M with simple singularities. From what we observed above, the sum of the indices of v is the same than that for $-v$. But the singularities of v and $-v$ are the same, and at such a point p, the index of v is the sign of the determinant $\det\left(\frac{\partial a_i}{\partial x_j}(x_0)\right)$, while the index of $-v$ at the same point is the sign of the determinant $\det\left(-\frac{\partial a_i}{\partial x_j}(x_0)\right) = (-1)^m \det\left(\frac{\partial a_i}{\partial x_j}(x_0)\right)$. Thus, $\chi(M) = (-1)^m\chi(M)$, whence $\chi(M) = 0$. $\qquad\square$

Example 1.50. If $M = M_1 \cup \cdots \cup M_k$ is a disjoint union of compact manifolds, then $\chi(M) = \chi(M_1) + \cdots + \chi(M_k)$.

Example 1.51. Let M^m and N^n be compact manifolds. Then $\chi(M \times N) = \chi(M)\chi(N)$. To see why this is true, let $v \in \mathfrak{X}(M)$ and $w \in \mathfrak{X}(N)$ be vector fields with simple singularities, and let $v \times w \in \mathfrak{X}(M \times N)$ be the vector field $(v \times w)(p,q) = (v(p), w(q))$. Each singularity of $v \times w$ is simple, by the criterion we gave earlier, for if $\varphi : U \to M$ and $\psi : V \to N$ are parametrisations covering neighbourhoods of the singularities $p \in M$ and $q \in N$, write

$$v(\varphi(x)) = -\sum_{i=1}^{m} a_i(x)\frac{\partial\varphi}{\partial x_i}(x)$$

and

$$w(\psi(y)) = -\sum_{i=1}^{n} b_i(x) \frac{\partial \psi}{\partial x_i}(y).$$

Then the expression of $v \times w$ on the coordinate system given by $\varphi \times \psi$: $(x, y) \mapsto (\varphi(x), \psi(y))$ is

$$(v \times w)(\varphi(x), \psi(y)) = -\sum_{i=1}^{m+n} c_i(x) \frac{\partial(\varphi \times \psi)}{\partial x_j}(x, y),$$

where $c_k = a_k$ for $k \in \{1, \ldots, m\}$ and $c_k = b_k$ for $k \in \{m+1, \ldots, m+n\}$. Consequently, the matrix $\left(\frac{\partial c_i}{\partial x_j}\right)$ takes the form

$$\begin{bmatrix} \left(\dfrac{\partial a_i}{\partial x_j}\right) & 0 \\ 0 & \left(\dfrac{\partial b_i}{\partial x_j}\right) \end{bmatrix},$$

whence $I_{v \times w}(p, q) = I_v(p) I_q(w)$ by Proposition 1.45. Thus,

$$\chi(M \times N) = \sum_{\substack{v(p)=0 \\ w(q)=0}} I_{(p,q)}(v \times w) = \sum_{v(p)=0} \sum_{w(q)=0} I_v(p) I_q(w)$$

$$= \sum_{v(p)=0} \left(\sum_{w(q=0)} I_q(w)\right) I_p(v) = \chi(N) \sum_{v(p)=0} I_p(v) = \chi(M)\chi(N).$$

So, for example, the n-dimensional torus $T^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ has Euler characteristic equal to zero. More generally, if M splits as a product $K \times N$, where K is odd dimensional, then $\chi(M) = 0$.

Translational manifolds

In this chapter we shall extend the definition of a Gauss map, now for hypersurfaces of a parallelisable manifold. Then, we define an associated curvature and prove a Gauss-Bonnet theorem. As an example, we study the case where the ambient space is an Euclidean sphere minus a point and obtain a topological rigidity theorem for its hypersurfaces. This, in turn, will serve to provide an alternative proof for a theorem of Qiaoling Wang and Changyu Xia. It asserts that if such an immersed hypersurface of the sphere is contained in an open hemisphere and has nowhere zero Gauss-Kronecker curvature, then it is diffeomorphic to a sphere. This chapter comprises the article [17], a joint work with Jaime Ripoll.

2.1 Translational structures

Let us start with some observations on parallelisable Riemannian manifolds.

Definition 2.1. *A Riemannian manifold $\overline{M}^{n+1}$ is called parallelisable if its tangent bundle $T\overline{M}$ is trivial, meaning that there exists a diffeomorphism $\lambda : T\overline{M} \to \overline{M} \times \mathbb{R}^{n+1}$, called a trivialisation, that maps each fibre $\{p\} \times T_p\overline{M} \cong T_p\overline{M}$ isomorphically onto the fibre $\{p\} \times \mathbb{R}^{n+1} \cong \mathbb{R}^{n+1}$.*

Definition 2.2. *A referential on a manifold $\overline{M}^{n+1}$ is a set of $n+1$ vector fields which are linearly independent at every point of $\overline{M}$.*

For a manifold $\overline{M}^{n+1}$ to be parallelisable, it is necessary and sufficient that it possesses a referential, for if $\lambda : T\overline{M} \to \overline{M} \times \mathbb{R}^{n+1}$ is a trivialisation, then

the vector fields $v_i(p) = \lambda^{-1}(p, e_i)$ form a referential, where $\{e_1, \ldots, e_{n+1}\}$ is a basis of $\mathbb{R}^{n+1}$. Reciprocally, if $\{v_1, \ldots, v_{n+1}\}$ are everywhere linearly independent vector fields on $\overline{M}$, then $\lambda : T\overline{M} \to \overline{M} \times \mathbb{R}^{n+1}$ defined by

$$\lambda \left(p, \sum_{i=1}^{n+1} a_i v_i(p) \right) = (p, a_1, \ldots, a_{n+1})$$

is a trivialisation.

Example 2.3. Every finite dimensional real vector space V is parallelisable. Indeed, if $\{e_1, \ldots, e_n\}$ is a basis for V, just consider the constant vector fields $v_i(x) = e_i \in T_x V \cong V$.

Example 2.4. More generally, every Lie group is parallelisable. To see why this is true, let G^n be a lie group and $\{e_1, \ldots, e_n\}$ be a basis for the lie algebra $\mathfrak{g}$ of G. It is immediate that the vector fields $\overline{v}_i(x) = DL_x(e) \cdot e_i$ are linearly independent at every point of G. This example includes the previous one, and the spheres $\mathbb{S}^1$ and $\mathbb{S}^3$. Although $\mathbb{S}^7$ is not a Lie group, it is the last parallelisable sphere, other than $\mathbb{S}^0$, of course! (See [3]).

Example 2.5. The product of two parallelisable manifold is another. For example, the n-torus $T^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ has this property.

Example 2.6. It is a remarkable fact that every compact orientable three-dimensional manifold is parallelisable (see [20]).

On a parallelisable Riemannian manifold we can introduce translational structures:

Definition 2.7. *A translational structure on a parallelisable Riemannian manifold $\overline{M}$ consists of a trivialisation $\Gamma : T\overline{M} \to \overline{M} \times V$, where V is an $(n+1)$-dimensional real vector space with an inner product, such that the maps $\Gamma_p : T_p M \to V$ implicitly defined by*

$$(p, v) \mapsto \Gamma(p, v) = (p, \Gamma_p(v))$$

are linear isometries for every point p of $\overline{M}$. The pair $\left(\overline{M}, \Gamma \right)$, or just $\overline{M}$, if Γ is understood from context, is called a translational Riemannian manifold, and $\overline{M}$ is said to be equipped with a translational structure.

The maps Γ_p are to be thought as translations, as means of identifying the tangent spaces to $\overline{M}$ with the vector space V. This will allow us to define the Gauss map of a hypersurface.

Let $(\overline{M}, \Gamma)$ be a translational Riemannian manifold and $f : M^n \to \overline{M}$ an immersion of an orientable manifold M into $\overline{M}$. The following constructions are purely local, so we identify small neighbourhoods of M with their images via f, and the tangent spaces to M with their images via Df. Let $\eta : M \to T\overline{M}$ be a unit normal vector field along f, and let $\mathbb{S}^n$ be the unit sphere of V.

Definition 2.8. *The Gauss map $\gamma : M \to \mathbb{S}^n$ associated to the normal vector field η is given by*

$$\gamma(p) = \Gamma_p(\eta(p)), \quad p \in M.$$

The tangent space of V at any point is canonically isomorphic to V, and the tangent space of $\mathbb{S}^n$ at a point x is just $\{x\}^\perp$, the orthogonal complement of x. Thus, the derivative $D\gamma(p)$ maps T_pM into $T_{\gamma(p)}\mathbb{S}^n = \{\gamma(p)\}^\perp$ and Γ_p^{-1} maps the latter back into T_pM. This makes possible the following:

Definition 2.9. *The Γ-curvature of M is the map $\kappa_\Gamma : M \to \mathbb{R}$ given by*

$$\kappa_\Gamma(p) = \det\left(\Gamma_p^{-1} \circ D\gamma(p)\right), \quad p \in M.$$

Next, we define a special type of vector field that will play an important role.

Definition 2.10. *Given a point $p \in \overline{M}$ and a vector $v \in T_p\overline{M}$, the vector field $\tilde{v} \in \mathfrak{X}(\overline{M})$ defined by*

$$\tilde{v}(q) = \left(\Gamma_q^{-1} \circ \Gamma_p\right)(v), \quad q \in \overline{M}$$

is called the Γ-invariant (or simply invariant) vector field of $\overline{M}$ associated with v.

Example 2.11 (The Euclidean translation)**.** If $\overline{M} = \mathbb{R}^{n+1}$ and $\Gamma : T\mathbb{R}^{n+1} \to \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ is the identity, then the Gauss map γ for an orientable hypersurface M is the ordinary one. The invariant vector fields of $\mathbb{R}^{n+1}$ are the constant vector fields and κ_Γ is the Gauss-Kronecker curvature of M.

Example 2.12 (Left translation on Lie groups)**.** More generally, let $\overline{M} = G$ be a Lie group and $V = \mathfrak{g}$ be the Lie algebra of G, considered as the tangent space of G at the identity. Choose a left invariant metric for G and define $\Gamma : TG \to G \times \mathfrak{g}$ by

$$\Gamma(g, v) = (g, DL_{g^{-1}}(g) \cdot v), \quad (g, v) \in TG,$$

where $L_x : y \mapsto xy$ is the left translation. Here, the Γ-invariant vector fields are the left invariant vector fields of G. This is the setting studied in [22].

