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Matrix Representation for partitions and Mock Theta Functions

Representação Matricial para partições e funções Mock Theta

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Thesis presented to the Institute of Mathematics of the Federal University of Rio Grande do Sul in partial fulfillment of the requirements for the degree of Doctor in Mathematics.

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A mathematician, like a painter or a poet,
is a maker of patterns. If his patterns are
more permanent than theirs, it is because
they are made with ideas.
(Godfrey H. Hardy)

Resumo

Neste trabalho, com base em representações por matrizes de duas linhas para alguns tipos de partição (algumas já conhecidas e outras novas), identificamos propriedades sugeridas por classificá-las de acordo com a soma dos elementos de sua segunda linha. Esta soma sempre fornece alguma propriedade da partição relacionada. Se considerarmos versões sem sinal de algumas funções Mock Theta, seu termo geral pode ser interpretado como função geradora para algum tipo de partição com restrições. Para retornar aos coeficientes originais, é possível definir um peso para cada matriz e depois somá-las para contá-los. Uma representação análoga para essas partições nos permite observar propriedades sobre elas, novamente por meio de uma classificação referente à soma dos seu elementos da segunda linha. Esta seriação é feita por meio de tabelas criadas pelo software matemático *Maple*, as quais nos sugerem padrões e identidades relacionadas com outros tipos de partições conhecidas e, muitas vezes, encontrando uma fórmula fechada para contá-las. Tendo as conjecturas obtidas, elas são provadas por meio de bijeções entre conjuntos ou por contagem.

Palavras-chave: Partições. Funções Mock Theta. Representação Matricial. Diagrama de Young. Bijeção.

Abstract

In this work, based on representations by matrices of two lines for some kind of partition (some already known and other new ones), we identify properties suggested by classifying them according to the sum of its second line. This sum always provides some properties of the related partition. If we consider unsigned versions of some Mock Theta Functions, its general term can be interpreted as generating function for some kind of partition with restrictions. To come back to the original coefficients, you can set a weight for each array and so add them to evaluate the coefficients. An analogous representation for partitions allows us to observe properties, again by classificating them according to the sum of its elements on the second row. This classification is made by means of tables created by mathematical software *Maple*, which suggest patterns, identities related to other known types of partitions and often, finding a closed formula to count them. Having established conjectured identities, all are proved by bijections between sets or counting methods.

Keywords: Partitions. Mock Theta Functions. Matrix Representation. Young Diagram. Bijection.

Contents

Introduction

Srinivasa Ramanujan is a well known Indian mathematician born in 1887. Since his childhood, it was remarkable his intelligence and facility in playing with numbers. He earned a scholarship, but lost it due to his english been considered not enough to keep studying. Despite of it, he kept doing his studies and researches in a self-education way and then he began attending the local university as a listener. His professors noted his abilities and advised him to send the works he had done to a brilliant English professor, Gordfrey Hardy. Impressed by the sophisticated mathematics, Hardy invited him to go to England. They gave several important contributions to Mathematics, being fundamental to the development of the Theory of Partitions.

Three months before his death in 1920 at the age of 32, Ramanujan send Hardy a letter that describes functions from what he called "Mock Theta Functions". He did not give a formal definition, but explained what properties a mock theta function should have. He illustrated them by plenty of examples and also gave some properties without proving them.

In the last years, some authors have considered another combinatorial way to see partitions as two-line arrays with non-negative integer entries, subject to some restrictions. This concept started with Santos, Mondek and Ribeiro in [15], where they described a new way of representing, as two-line matrices, unrestricted partitions and several identities from Slater's list [16], including Rogers-Ramanujan Identities, and Lebesgue's Partition Identity. This new representation has been useful to prove many identities, as in [8], which are done by showing bijections between two sets that count specific partitions. Another benefit from this is that the second line describes some property of the related partition, such as number of parts, rank, number of parts below the Durfee square and other ones.

One representation for unrestricted partitions in terms of two-line matrices, given in [15], had a significant contribution to discover other identities by classifying them according to the sum on the second line. Organizing those numbers in a table, it suggested plenty of patterns and results, as we can find in [1] and [11]. After discovering and writing such identities formally, most of them can be proven by showing a bijection between both considered sets. In [10], the authors also used this to solve a problem stated in [3], which works with the concept of *lower odd parity index* of a partition. Its formal definition can be found in that paper too.