Example 2.13 (Parallel transport)**.** Assume $\overline{M}$ is a Cartan-Hadamard manifold, that is, a complete, connected and simply connected Riemannian manifold with nonpositive sectional curvature. Given a point $p_0 \in \overline{M}$, the exponential map at p_0 is, by Hadamard's Theorem, a diffeomorphism from $T_{p_0}\overline{M}$ onto $\overline{M}$, so that every point p can be joined to p_0 by a unique geodesic. Setting $V = T_{p_0}\overline{M}$, we may then define $\Gamma_p : T_p\overline{M} \to V$ by choosing $\Gamma_p(v)$ as being the parallel transport of $v \in T_p\overline{M}$ to $T_{p_0}\overline{M}$ along this geodesic. Thus, the invariant vector fields here are the parallel vector fields along the geodesic rays issuing from p_0.

More generally, given any complete Riemannian manifold $\overline{M}$ and a point p_0 in $\overline{M}$, we can define the parallel transport to $T_{p_0}\overline{M}$ on $\overline{M} \setminus C_{p_0}$ as above, where C_{p_0} is the cut locus of p_0 (see [26]). We study this case in detail on the sphere (see next section).

We next describe the geometry of the Gauss map. Let $\overline{\nabla}$ be the Riemannian connection of $\overline{M}$. Recall that the shape operator of M is the section A of the vector bundle $\mathrm{End}(TM)$ of endomorphisms of TM given by

$$A_p(v) = -\overline{\nabla}_v \eta, \quad p \in M, v \in T_p M,$$

where $\eta : M \to T\overline{M}$ is a normal vector field for M.

Similarly, we define another section of $\mathrm{End}(TM)$, which depends additionally on the choice of the translation Γ.

Definition 2.14. *The invariant shape operator of M is the section α of the bundle* $\mathrm{End}(TM)$ *given by*

$$\alpha_p(v) = \overline{\nabla}_v \widetilde{\eta(p)}, \quad p \in M, v \in T_p M,$$

where $\widetilde{\eta(p)}$ is the invariant vector field associated with $\eta(p)$.

In the next section we will calculate α when $\overline{M}$ is the sphere minus one point endowed with the translational structure induced by parallel transport. For now, we give the following example.

Example 2.15. If M is a hypersurface of $\overline{M} = \mathbb{R}^{n+1}$ with the translational structure of Example 2.11, then α_p is identically zero for each point p of M, since the invariant vector fields are constant. More generally, α_p is zero for any hypersurface M of a commutative Lie group G equipped with the translational structure of Example 2.12. To see why, recall Koszul's formula for the connection in terms of the metric:

$$2\langle \overline{\nabla}_X Y, Z \rangle = X\langle Y, Z \rangle + Y\langle X, Z \rangle - Z\langle X, Y \rangle$$
$$+ \langle [X, Y], Z \rangle - \langle [X, Z], Y \rangle - \langle [Y, Z], X \rangle,$$

for $X, Y, Z \in \mathfrak{X}(G)$. Given $v, w \in T_p M$, we have

$$2\langle \overline{\nabla}_v \widetilde{\eta(p)}, w \rangle = v\langle \widetilde{\eta(p)}, \tilde{w} \rangle + \eta(p)\langle \tilde{v}, \tilde{w} \rangle - w\langle \widetilde{\eta(p)}, \tilde{v} \rangle$$
$$+ \langle [\tilde{v}, \widetilde{\eta(p)}](p), w \rangle - \langle [\tilde{v}, \tilde{w}](p), \eta(p) \rangle - \langle [\widetilde{\eta(p)}, \tilde{w}](p), v \rangle.$$

Notice, however, that the inner product of two invariant vector fields is a constant function throughout G, for Γ is an isometry in each fibre. So, the first three terms above vanish. Moreover, since G is commutative, the Lie bracket of any two invariant fields is zero. Thus,

$$2\langle \overline{\nabla}_v \widetilde{\eta(p)}, w \rangle = 0 \tag{2.1}$$

for each $v, w \in T_p M$, that is to say, $\alpha_p \equiv 0$. It is worth noticing that in this case, the Γ-curvature of M is also the Gauss-Kronecker curvature of M.

The proposition below establishes a relationship between γ and the extrinsic geometry of M.

Proposition 2.16. *For any $p \in M$, the following identity holds:*

$$\Gamma_p^{-1} \circ D\gamma(p) = -\left(A_p + \alpha_p\right).$$

Proof. Fix $p \in M$ and an orthonormal basis $\{v_1, \ldots, v_{n+1}\}$ of $T_p\overline{M}$ such that $v_{n+1} = \eta(p)$. The vector fields $\tilde{v}_1, \ldots, \tilde{v}_{n+1}$ form a global orthonormal referential of $\overline{M}$, so that we can write

$$\eta = \sum_{i=1}^{n+1} a_i \tilde{v}_i \tag{2.2}$$

for certain functions $a_i \in C^\infty(M)$. Notice that $a_i(p) = 0$ for $i \in \{1, \dots, n\}$ and $a_{n+1}(p) = 1$.

For $y \in M$ we have

$$\gamma(y) = \Gamma_y(\eta(y)) = \Gamma_y \left(\sum_{i=1}^{n+1} a_i(y)\tilde{v}_i(y) \right) = \sum_{i=1}^{n+1} a_i(y)\Gamma_p(v_i).$$

Therefore, if $v \in T_pM$,

$$\Gamma_p^{-1}(D\gamma(p) \cdot v) = \Gamma_p^{-1} \left(\sum_{i=1}^{n+1} v(a_i)\Gamma_p(v_i) \right) = \sum_{i=1}^{n+1} v(a_i)v_i. \tag{2.3}$$

From (2.2) and (2.3) we obtain

$$-A_p(v) = \overline{\nabla}_v \eta = \sum_{i=1}^{n+1} \overline{\nabla}_v(a_i\tilde{v}_i) = \sum_{i=1}^{n+1} \left[a_i(p)\overline{\nabla}_v\tilde{v}_i + v(a_i)\tilde{v}_i(p) \right]$$

$$= \overline{\nabla}_v\tilde{v}_{n+1} + \sum_{i=1}^{n+1} v(a_i)v_i = \alpha_p(v) + \Gamma_p^{-1}(D\gamma(p) \cdot v),$$

which gives the desired result. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Before proving our Gauss-Bonnet theorem, we state the change of variables formula, which will be used in the proof. The reader can check [32].

Theorem 2.17 (Change of Variables Formula). *Let M^n and N^n be connected and oriented manifolds, and consider a proper map $f : M \to N$ (for example, if M is compact). Given an n-form ω with compact support on N, we have*

$$\int_M f^*\omega = \deg(f) \int_N \omega.$$

We now prove the main theorem of this section.

Theorem 2.18 (Theorem 1.2, [17])**.** *Let* $\left(\overline{M}, \Gamma\right)$ *be a translational Riemannian manifold and* M^n *a compact, connected and orientable immersed hypersurface of even dimension of* $\overline{M}$*, and denote by* ω *the volume element of* M *induced by the metric of* $\overline{M}$*. Then*

$$\int_M \kappa_\Gamma \, \omega = \frac{c_n}{2} \chi(M),$$

where c_n *is the volume of* $\mathbb{S}^n \subset V$ *and* $\chi(M)$ *is the Euler characteristic of* M*.*

Proof. Firstly, orient V arbitrarily. This induces an orientation on $\overline{M}$ by requiring the maps Γ_p to preserve orientation. We also orient M and $\mathbb{S}^n$ as follows: an oriented basis $\{w_1, \ldots, w_n\}$ of T_pM is positive if $\{\eta(p), w_1, \ldots, w_n\}$ is a positive basis of $T_p\overline{M}$, and the ordered basis $\{x_1, \ldots, x_n\}$ of $T_x\mathbb{S}^n$ is positive precisely if $\{x, x_1, \ldots, x_n\}$ is a positive basis of V.

Let σ be the volume form of $\mathbb{S}^n$ induced by the metric on V. From the fact that Γ restricts to isometries in each fibre and from the definition of κ_Γ, it follows that $\gamma^*\sigma = \kappa_\Gamma \, \omega$. Then, the change of variables formula yields

$$\int_M \kappa_\Gamma \, \omega = \int_M \gamma^*\sigma = \deg(\gamma) \int_{\mathbb{S}^n} \sigma = c_n \deg(\gamma).$$

It remains to show that $\deg(\gamma) = \frac{1}{2}\chi(M)$.

For this, let $\{a, b\}$ be a pair of antipodal points on $\mathbb{S}^n$ which are regular values for γ (consider a regular value for the composite $M \to \mathbb{S}^n \to \mathbb{R}P^n$, where the last map is the natural projection onto the projective space). Let u be a vector field on $\mathbb{S}^n$ having only a and b as singularities, and suppose they are simple and of index equal to one (rotate the vector field of Example 1.43). Here is where the hypothesis on the dimension of M is important.

The singularities of v are $\{a_1, \ldots, a_r\} \cup \{b_1, \ldots, b_s\}$, with $\gamma(a_i) = a$ and $\gamma(b_i) = b$. Firstly, we will show that they are simple. For this, let $\varphi : U \to M$ a parametrisation of M around a_i, with $\varphi(0) = a_i$, so that γ is a diffeomorphism from $\varphi(U)$ to the open set $V \subset \mathbb{S}^n$. Then $\psi = \gamma|_{\varphi(U)} \circ \varphi : U \to V$ is a parametrisation of $\mathbb{S}^n$ around a. Write

$$u(\psi(x)) = -\sum_{i=1}^n u_i(x)\frac{\partial\psi}{\partial x_i}(x)$$

and

$$v(\varphi(x)) = -\sum_{i=1}^{n} v_i(x)\frac{\partial \varphi}{\partial x_i}(x)$$

for some functions u_i and v_i on U. If $\varphi(x) = p$, then

$$-\sum_{i=1}^{n} v_i(x)\frac{\partial \varphi}{\partial x_i}(x) = v(\varphi(x)) = v(\gamma|_U^{-1} \circ \psi(x)) = \Gamma_p^{-1}(u(\psi(x)))$$

$$= -\sum_{i=1}^{n} u_i(x)\Gamma_p^{-1}\left(\frac{\partial \psi}{\partial x_i}(x)\right)$$

$$= -\sum_{i=1}^{n} u_i(x)\left(\Gamma_p^{-1} \circ D\gamma(p)\right)\left(\frac{\partial \varphi}{\partial x_i}(x)\right). \qquad (2.4)$$

So, setting

$$\left(\Gamma_p^{-1} \circ D\gamma(p)\right)\left(\frac{\partial \varphi}{\partial x_i}(x)\right) = \sum_{j=1}^{n} f_{ji}(x)\frac{\partial \varphi}{\partial x_j}(x),$$

for some functions f_{ji} on U and comparing terms on (2.4), we have

$$v_i(x) = \sum_{j=1}^{n} f_{ij}(x)u_j(x).$$

This way,

$$\frac{\partial v_i}{\partial x_j}(0) = \sum_{k=1}^{n}\left[\frac{\partial f_{ik}}{\partial x_j}(0)u_k(0) + f_{ik}(0)\frac{\partial u_k}{\partial x_j}(0)\right] = \sum_{k=1}^{n} f_{ik}(0)\frac{\partial u_k}{\partial x_j}(0),$$

and, as matrices,

$$\left(\frac{\partial v_i}{\partial x_j}(0)\right) = (f_{ij}(0))\left(\frac{\partial u_i}{\partial x_j}(0)\right). \qquad (2.5)$$

Thus, since both matrices on the right-hand side are invertible, so is the one on the left-hand side. By Proposition 1.42, each a_i is a simple singularity for v, and the same argument guarantees this property for each b_i.

Now notice that since $(f_{ij}(0))$ is the matrix of $\Gamma_{a_i}^{-1} \circ D\gamma(a_i)$ on a certain basis and a is a singularity of index one for u, (2.5) implies that $\det\left(\frac{\partial v_i}{\partial x_j}(0)\right) > 0$ if and only if $\Gamma_{a_i}^{-1} \circ D\gamma(a_i)$ preserves orientation. But the maps Γ_p preserve orientation. Consequently, the index of v at a_i is one precisely when $D\gamma(a_i)$ preserves orientation, that is to say, $I_{a_i}(v) = \text{sign}_{a_i}(\gamma)$ holds (and analogously for the b_i). Hence,

$$2\deg(\gamma) = \sum_{i=1}^{r} \text{sign}_{a_i}(\gamma) + \sum_{i=1}^{s} \text{sign}_{b_i}(\gamma) = \sum_{v(p)=0} I_p(v) = \chi(M).$$

$\square$

A relevant question concerning a compact, connected and oriented immersed hypersurface M^n of $\overline{M}^{n+1}$ with normal $\eta : M \to T\overline{M}$ is the following: what values can the normal degree $\deg\gamma$ assume when we vary the immersion? This question has been raised and answered by Hopf when $\overline{M} = \mathbb{R}^{n+1}$ and n is even (see [14] and [15]): the degree is uniquely determined by the formula $\deg\gamma = \frac{1}{2}\chi(M)$. When n is odd, Milnor proved in [19] that if M can be immersed in $\mathbb{R}^{n+1}$ with normal degree d then it can also be immersed with any degree d' which is congruent to d modulo 2. We have actually shown the analogue of Hopf's theorem in the proof of Theorem 2.18. As to what are the possible values of $\deg\gamma$ when n is odd, we conjecture that the same result of Milnor holds. We hope to demonstrate it in a future work.

2.2   An example and some consequences

In this section we will investigate the earlier constructions when the ambient manifold is the sphere with a point deleted. Later, we derive a rigidity theorem for hypersurfaces of the sphere and give an alternative proof for Theorem 1.1 in [31].

Let $\overline{M}$ be the unit sphere $\mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$ with a point $-p_0$ removed, which we denote by $\mathbb{S}^{n+1}_{-p_0}$, and let V be the tangent space of the sphere at p_0. The metrics of $\mathbb{S}^{n+1}$ and V are those induced from $\mathbb{R}^{n+2}$. Given two non-antipodal points p, q in the sphere, let $\tau_p^q : T_p\mathbb{S}^{n+1} \to T_q\mathbb{S}^{n+1}$ be the parallel transport along the unique geodesic joining p to q (we use the convention that that τ_p^p is the identity of $T_p\mathbb{S}^{n+1}$). Since this map is a linear isometry, we define $\Gamma : T\mathbb{S}^{n+1}_{-p_0} \to \mathbb{S}^{n+1}_{-p_0} \times V$ by

$$\Gamma(p, v) = \left(p, \tau_p^{p_0}(v)\right), \quad (p, v) \in T\mathbb{S}^{n+1}_{-p_0}.$$

If M^n is an orientable immersed hypersurface of $\mathbb{S}^{n+1}_{-p_0}$ and $\eta : M \to \mathbb{R}^{n+2}$ is a unit normal vector field (tangent to the sphere), the Gauss map γ at a point p is just the parallel transport of the normal $\eta(p)$ to M along the geodesic joining p to p_0. The next proposition contains the relevant information we will need.

Proposition 2.19. *Let p and q be non-antipodal points in $\mathbb{S}^{n+1}$. With the above notations, the following formulae hold:*

(i)

$$\tau_p^q(v) = -\left[\frac{\langle v, q \rangle}{1 + \langle q, p \rangle}\right](q + p) + v, \quad v \in T_p\mathbb{S}^{n+1}.$$

(ii)

$$\gamma(p) = -\left[\frac{\langle \eta(p), p_0 \rangle}{1 + \langle p, p_0 \rangle}\right](p + p_0) + \eta(p).$$

(iii)

$$\alpha_p(X) = \left[\frac{\langle \eta(p), p_0 \rangle}{1 + \langle p, p_0 \rangle}\right]X, \quad X \in T_pM.$$

Proof. To prove the first two items, let $\beta : [0, t_q] \to \mathbb{S}^{n+1}$ be the unit speed geodesic joining p to q, given by

$$\beta(t) = (\cos t)p + (\sin t)\overline{q}, \quad t \in [0, t_q],$$

where

$$\overline{q} = \frac{q - \langle q, p \rangle p}{\|q - \langle q, p \rangle p\|} = \frac{q - \langle q, p \rangle p}{\sqrt{1 - \langle q, p \rangle^2}}.$$

For fixed $v \in T_p\mathbb{S}^{n+1}$, let $X : [0, t_q] \to \mathbb{R}^{n+2}$ be the parallel vector field along β and tangent to the sphere with prescribed initial value $X(0) = v$. Differentiating $\langle X, \beta \rangle \equiv 0$, we obtain

$$-\langle X', \beta \rangle \equiv \langle X, \beta' \rangle, \tag{2.6}$$

and since X and β' are parallel along β, $\langle X, \beta' \rangle$ is constant, equal to $C \in \mathbb{R}$, say, with

$$C = \langle X(0), \beta'(0) \rangle = \langle v, \overline{q} \rangle = \frac{\langle v, q \rangle}{\sqrt{1 - \langle q, p \rangle^2}}.$$

The equation for X to be a parallel vector field is $X' - \langle X', \beta \rangle \beta \equiv 0$. Writing $X = (x_1, \ldots, x_{n+2})$, using (2.6) and the expression for β, we have

$$X'(t) = -C \left[(\cos t)p + (\sin t)\overline{q} \right], \quad t \in [0, t_q].$$

The solution satisfying $X(0) = v$ is then

$$X(t) = C \left[(\cos t - 1)\overline{q} - (\sin t)p \right] + v, \quad t \in [0, t_q].$$

Noticing that $\cos t_q = \langle q, p \rangle$ and $\sin t_q = \sqrt{1 - \langle q, p \rangle^2}$, we finally obtain

$$\tau_p^q(v) = X(t_p) = - \left[\frac{\langle v, q \rangle}{1 + \langle q, p \rangle} \right] (q + p) + v$$

and

$$\gamma(p) = \tau_p^{p_0}(\eta(p)) = - \left[\frac{\langle \eta(p), p_0 \rangle}{1 + \langle p, p_0 \rangle} \right] (p + p_0) + \eta(p),$$

as required.

For the last item, let $w = \gamma(p)$. Recall that $\tilde{w} \in \mathfrak{X}\left(\mathbb{S}_{-p_0}^{n+1} \right)$ is the invariant vector field associated with w, and $\tilde{w} = \widetilde{\eta(p)}$. From (i) we have

$$\tilde{w}(q) = \tau_{p_0}^q(w) = - \left[\frac{\langle w, q \rangle}{1 + \langle q, p_0 \rangle} \right] (q + p_0) + w, \quad q \in \mathbb{S}_{-p_0}^{n+1}. \qquad (2.7)$$

If $\widetilde{\nabla}$ denotes the Riemannian connection of $\mathbb{R}^{n+2}$, then

$$\alpha_p(v) = \overline{\nabla}_v \tilde{w} = \widetilde{\nabla}_v \tilde{w} - \langle \widetilde{\nabla}_v \tilde{w}, p \rangle p, \quad v \in T_p M.$$

A straightforward calculation shows that

$$\widetilde{\nabla}_v \tilde{w} = \left[\frac{-\langle w, v \rangle (1 + \langle p, p_0 \rangle) + \langle w, p \rangle \langle v, p_0 \rangle}{(1 + \langle p, p_0 \rangle)^2} \right] (p + p_0) - \left[\frac{\langle w, p \rangle}{1 + \langle p, p_0 \rangle} \right] v.$$
$$(2.8)$$

Notice that $\langle v, \tilde{w}(p) \rangle = \langle v, \eta(p) \rangle = 0$, since $v \in T_p M$. Using (2.7), we have

$$-\langle w, v \rangle (1 + \langle p, p_0 \rangle) + \langle w, p \rangle \langle v, p_0 \rangle = 0.$$

Substituting this in (2.8), we obtain

$$\widetilde{\nabla}_v \tilde{w} = - \left[\frac{\langle w, p \rangle}{1 + \langle p, p_0 \rangle} \right] v.$$

Then, using formula (ii) for $\gamma(p)$,

$$\langle w, p \rangle = \langle \gamma(p), p \rangle = - \left[\frac{\langle \eta(p), p_0 \rangle}{1 + \langle p, p_0 \rangle} \right] \langle p + p_0, p \rangle + \langle \eta(p), p \rangle = -\langle \eta(p), p_0 \rangle.$$

Hence,

$$\alpha_p(v) = \widetilde{\nabla}_v \tilde{w} = \left[\frac{\langle \eta(p), p_0 \rangle}{1 + \langle p, p_0 \rangle} \right] v, \quad v \in T_p M.$$

$\square$

We now state our rigidity theorem.