Matrix representation for partitions was also useful to find nice information about partitions into parts congruent to $\pm 1 \pmod 5$, as set in [6]. Once there is a representation for partitions into 2-distinct parts in [15], the authors found one to the other considered set of partitions in the first Rogers-Ramanujan identity. By classifying them according to the sum of second line of the related matrix, some identities were set and proven.

Another application was by showing a straight relationship between two-line matrices and coefficients of some Mock Theta Functions. By considering the general unsigned terms as generating functions for specific partitions, in [9], the authors first established a bijection between them and restricted matrices. In order to get the coefficient, they settled a weight for each matrix, whose second line also reveals characteristics about the parts of such partition. In her PhD thesis, Andrade [11] constructed tables that express the coefficients for the unsigned version, now considering the defined weight for each Mock Theta Function when we sum them.

In this work we intend to extend that representation into two-line matrices for other types of partitions, but first we need to remember some definitions, theorems and notations. They are stated in the first chapter, called *Background*, then followed by combinatorial interpretations for partitions into distinct parts, odd, even and parts which are Fibonacci numbers in the next chapter. In some cases, we built a table that classifies those partitions according to the sum on the second line for the respective matrices and get some new and interesting identities.

The third chapter, taking advantage of the representation of its unsigned version given in [9], we consider four Mock Theta Functions and construct a table similarly to the ones we did before for each one of them. Although the next chapter consider some Mock Theta Functions that can be seen as generating functions for partitions into two colors, its structure is basicaly the same.

Throughout the text, when we do not say where the theorems and propositions we present came from, it means they are new and proved by us.

1 Background

In this chapter, we intend to summarize fundamental definitions, results and identities which are going to serve as basis for all subsequent theory. Through the text, these concepts are going to be considered as known.

1.1 Partitions

We start with the main definition that goes along through the text:

Definition 1.1. *A partition of* n *is a decomposition of* n *as sum of positive integers that does not consider the order of the summands. We can write it as*

$$n = \lambda_1 + \lambda_2 + \cdots + \lambda_k \text{ or } \lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k),$$

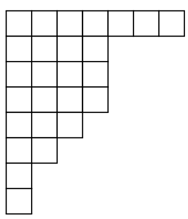
where $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k \geqslant 1$*. Each* λ_i *is called a part of the partition. We call the largest part* λ_1 *and the number of parts* $l(n)$*.*

We denote the number of partitions of n *by* $p(n)$*.*

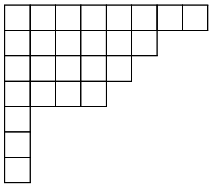
Example 1.1. *We have seven partition of* 5 *that are listed below:*

$$(5)$$
$$(4, 1)$$
$$(3, 2)$$
$$(3, 1, 1)$$
$$(2, 2, 1)$$
$$(2, 1, 1, 1)$$
$$(1, 1, 1, 1, 1)$$

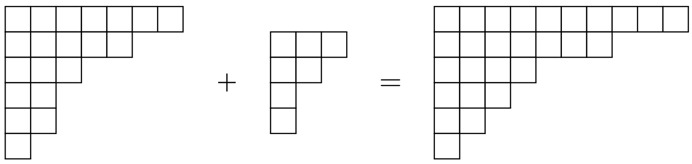
A *Young diagram* of a partition λ of n is a collection of n 1×1 squares (i; j) on a square grid $\mathbb{Z}^2$, with $1 \leqslant i \leqslant l(\lambda)$ and $1 \leqslant j \leqslant \lambda_i$. It is an important tool to see properties of partitions and a very useful representation to prove identities by doing modifications on its structure and construct bijections between sets of different kinds of partitions. Pictorially, the first coordinate i increases downward, while the second coordinate j increases from left to right. The partition $\lambda = (7, 4, 4, 4, 3, 2, 1, 1)$ of 26 has the following Young iagram