Theorem 2.20 (Theorem 1.3, [17]). *Let M^n be a compact, connected and oriented immersed hypersurface of $\mathbb{S}^{n+1}$, $n \geq 2$, and let R be the radius of the smallest geodesic ball containing M. If the principal curvatures $\lambda_1, \ldots, \lambda_n$ of M satisfy*

$$|\lambda_i| > \tan \left(\frac{R}{2} \right), \quad \forall i \in \{1, \ldots, n\},$$

then M is diffeomorphic to $\mathbb{S}^n$. Moreover, for any $\varepsilon \in (0, \sqrt{2} - 1)$ there exists a compact, connected and oriented immersed hypersurface M_ε of $\mathbb{S}^{n+1}$ whose principal curvatures satisfy

$$|\lambda_i| > \varepsilon \tan \left(\frac{R}{2} \right), \quad \forall i \in \{1, \ldots, n\}, \tag{2.9}$$

but M_ε is not homeomorphic to a sphere.

The above mentioned hypersurfaces M_ε are given by the following lemma, as we shall see later.

Lemma 2.21. *For a parameter $r \in (0, 1)$, let*

$$M_r = \mathbb{S}^1(r) \times \mathbb{S}^{n-1}(s) = \left\{ (x, y) \in \mathbb{R}^2 \times \mathbb{R}^n : \|x\| = r, \|y\| = s \right\} \subset \mathbb{S}^{n+1},$$

where $s = \sqrt{1 - r^2}$. If R is the radius of the largest open geodesic ball of $\mathbb{S}^{n+1}$ which does not intersect M_r, then

$$\cos R = \min\{r, s\}.$$

Proof. Recall that the distance between two points p, q in the sphere $\mathbb{S}^{n+1}$ is given by $\arccos\langle p, q \rangle$, so that

$$\cos R = \inf \left\{ \sup\{ \langle p, q \rangle : q \in M_r \} : p \in \mathbb{S}^{n+1} \right\}.$$

Writing $p = (x, y) \in \mathbb{R}^2 \times \mathbb{R}^n$, we have

$$
\begin{aligned}
\sup\{ \langle p, q \rangle : q \in M_r \} &= \sup\{ \langle x, u \rangle + \langle y, v \rangle : (u, v) \in M_r \} \\
&= r \|x\| + s \|y\|.
\end{aligned}
$$

Thus,

$$\cos R = \inf \left\{ r \|x\| + s \|y\| : (x, y) \in \mathbb{S}^{n+1} \right\} = \min\{r, s\}.$$

$\square$

Proof of Theorem 2.20. Let $\eta : M \to \mathbb{R}^{n+2}$ be the unit normal vector field which gives rise to the orientation of M and let p_0 be the center of a geodesic ball of radius R containing M. Define a function $c : M \to \mathbb{R}$ by

$$c(p) = \frac{\langle \eta(p), p_0 \rangle}{1 + \langle p, p_0 \rangle}, \quad p \in M$$

and a vector field $E \in \mathfrak{X}(\mathbb{S}^{n+1})$ by

$$E(p) = p_0 - \langle p, p_0 \rangle p, \quad p \in M.$$

Note that $\langle \eta(p), E(p) \rangle = \langle \eta(p), p_0 \rangle$ for p in M. Then, using Cauchy-Schwarz inequality, we have the following estimate for c:

$$|c(p)| \leq \frac{\|\eta(p)\| \, \|E(p)\|}{1 + \langle p, p_0 \rangle} = \frac{\sqrt{1 - \langle p, p_0 \rangle^2}}{1 + \langle p, p_0 \rangle} = \sqrt{\frac{1 - \langle p, p_0 \rangle}{1 + \langle p, p_0 \rangle}}, \quad \forall p \in M.$$

Thus,

$$|c(p)| \leq \sqrt{\frac{1 - \cos d(p, p_0)}{1 + \cos d(p, p_0)}} = \tan\left(\frac{d(p, p_0)}{2}\right) \leq \tan\left(\frac{R}{2}\right), \quad \forall p \in M.$$

Let $p \in M$. Choosing an orthonormal basis of T_pM that diagonalises the shape operator A_p, the matrix of $-\Gamma_p^{-1} \circ D\gamma(p)$ with respect to this basis is diagonal with entries $\lambda_i(p) + c(p) \neq 0$. Therefore, this map is an isomorphism for each $p \in M$, and so is $D\gamma(p)$. Since M is compact, γ is a covering map, and as M is connected with $n \geq 2$, γ is a diffeomorphism. This proves the first statement of the theorem.

For the second part, let $\varepsilon \in (0, \sqrt{2} - 1)$. We will show that it is possible to choose $r \in I = \left(0, \frac{1}{\sqrt{2}}\right]$ so that the principal curvatures of the hypersurface $M_r \subset \mathbb{S}^{n+1}$ from Lemma 2.21 satisfy (2.9).

For any $r \in (0, 1)$, the principal curvatures λ_i of M_r are constant, with

$$\lambda_1 = -\frac{\sqrt{1 - r^2}}{r}$$

and

$$\lambda_2 = \cdots = \lambda_n = \frac{r}{\sqrt{1 - r^2}}.$$

If $r \in I$ then $r \leq \sqrt{1 - r^2}$ and, according to Lemma 2.21, $\cos R = r$. A simple calculation then shows that (2.9) holds if and only if $r \in J_\varepsilon = \left(\frac{\varepsilon}{1-\varepsilon}, \frac{1}{1+\varepsilon}\right)$. Since $\varepsilon \in (0, \sqrt{2} - 1)$, we have $J_\varepsilon \neq \emptyset$ and $I \cap J_\varepsilon \neq \emptyset$. Thus, any r in this intersection is suitable for our purposes. $\qquad \square$

Remark 2.22. It is an open question to know whether there exists a manifold M_ε not homeomorphic to a sphere for which inequality 2.9 holds true for any ε close to 1. This would show that the lower bound $\tan(R/2)$ in the theorem is optimal.

Lastly, we provide an alternative proof for Theorem 1.1 in [31]:

Theorem 2.23. *Let M^n be a compact, connected and oriented immersed hypersurface of $\mathbb{S}^{n+1}$, $n \geq 2$, with nowhere vanishing Gauss-Kronecker curvature. If M is contained in an open hemisphere, then M is diffeomorphic to $\mathbb{S}^n$.*

To begin with, we introduce some ingredients that will be used in the proof of Theorem 2.23. Let p_0 be the north pole of $\mathbb{S}^{n+1}$ and let $\mathbb{S}^{n+1}_+$ be the open hemisphere centered at p_0. The Beltrami map $B : \mathbb{S}^{n+1}_+ \to \mathbb{R}^{n+1} \cong T_{p_0}\mathbb{S}^{n+1}$ is the diffeomorphism obtained by central projection:

$$B(p) = \left(\frac{p_1}{p_{n+2}}, \ldots, \frac{p_{n+1}}{p_{n+2}} \right), \quad p = (p_1, \ldots p_{n+2}) \in \mathbb{S}^{n+1}_+.$$

For $t > 0$, let $H_t : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be the homothety $x \mapsto tx$. The map we are interested in is $C_t = B^{-1} \circ H_t \circ B$. It can be shown that

$$C_t(p) = \frac{m_t(p)}{\|m_t(p)\|}, \quad p \in \mathbb{S}^{n+1}_+,$$

where $m_t : S^{n+1}_+ \to \mathbb{R}^{n+2} \setminus \{0\}$ is defined by

$$m_t(p) = \left(p_1, \ldots, p_{n+1}, \frac{p_{n+2}}{t} \right), \quad p = (p_1, \ldots p_{n+2}) \in \mathbb{S}^{n+1}_+. \tag{2.10}$$

It holds that

$$DC_t(p) \cdot v = \frac{1}{\|m_t(p)\|} \left\{ \left[\frac{(t-1)\langle v, p_0 \rangle}{t^2 \|m_t(p)\|^2} \right] [(t+1)\langle p, p_0 \rangle p - tp_0] + v \right\},$$

for $(p, v) \in T\mathbb{S}^{n+1}_+$.

Let M be an oriented hypersurface of $\mathbb{S}^{n+1}$ with unit normal vector field $\eta : M \to \mathbb{R}^{n+2}$. Recall that the second fundamental form of M at a point p (in the direction of η) is the quadratic form $\mathrm{II}_p : T_pM \to \mathbb{R}$ induced by the shape operator A_p, that is,

$$\mathrm{II}_p(v) = \langle A_p(v), v \rangle, \quad v \in T_pM.$$

Alternatively, if $\alpha : (-\varepsilon, \varepsilon) \to M$ is a curve with $\alpha(0) = p$ and $\alpha'(0) = v$, then

$$\text{II}_p(v) = \langle \alpha''(0), \eta(p) \rangle,$$

where the double prime indicates the usual second derivative, regarding α as a curve in $\mathbb{R}^{n+2}$.

Proof of Theorem 2.23. After a rotation, we may suppose M is contained in $\mathbb{S}^{n+1}_+$. By Theorem 2.20 (with $R = \frac{\pi}{2}$), M would be diffeomorphic to $\mathbb{S}^n$ if all its principal curvatures were bigger than 1 in absolute value. This is not necessarily true. However, defining $M_t = C_t(M)$, we will show that if t is sufficiently small, then this bound on the principal curvatures holds for M_t. So, M_t, and hence M, will be difeomorphic to $\mathbb{S}^n$

Let $\eta : M \to \mathbb{R}^{n+2}$ be the unit normal vector field that induces the orientation of M. One can directly check that the vector field $\eta_t : M_t \to \mathbb{R}^{n+2}$ given by

$$\eta_t(C_t(p)) = \frac{\eta(p) + (t-1)\langle \eta(p), p_0 \rangle p_0}{\sqrt{1 + (t^2-1)\langle \eta(p), p_0 \rangle^2}}, \quad p \in M, \tag{2.11}$$

is normal to M_t and has unit length.

Claim. *The following relation between the second fundamental forms* II *and* II^t *of M and M_t with respect to the normals η and η_t holds:*

$$\text{II}^t_q \left(\frac{w}{\|w\|} \right) = F_t(p,v) \text{II}_p(v),$$

where

$$F_t(p,v) = \frac{\left[(1-t^2)\langle p, p_0 \rangle^2 + t^2 \right]^{3/2}}{t \left[(1-t^2)(\langle p, p_0 \rangle^2 + \langle v, p_0 \rangle^2) + t^2 \right] \left[1 + (t^2-1)\langle \eta(p), p_0 \rangle^2 \right]^{1/2}}.$$

Proof of the Claim. Let $\alpha : (-\varepsilon, \varepsilon) \to M$ be a curve with $\alpha(0) = p$ and $\alpha'(0) = v$, with $\|v\| = 1$. Consider $\beta = C_t \circ \alpha$ the corresponding curve in M_t, with $\beta(0) = q$ and $\beta'(0) = w$.

Introducing the functions $y_t, z_t : M \to \mathbb{R}$ given by

$$y_t(p) = \frac{(t+1)\langle p, p_0 \rangle}{t \|m_t(p)\|}, \quad p \in M$$

and

$$z_t(p) = \frac{1}{\|m_t(p)\|}, \quad p \in M,$$

one has, after rearranging,

$$\beta'(s) = z_t(\alpha(s)) \left\{ \left[\frac{(t-1)\langle \alpha'(s), p_0 \rangle}{t} \right] [y_t(\alpha(s))\beta(s) - p_0] + \alpha'(s) \right\}.$$

Differentiating and evaluating at $s = 0$, we obtain

$$\beta''(0) = (Dz_t(p) \cdot v) \, \|m_t(p)\| \, w + z_t(p) \left\{ \left[\frac{(t-1)\langle \alpha''(0), p_0 \rangle}{t} \right] [y_t(p)q - p_0] \right.$$
$$\left. + \left[\frac{(t-1)\langle v, p_0 \rangle}{t} \right] [(Dy_t(p) \cdot v)\, q + y_t(p)w] + \alpha''(0) \right\}.$$

Since $\langle q, \eta_t(q) \rangle = \langle w, \eta_t(q) \rangle = 0$, we have

$$\langle \beta''(0), \eta_t(q) \rangle = z_t(p) \left[\langle \alpha''(0), \eta_t(q) \rangle - \frac{(t-1)\langle \alpha''(0), p_0 \rangle \langle \eta_t(q), p_0 \rangle}{t} \right].$$

Using expression (2.11) for η_t we arrive at

$$\mathrm{II}_q^t(w) = \langle \beta''(0), \eta_t(q) \rangle = \frac{\mathrm{II}_p(v)}{\|m_t(p)\| \, [1 + (t^2 - 1)\langle \eta(p), p_0 \rangle^2]^{1/2}}.$$

Furthermore,

$$\|w\|^2 = \frac{1}{\|m_t(p)\|^2} \left[\frac{(1 - t^2)(\langle p, p_0 \rangle^2 + \langle v, p_0 \rangle^2) + t^2}{t^2 \, \|m_t(p)\|^2} \right].$$

Thus, these two last equations and the value of $\|m_t(p)\|$ obtainable from (2.10) yield the desired relation between II_p and II_q^t $\square$

Since M is compact and contained in $\mathbb{S}_+^{n+1}$ we may choose $h, \varepsilon \in (0, 1)$ such that $\langle x, p_0 \rangle^2 \geq h$ and $\langle \eta(x), p_0 \rangle^2 < 1 - \varepsilon^2$ for all $x \in M$. We have the following estimates if $0 < t < \frac{1}{\sqrt{2}}$:

$$(1 - t^2)\langle p, p_0 \rangle^2 + t^2 \geq \frac{h}{2},$$
$$(1 - t^2)(\langle p, p_0 \rangle^2 + \langle v, p_0 \rangle^2) + t^2 \leq 3,$$
$$1 + (t^2 - 1)\langle \eta(p), p_0 \rangle^2 \leq 1.$$

This way,

$$F_t(p, v) \geq \frac{K}{t}, \quad \forall (p, v) \in TM, \ \|v\| = 1,$$

for $K = \frac{h^{3/2}}{6\sqrt{2}}$.

Let $\lambda_1 \leq \cdots \leq \lambda_n$ and $\mu_{1,t} \leq \cdots \leq \mu_{n,t}$ be the principal curvatures of M and M_t, respectively. The variational principle for eigenvalues gives

$$\lambda_j(p) = \min_{\substack{V \subseteq T_p M \\ \dim V = j}} \max_{\substack{v \in V \\ \|v\| = 1}} \mathrm{II}_p(v)$$

and

$$\mu_{j,t}(C_t(p)) = \min_{\substack{V \subseteq T_p M \\ \dim V = j}} \max_{\substack{v \in V \\ \|v\| = 1}} F_t(p, v) \mathrm{II}_p(v).$$

Note that M must contain an elliptic point, that is, a point where all principal curvatures have the same sign, which we assume to be positive. By hypothesis, all principal curvatures must be then everywhere positive. So, for every point $p \in M$ and subspace V of $T_p M$, we have

$$\max_{\substack{v \in V \\ \|v\| = 1}} F_t(p, v) \mathrm{II}_p(v) \geq F_t(p, v(V)) \, \mathrm{II}_p(v(V)) \geq \frac{K}{t} \mathrm{II}_p(v(V)),$$

where $v(V) \in V$ satisfies $\|v(V)\| = 1$ and

$$\mathrm{II}_p(v(V)) = \max_{\substack{v \in V \\ \|v\| = 1}} \mathrm{II}_p(v) > 0.$$

Hence,

$$\mu_{j,t}(C_t(p)) \geq \min_{\substack{V \subseteq T_pM \\ \dim V = j}} \frac{K}{t}\, \mathrm{II}_p(v(V))$$

$$= \frac{K}{t} \min_{\substack{V \subseteq T_pM \\ \dim V = j}} \max_{\substack{v \in V \\ \|v\|=1}} \mathrm{II}_p(v)$$

$$= \frac{K}{t}\, \lambda_j(p).$$

Setting

$$\lambda = \min_{1 \leq j \leq n}\ \min_{p \in M}\ \lambda_j(p) > 0,$$

we have

$$\mu_{j,t}(C_t(p)) \geq \frac{K}{t}\, \lambda$$

for every $p \in M$ and $0 < t < \frac{1}{\sqrt{2}}$. Thus, provided that t is sufficiently small, all principal curvatures of $\tilde{M}_t$ are bigger than 1 in absolute value, as we wanted. $\qquad\square$

Remark 2.24. We observe that the same constructions done in the sphere can be carried out in the hyperbolic space using the Lorentzian model. In particular, one can prove using the same technique that a compact, connected and orientable hypersurface of the hyperbolic space having everywhere nonzero Gauss-Kronecker curvature is diffeomorphic to a sphere. However, if the Gauss-Kronecker curvature of a hypersurface in hyperbolic space is nowhere zero, then its principal curvatures necessarily have the same sign, and then the result follows from the Proposition of [23].

Some topological invariants

In this chapter we consider closed and orientable immersed hypersurfaces of translational manifolds. Given a vector field on such a hypersurface, we define a perturbation of its Gauss map, which allows us to obtain topological invariants for the immersion that depends on the geometry of the manifold and the ambient space. Later, we use these quantities to find obstructions to the existence of certain codimension-one foliations. Apart from the first section, this chapter constitutes the article [12], in cooperation with Ícaro Gonçalves.

3.1 Foliations

This section will serve to introduce the concept of foliation and to give some examples to sharpen the reader's intuition. We recommend [6] for an excellent exposition. All manifolds in this chapter are boundaryless.

Given an integer $0 \leq k \leq n$, the Euclidean space $\mathbb{R}^n$ can be partitioned into the horizontal hyperplanes $\mathbb{R}^{n-k} \times \{c\}$, where c ranges over $\mathbb{R}^k$. Each of these is a submanifold of $\mathbb{R}^n$ of codimension k, and each point p of $\mathbb{R}^n$ is contained in only one such submanifold. This serves as a model for the general definition of a foliation.

Definition 3.1 (Foliations, I). *A (smooth) foliation of codimension k of a manifold M^n is a maximal (smooth) atlas $\mathcal{F} = \{\varphi_\alpha : U_\alpha \to M\}_\alpha$ satisfying the following conditions:*

(i) *The domain of φ_α is a product $U_\alpha = U_{\alpha,1} \times U_{\alpha,2} \subset \mathbb{R}^{n-k} \times \mathbb{R}^k$, where $U_{\alpha,1}$ and $U_{\alpha,2}$ are open balls of $\mathbb{R}^{n-k}$ and $\mathbb{R}^k$, respectively.*

(ii) *If $W = \varphi_\alpha(U_\alpha) \cap \varphi_\beta(U_\beta) \neq \emptyset$, then the change of coordinates $\varphi_\beta^{-1} \circ \varphi_\alpha : \varphi_\alpha^{-1}(W) \to \varphi_\beta^{-1}(W)$ preserves the foliation $\mathbb{R}^n = \mathbb{R}^{n-k} \times \mathbb{R}^k$, that is, it has the form*

$$(\varphi_\beta^{-1} \circ \varphi_\alpha)(x,y) = (\xi_1(x,y), \xi_2(y)), \quad (x,y) \in \mathbb{R}^{n-k} \times \mathbb{R}^k$$

The sets $\varphi_\alpha(U_{\alpha,1} \times \{c\})$ for $c \in U_{\alpha,2}$ are called plaques of $\mathcal{F}$. They are $(n-k)$-dimensional embedded submanifolds of M.
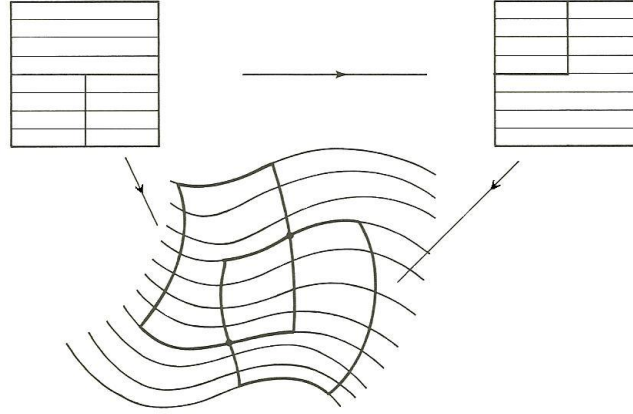

Figure 3.1: The change of coordinates of a foliation.

Alternatively, we have the following definition.

Definition 3.2 (Foliations, II)**.** *A (smooth) foliation of codimension k of a manifold M^n is a maximal collection of (smooth) maps $\mathcal{F} = \{f_\alpha : V_\alpha \to \mathbb{R}^k\}_\alpha$ satisfying the following conditions:*

(i) *Each f_α is a submersion and their domains cover M.*

(ii) *If $V_\alpha \cap V_\beta \neq \emptyset$, then there exists a local diffeomorphism $g_{\alpha\beta}$ of $\mathbb{R}^k$ such that $f_\alpha = g_{\alpha\beta} \circ f_\beta$ on $V_\alpha \cap V_\beta$.*

The maps f_α are called the distinguished maps of $\mathcal{F}$. A plaque of $\mathcal{F}$ is a connected component of $f_\alpha^{-1}(c)$, for $c \in \mathbb{R}^k$.

The curious reader is invited to check [6] for the equivalence between these definitions.

On a foliated manifold M we define the following equivalence relation: two points p and q are equivalent if there is a sequence of plaques $\alpha_1, \ldots, \alpha_r$ such that $\alpha_i \cap \alpha_{i+1} \neq \emptyset$, $p \in \alpha_1$ and $q \in \alpha_r$. The equivalence classes are called the leaves of the foliation. It is possible to prove that each leaf is a (path-connected) immersed submanifold of M. For each point p of M, we denote by $\mathcal{F}_p$ the leaf of $\mathcal{F}$ that contains p.

As the next proposition shows, each submersion gives rise to a foliation.

Proposition 3.3. *Let $f : M^m \to N^n$ be a submersion. Then there is a foliation $\mathcal{F}$ of codimension n of M whose fibres are the connected components of the level sets $f^{-1}(q)$, $q \in N$.*

Proof. By the Local Submersion Theorem, for each point p of M there exists parametrisations $\varphi : U \to M$ and $\psi : V \to N$ around p and $f(p)$ such that

(i) $U = B(0,1) \times B(0,1) \subset \mathbb{R}^{m-n} \times \mathbb{R}^n$, where $B(0,1)$ is the open unit ball in the correspondent space and $\varphi(0,0) = p$.

(ii) $V = B(0,1) \subset \mathbb{R}^n$ and $\psi(0) = f(p)$.

(iii) $\psi^{-1} \circ f \circ \varphi : B(0,1) \times B(0,1) \to B(0,1)$ coincides with the second projection $\pi_2 : (x,y) \mapsto y$.

Then the collection $\mathcal{F}$ of all parametrisations like φ form a foliation on M. Indeed, if φ_α and φ_β are elements of $\mathcal{F}$ with corresponding parametrisations ψ_α and ψ_β, then

$$\pi_2 \circ \varphi_\beta^{-1} \circ \varphi_\alpha = (\psi_\beta^{-1} \circ f \circ \varphi_\beta^{-1}) \circ \varphi_\beta \circ \varphi_\alpha = \psi_\beta^{-1} \circ f \circ \varphi_\alpha$$
$$= \psi_\beta^{-1} \circ \psi_\alpha \circ (\psi_\alpha^{-1} \circ f \circ \varphi_\alpha) = \psi_\beta^{-1} \circ \psi_\alpha \circ \pi_2,$$

that is, if

$$(\varphi_\beta^{-1} \circ \varphi_\alpha)(x,y) = (\xi_1(x,y), \xi_2(x,y)),$$

then $\xi_2(x,y) = \psi_\beta^{-1} \circ \psi_\alpha(y)$ only depends on y. This shows $\mathcal{F}$ is a foliation of M. Moreover, for any $c \in B(0,1)$, we have

$$f(\varphi(x_1,c)) = \psi(\pi_2(x_1,c)) = \psi(c) = \psi(\pi_2(x_2,c)) = f(\varphi(x_2,c)),$$

for all $x_1, x_2 \in B(0,1)$. Thus, the plaques (and hence, the leaves) of $\mathcal{F}$ are contained in the level sets of f. This completes the proof. $\square$

Example 3.4. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the map $f(x, y) = y - \alpha x$, where α is a real number. The foliation $\mathcal{F}_\alpha$ of $\mathbb{R}^2$ induced by f is by the parallel lines $y = \alpha x + c$, $c \in \mathbb{R}$.

Notice that the translations $(x, y) \mapsto (x + k, y + l)$, where $(k, l) \in \mathbb{Z}^2$, maps each leaf into another. Indeed, if $(x, y) \in f^{-1}(c)$, then

$$f(x + k, y + l) = y + l - \alpha(x + k) = (y - \alpha x) + l - \alpha k = c + l - \alpha k,$$

and so $(x + k, y + l) \in f^{-1}(c + l - \alpha k)$. Thus, $\mathcal{F}_\alpha$ descends to a foliation $\overline{\mathcal{F}}_\alpha$ on the 2-torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ (see Figure 3.2). If α is rational, all the leaves of $\overline{\mathcal{F}}_\alpha$ are homeomorphic to circles. However, if α is irrational then they are all homeomorphic to lines. Actually, in this case every leaf is dense on T^2. This shows that, in general, the leaves of a foliation are only immersed submanifolds of the ambient space.
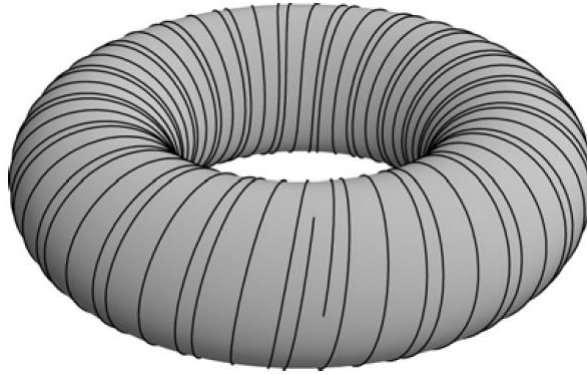

Figure 3.2: The linear foliation on the 2-torus.

Example 3.5. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be defined by $f(x, y, z) = \alpha(r^2)e^z$, where $r^2 = x^2 + y^2$ and $\alpha : \mathbb{R}_+ \to \mathbb{R}$ is a smooth function satisfying $\alpha(0) = 1$, $\alpha(1) = 0$ and $\alpha'(t) < 0$ for $t > 0$. Then f is a submersion, since

$$\nabla f(x, y, z) = (2\alpha'(r^2)xe^z, 2\alpha'(r^2)ye^z, \alpha(r^2)e^z)$$

never vanishes. So, by the above proposition, the level sets $f^{-1}(c)$ foliate $\mathbb{R}^3$. Notice that if $c = 0$, then $\alpha(r^2)e^z = 0$, that is, $r = 1$. Thus, the cylinder $C = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$ is a leaf. If $c > 0$, then

$$\alpha(r^2)e^z = c \implies e^z = \frac{c}{\alpha(r^2)} \implies z = K - ln(\alpha(r^2)),$$

for $K = ln(c)$. Consequently, the leaves inside the cylinder C are the graphs of the functions $(x, y) \mapsto K - ln(\alpha(r^2))$. Since the vertical translations by an integer amount $(x, y, z) \mapsto (x, y, z + n)$ preserve the foliation inside C, this descends to a foliation on the solid torus $\mathbb{D}^2 \times \mathbb{S}^1$, called the (orientable) Reeb foliation, see Figure 3.3. Its leaves are all homeomorphic to $\mathbb{R}^2$ and accumulate on the boundary torus $\mathbb{S}^1 \times \mathbb{S}^1$

A Reeb component of a codimension-one foliation $\mathcal{F}$ of a 3-manifold M^3 is a solid torus T inside M which is the union of leaves of $\mathcal{F}$, and such that $\mathcal{F}$ restricted to T is equivalent to the Reeb foliation of $\mathbb{D}^2 \times \mathbb{S}^1$ above.
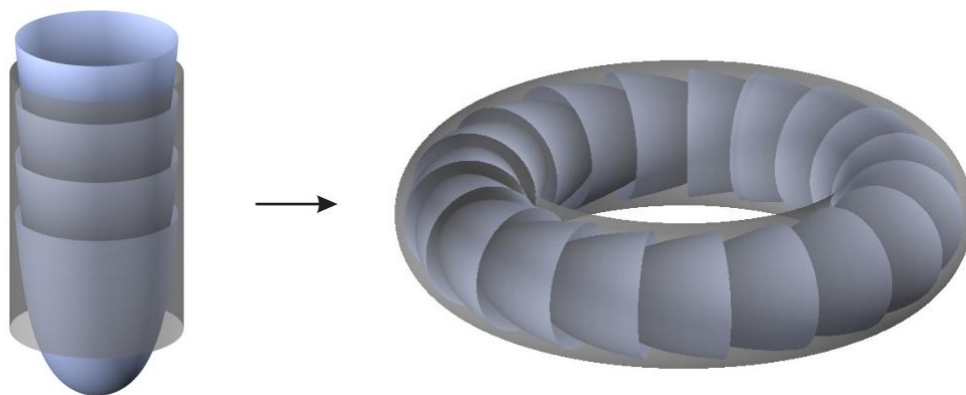

Figure 3.3: The Reeb foliation on $\mathbb{D}^2 \times \mathbb{S}^1$.

Example 3.6. Let v be a vector field without singularities on M. Then the ODE existence theorem guarantees that the integral curves of v are the leaves of a foliation on M.

Example 3.7. Let $\mathbb{S}^{2n+1}$ be the unit sphere of $\mathbb{R}^{2n+2} \cong \mathbb{C}^{n+1}$. The circle $\mathbb{S}^1 \subset \mathbb{C}$ acts on this sphere by complex multiplication:

$$\lambda \cdot (z_1, \ldots, z_{n+1}) \mapsto (\lambda z_1, \ldots, \lambda z_{n+1}).$$

The complex projective space $\mathbb{CP}^n$ is the quotient of $\mathbb{S}^{2n+1}$ by this action. The quotient map $\pi : \mathbb{S}^{2n+1} \to \mathbb{CP}^n$ is a submersion when we induce on $\mathbb{CP}^n$ the differentiable structure from the sphere. Then $\mathbb{S}^{2n+1}$ is foliated by the level sets of π, which are all homeomorphic to circles.

Example 3.8. More generally, let (E, π, B, F) be a fibre bundle. The manifolds E, B and F are called the total space, the base and the fibre, respectively. The map $\pi : E \to B$ is a submersion subject to the following local

triviality condition: for every point $b \in B$ there exists an open neighbourhood U of b and a diffeomorphism $\varphi : \pi^{-1}(U) \to U \times F$ satisfying $\pi_1 \circ \varphi = \pi|_{\pi^{-1}(U)}$, where $\pi_1 : U \times F \to U$ is the projection. The leaves of the foliation induced by π are homeomorphic to the connected components of F.

We now introduce the concept of orientability for foliations.

Definition 3.9. *Let M^n be a manifold and k a positive integer less than or equal to n. A k-referential on M is a set of k vector fields on M which are linearly independent at every point.*

Definition 3.10. *A field of k-planes on a manifold M^n is a smooth map P that assigns to each point p of M a k-dimensional subspace $P(p)$ of T_pM. The smoothness condition means that P is locally spanned by k-referentials. Explicitly, for every point p of M there exist an open neighbourhood U of p and a k-referential $\{v_1, \ldots, v_k\}$ on U such that $P(q) = \mathrm{span}\{v_1(q), \ldots, v_k(q)\}$, for every $q \in U$. A field of 1-planes is also called a line field.*

Definition 3.11. *Let P be a field of k-planes on M^n. A complementary field of planes for P is a field of $(n-k)$-planes P' on M such that $P(p) \oplus P'(p) = T_pM$ for every point p of M.*

Definition 3.12. *A k-plane field on a manifold M^n is called orientable if there exist an open covering $\{U_\alpha\}_\alpha$ of M and ordered k-referentials $v^\alpha = \{v_1^\alpha, \ldots, v_k^\alpha\}$ on U_α such that:*

(i) $P|_{U_\alpha}$ is spanned by v^α.

(ii) If $U_\alpha \cap U_\beta \neq \emptyset$ then the orientations of $P|_{U_\alpha \cap U_\beta}$ determined by $v^\alpha|_{U_\alpha \cap U_\beta}$ and $v^\beta|_{U_\alpha \cap U_\beta}$ agree.

The k-plane field is called transversely orientable if it admits an orientable complementary field of $(n-k)$-planes.

A foliation $\mathcal{F}$ of dimension k on M naturally defines a field of k-planes $T\mathcal{F}$, which assigns to every point p of M the tangent space of the leaf through p at p: $T\mathcal{F}(p) = T_p\mathcal{F}_p$. The foliation is called orientable or transversely orientable according to whether this field of k-planes is orientable or transversely orientable.

Proposition 3.13. *A line field P on a manifold M is orientable if and only if P is globally spanned by a vector field v.*

Proof. It is obvious that P is orientable if it spanned by a vector field. Reciprocally, suppose P is orientable and let $\{U_\alpha\}_\alpha$ and $\{v^\alpha\}$ be an open covering of M and vector fields on U_α satisfying conditions (i) and (ii) of Definition 3.12. Since the orientations agree on the overlaps $U_\alpha \cap U_\beta$, there are maps $h_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{R}$ such that $v^\alpha = h_{\alpha\beta} v^\beta$ and $h_{\alpha\beta} > 0$. Introduce a Riemannian metric on M and define a vector field v as follows: for a point p, let α such that $p \in U_\alpha$ and set $v(p) = \frac{v^\alpha(p)}{\|v^\alpha(p)\|}$. If $p \in U_\alpha \cap U_\beta$ then

$$\frac{v^\alpha(p)}{\|v^\alpha(p)\|} = \frac{h_{\alpha\beta}(p)v^\beta(p)}{\|h_{\alpha\beta}(p)v^\beta(p)\|} = \frac{v^\beta(p)}{\|v^\beta(p)\|},$$

so v is globally defined and clearly spans P. $\qquad\qquad\qquad\qquad\square$

Corollary 3.14. *A codimension-one foliation $\mathcal{F}$ on the Riemannian manifold M is transversely orientable if and only if there exists a unitary vector field η on M which is orthogonal to every leaf of $\mathcal{F}$.*

3.2   Some topological invariants

Throughout this section, a Riemannian manifold $\overline{M}^{n+2}$ endowed with a translational structure $\Gamma : T\overline{M} \to \overline{M} \times V$ will be fixed. Let M^{n+1} be a compact, connected and orientable manifold and let $f : M \to \overline{M}$ be an immersion of M into $\overline{M}$, with a normal unitary vector field $\eta : M \to T\overline{M}$. We identify small neighbourhoods of M and their images via f and the tangent spaces to M with their images via Df. Denote by $\overline{\nabla}$ and ∇ the Riemannian connections of $\overline{M}$ and M, respectively. Moreover, let $\mathbb{S}^{n+1}(r)$ be the sphere of radius r of V, centred at the origin; $\mathbb{S}^{n+1}$ will denote the unit sphere of V.

We prove the following theorem:

Theorem 3.15 (Theorem 1.1, [12]). *Suppose $\chi(M) = 0$. For a unitary vector field v on M and positive t, consider the map $\varphi_t^v : M \to V$ defined by*

$$\varphi_t^v(p) = \Gamma_p(\eta(p) + tv(p)), \quad p \in M.$$

Furthermore, define functions $\mu_k : M \to \mathbb{R}$ by

$$\det(D\varphi_t^v) = \sqrt{1 + t^2} \sum_{k=0}^{n} \mu_k t^k.$$

Then, the following formula holds:

$$\int_M \mu_k = \begin{cases} \deg(\gamma)c_{n+1}\binom{n/2}{k/2}, & \text{if } n \text{ and } k \text{ are even} \\ 0, & \text{if } n \text{ or } k \text{ is odd} \end{cases} \quad (3.1)$$

where $\gamma : M \to \mathbb{S}^{n+1}$ is the Gauss map associated to η and c_{n+1} is the volume of $\mathbb{S}^{n+1}$.

We begin with a lemma.

Lemma 3.16. *Let $\overline{v} : M \to \mathbb{S}^{n+1}$ be defined by $\overline{v}(p) = \Gamma_p(v(p))$. Then, the following formula holds for every point $p \in M$ and $w \in T_pM$:*

$$\Gamma_p^{-1}\left(D\overline{v}(p) \cdot w\right) = \overline{\nabla}_w v - \overline{\nabla}_w \widetilde{v(p)}$$
$$= \nabla_w v + \langle v(p), A_p(w)\rangle \eta(p) - \overline{\nabla}_w \widetilde{v(p)},$$

where $\widetilde{v(p)}$ is the invariant vector field associated with $v(p)$.

Proof. Imitate the proof of Proposition 2.16 using an orthonormal basis $\{v_1, \ldots, v_{n+2}\}$ of $T_p\overline{M}$ such that $v_{n+1} = v(p)$ and $v_{n+2} = \eta(p)$. $\qquad\square$

Proof of Theorem 3.15. The Change of Variables Formula gives

$$\int_M (\varphi_t^v)^* \omega = \deg(\varphi_t^v) \int_{\mathbb{S}^{n+1}(\sqrt{1+t^2})} \omega = \deg(\varphi_t^v) c_{n+1} \left(\sqrt{1+t^2}\right)^{n+1}, \quad (3.2)$$

where ω is the volume form of $\mathbb{S}^{n+1}(\sqrt{1+t^2})$. But $(\varphi_t^v)^* \omega = \det(D\varphi_t^v)\omega_M$, for ω_M the volume form of M. In the sequel, we will calculate this determinant.

From Lemma 3.16 and Proposition 2.16, it follows that

$$\Gamma_p^{-1}\left(D\varphi_t^v(p) \cdot w\right) = -\left(A_p(w) + \alpha_p(w)\right)$$
$$+ t\left[\nabla_w v + \langle v(p), A_p(w)\rangle \eta(p) - \overline{\nabla}_w \widetilde{v(p)}\right]. \quad (3.3)$$

Consider an orthonormal basis $\{e_1, \ldots, e_n, e_{n+1} = v(p)\}$ of T_pM. Defining u by

$$u = \frac{v(p)}{\sqrt{1+t^2}} - \frac{t\eta(p)}{\sqrt{1+t^2}},$$

one can directly check that $\{\Gamma_p(e_1), \ldots, \Gamma_p(e_n), \Gamma_p(u)\}$ is an orthonormal basis of $T_{\varphi_t^v(p)}\mathbb{S}^{n+1}(\sqrt{1+t^2})$. We will express the matrix of $D\varphi_t^v(p)$ relative to these basis.

As a matter of convention, upper case indices will run from 1 to $n+1$, whereas lower case ones will vary from 1 to n. This said, we define the following quantities:

$$h_{AB} = \langle A_p(e_B), e_A \rangle \qquad a_{ij} = \langle \nabla_{e_j} v, e_i \rangle \qquad v_i = \langle \nabla_{v(p)} v, e_i \rangle$$
$$\alpha_{AB} = \langle \alpha_p(e_B), e_A \rangle \qquad \tilde{a}_{ij} = \langle \overline{\nabla}_{e_j} \widetilde{v(p)}, e_i \rangle \qquad \tilde{v}_i = \langle \overline{\nabla}_{v(p)} \widetilde{v(p)}, e_i \rangle$$

Using (3.3) and the fact that Γ_p is an isometry, it is direct to check that (see also [4]):

$$\langle D\varphi_t^v(p) \cdot e_j, \Gamma_p(e_i) \rangle = -(h_{ij} + \alpha_{ij}) + t(a_{ij} - \tilde{a}_{ij})$$
$$\langle D\varphi_t^v(p) \cdot e_j, \Gamma_p(u) \rangle = -\sqrt{1+t^2}\,(h_{n+1,j} + \alpha_{n+1,j})$$
$$\langle D\varphi_t^v(p) \cdot v(p), \Gamma_p(e_i) \rangle = -(h_{i,n+1} + \alpha_{i,n+1}) + t(v_i - \tilde{v}_i)$$
$$\langle D\varphi_t^v(p) \cdot v(p), \Gamma_p(u) \rangle = -\sqrt{1+t^2}\,(h_{n+1,n+1} + \alpha_{n+1,n+1})$$

Defining line vectors by

$$V_i = (a_{i1} - \tilde{a}_{i1}, \ldots, a_{i,n} - \tilde{a}_{i,n}, v_i - \tilde{v}_i)$$
$$H_A = (h_{A1} + \alpha_{A1}, \ldots, h_{A,n+1} + \alpha_{A,n+1}),$$

the matrix of $D\varphi_t^v(p)$ is

$$D\varphi_t^v(p) = \begin{bmatrix} -H_1 + tV_1 \\ \vdots \\ -H_n + tV_n \\ -\sqrt{1+t^2}\,H_{n+1} \end{bmatrix}.$$

The multilinearity of the determinant gives

$$\det\left(D\varphi_t^v(p)\right) = \sqrt{1+t^2} \sum_{k=0}^{n} \mu_k t^k,$$

where

$$\mu_0 = (-1)^{n+1} \det\left(H_1, \ldots, H_n, H_{n+1}\right)$$

$$\mu_1 = (-1)^n \sum_{1 \le i \le n} \det\left(H_1, \ldots, V_i, \ldots, H_n, H_{n+1}\right)$$

$$\mu_2 = (-1)^{n-1} \sum_{1 \le i < j \le n} \det\left(H_1, \ldots, V_i, \ldots, V_j, \ldots, H_n, H_{n+1}\right)$$

$$\vdots$$

$$\mu_n = -\det\left(V_1, \ldots, V_n, H_{n+1}\right).$$

From this and (3.2), we conclude that

$$\sum_{k=0}^{n} t^k \int_M \mu_k = \begin{cases} \deg\left(\varphi_t^v\right) c_{n+1} \sum_{k=0}^{n/2} \binom{n/2}{k} t^{2k}, & \text{if } n \text{ is even} \\ \deg\left(\varphi_t^v\right) c_{n+1} \sqrt{1+t^2} \sum_{k=0}^{(n-1)/2} \binom{(n-1)/2}{k} t^{2k}, & \text{if } n \text{ is odd.} \end{cases}$$

When n is even, both sides of the above equation are polynomials in t, and the powers in the right hand side are all even, which implies that the coefficients multiplying odd powers in the left hand side all vanish. When n is odd, the presence of the factor $\sqrt{1+t^2}$ forces all coefficients in the left hand side to be zero. Furthermore, notice that $\deg\left(\varphi_t^v\right) = \deg(\gamma)$. To see why this is true, let i and i_t be the inclusions of $\mathbb{S}^{n+1}$ and $\mathbb{S}^{n+1}(\sqrt{1+t^2})$ into V. The maps $i_t \circ \varphi_t^v$ and $i \circ \gamma$ are (linearly) homotopic, so that $\deg(i_t) \deg\left(\varphi_t^v\right) = \deg(i) \deg(\gamma)$. But the degrees of both i and i_t are obviously equal to 1. Hence,

$$\int_M \mu_k = \begin{cases} \deg(\gamma) c_{n+1} \binom{n/2}{k/2}, & \text{if } n \text{ and } k \text{ are even} \\ 0, & \text{if } n \text{ or } k \text{ is odd} \end{cases}$$

completing the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3.3 Applications to foliation theory

Firstly, recall the definition of the shape operator in arbitrary codimension.

Definition 3.17. *Let L^l be an immersed submanifold of $\overline{M}^{n+2}$ with positive codimension and let $\eta : L \to T\overline{M}$ be a unit normal vector field along L. The shape operator of L at the point p in the direction of $\eta(p)$ is the linear operator $A_N : T_pL \to T_pL$ given by*

$$A_{\eta(p)}(w) = -\left(\overline{\nabla}_w \eta\right)^\top, \quad w \in T_p L,$$

where $(\cdot)^\top$ indicates projection onto $T_p L$.

Inspired by this, we extend Definition 2.14.

Definition 3.18. *The invariant shape operator of L at the point p in the direction of $\eta(p)$ is the linear operator $\alpha_{\eta(p)} : T_p L \to T_p L$ given by*

$$\alpha_{\eta(p)}(w) = \left(\overline{\nabla}_w \widetilde{\eta(p)}\right)^\top, \quad w \in T_p L.$$

Notice that $\overline{\nabla}_x \widetilde{\eta(p)} \in \{\eta(p)\}^\perp$, so that this really coincides with Definition 2.14 when the codimension of L in $\overline{M}$ is 1.

Now, let $\mathcal{F}$ be a transversely oriented codimension-one foliation of the compact, connected and oriented immersed hypersurface M^{2n+1} of $\overline{M}^{2n+2}$. Consider a unit vector field $v \in \mathfrak{X}(M)$ normal to the leaves of $\mathcal{F}$ (see Corollary 3.14). In this case, the matrices $(-a_{ij})$ and $(\tilde{a}_{ij})$ from Theorem 3.15 at a point p are the matrices of $A_{v(p)}$ and $\alpha_{v(p)}$ — the shape and invariant shape operators of the leaf $\mathcal{F}_p$ of $\mathcal{F}$ containing p — with respect to a chosen basis of $T_p \mathcal{F}_p$. Furthermore,

$$\mu_{2n} = -\det \begin{bmatrix} & & & v_1 - \tilde{v}_1 \\ & a_{ij} - \tilde{a}_{ij} & & \vdots \\ & & & v_{2n} - \tilde{v}_{2n} \\ \hline h_{2n+1,1} \cdots h_{2n+1,2n} & & h_{2n+1,2n+1} \end{bmatrix}.$$

If $\mu_{2n} = 0$, then $\deg(\gamma) = 0$ due to (3.1). This, in turn, implies that M itself is parallelisable. Indeed, consider the vector bundle map $\widetilde{\gamma}(p, v) = (\gamma(p), \Gamma_p(v))$, which covers γ:

$$\begin{array}{ccc} TM & \xrightarrow{\tilde{\gamma}} & T\mathbb{S}^{2n+1} \\ \downarrow & & \downarrow \\ M & \xrightarrow{\gamma} & \mathbb{S}^{2n+1} \end{array}$$

Notice that TM is the pullback of $T\mathbb{S}^{2n+1}$ by γ. If the degree of γ is zero, then TM is trivial, since homotopic maps induce isomorphic pullback bundles (see [32] for more details). Thus, we proved:

Theorem 3.19 (Theorem 4.2, [12])**.** *Let $\mathcal{F}$ be a transversely orientable codimension-one foliation of the compact, connected and oriented immersed hypersurface M^{2n+1} of $\overline{M}^{2n+2}$, and let v a unit length vector field tangent to M and normal to the leaves of $\mathcal{F}$. If the operator $A_v + \alpha_v$ has rank less than or equal to $2(n-1)$ along the leaves, then the degree of the Gauss map $\gamma : M \to \mathbb{S}^{2n+1}$ is equal to zero. In particular, M is parallelisable.*

Remark 3.20. Instead of a foliation, we may assume that only a transversely oriented codimension-one plane field has been given (not necessarily integrable). The hypothesis on the rank of $A_v + \alpha_v$ is the same, but now these operators lack some geometrical meaning.

Corollary 3.21 (Corollary 4.4, [12])**.** *Let G^{2n+2} be a Lie group with a left invariant metric and equipped with the translational structure of Example 2.12. Consider a compact, connected and oriented immersed hypersurface M^{2n+1} of G^{2n+2} together with a transversely orientable codimension-one foliation $\mathcal{F}$ and a unit length vector field v tangent to M and normal to the leaves of $\mathcal{F}$. If G is commutative and the rank of the shape operator of the leaves is smaller than or equal to $2(n-1)$, then the degree of the Gauss map $\gamma : M \to \mathbb{S}^{2n+1}$ is zero. In particular, M is parallelisable.*

Proof. From Example 2.15 we obtain that α_v is identically zero. The result then immediately follows from Theorem 3.19. $\qquad\square$

Corollary 3.22 (Corollary 4.5, [12])**.** *Let G and M be as in Corollary 3.21. If M is **not** parallelisable, then it does not admit transversely orientable codimension-one foliations with totally geodesic leaves.*

Remark 3.23. It is a standard fact in Lie group theory that a commutative and connected Lie group is isomorphic to $\mathbb{R}^n \times T^k$, where T^k is the k-torus.

Remark 3.24. É. Ghys classified the compact and orientable Riemannian manifolds that admit a transversely orientable codimension-one foliation with totally geodesic leaves in [11]. Such a foliation must be transverse to a locally free action of the circle or M must be a specific manifold and the foliation must be conjugated to a model, both described in this article.

The last corollary implies that the only Euclidean spheres that have a chance to carry a transversely orientable codimension-one foliation with totally geodesic leaves are those of even dimension, $\mathbb{S}^3$ and $\mathbb{S}^7$. Theorem 1 in [30], however, states that a closed manifold has a codimension-one foliation if and only if its Euler characteristic vanishes. This rules out the spheres of even dimension.

As for $\mathbb{S}^3$, we reason as follows. A corollary of Novikov's theorem (see [25], Theorem 6.4.1) states that any transversely orientable codimension-one foliation of $\mathbb{S}^3$ has a Reeb component (recall Example 3.5). Moreover, a theorem by Sullivan (see [28], Corollary 3) asserts that there exists a Riemannian metric on an oriented manifold for which the leaves of a transversely orientable codimension-one foliation are minimal if and only if every compact leaf is cut by a closed transversal curve. Since a closed curve intersecting the boundary torus of the Reeb component cannot be everywhere transversal to the foliation, there is no metric on $\mathbb{S}^3$ making the leaves of the foliation totally geodesic.

Finally, the sphere $\mathbb{S}^7$ can be discarded due to Corollary 8.3 of [11], which states that if a manifold admits a Riemannian metric of strictly positive sectional curvature, then there is no totally geodesic foliation defined on the manifold, Sfor any Riemannian metric.

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