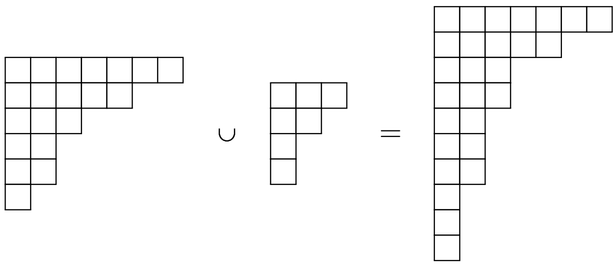


The conjugate partition λ' is obtained by exchanging rows with columns. For the previous example, we have $\lambda' = (8, 6, 5, 4, 1, 1, 1)$ and its Young diagram is



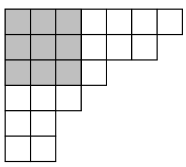
Consider $\lambda = (\lambda_1, \lambda_2, ...)$ and $\mu = (\mu_1, \mu_2, ...)$ two partitions. We define $\lambda + \mu = (\lambda_1 + \mu_1, \lambda_2 + \mu_2, ...)$ and $\lambda \cup \mu = (\lambda_1, \mu_1, \lambda_2, \mu_2, ...)$ then rearranging in non increasing order. In Young Diagrams, for example,





Definition 1.2. *We define the rank of a partition as the largest part minus the number of parts. We denote it as* $r(\lambda)$ *and we have* $r(\lambda) = \lambda_1 - l(\lambda)$.

Another definition that also has been used many times to prove identities is called the *Durfee Square*. It is the largest square that fits inside the Young Diagram. The partition $(7, 6, 4, 3, 2, 2)$ has Durfee square of size 3, as we can see in its Young Diagram



1.2 Generating Functions

Generating Function is a mathematical tool that describes an infinite sequence of numbers a_n by treating them like the coefficients of a series expansion. Unlike an ordinary series, this formal series is allowed to diverge. Although it were first introduced by Abraham de Moivre in 1730, in order to solve the general linear recurrence problem, it was Leonhard Euler who first used to solve partitions problems.

Definition 1.3. *We say that* $F(q)$ *is a generating function for a sequence* $(a_0, a_1, ...)$ *if its expansion into power series has* a_n *as coefficient for* q^n.

In the context of partitions, Generating Functions help us to count how many of them exists, as its coefficients represent the wished number. Sometimes we can not find a formula to express it, once the sequence does not fit into a representation in terms of products or sums. Although it normally does not give a closed formula for the coefficients, by using a mathematical software, we can expand it into power series and select the aimed one.

Next we present three Generating Functions for unrestricted partitions and into odd and distinct parts.

Example 1.2. *We denote by $p(n)$ the number of unrestricted partitions of n. The generating function can be expressed by*

$$(1+q+q^2+\cdots)(1+q^{2\cdot 1}+q^{2\cdot 2}+\cdots)(1+q^{3\cdot 1}+q^{3\cdot 2}+\cdots)\cdots,$$

knowing that each exponent represents how many times such part appears. For example, $q^{3\cdot 2}$ means that the part 3 appears twice. The previous expression can be rewritten as:

$$\frac{1}{(1-q)}\frac{1}{(1-q^2)}\frac{1}{(1-q^3)}\cdots = \prod_{n\geqslant 1}\frac{1}{1-q^n}.$$

Consider the first terms on its expansion

$$\sum_{n\geqslant 1} p(n)q^n = 1+q^1+2q^2+3q^3+5q^4+\cdots.$$

It allows us to see that $p(0)=1$, $p(1)=1$, $p(2)=2$, $p(3)=3$ e $p(4)=5$. In case of $n=3$ we have its three partitions 3, $2+1$ and $1+1+1$. For $n=4$, they are 4, $3+1$, $2+2$, $2+1+1$ and $1+1+1+1$.

Example 1.3. *Let $p_d(n)$ be the number of partitions of n into distinct parts. The generating function is given by*

$$\begin{aligned}\sum_{n\geqslant 0} p_d(n)q^n &= (1+q^1)\cdot(1+q^2)\cdot(1+q^3)\cdots \\ &= \prod_{n\geqslant 1}(1+q^n) \\ &= 1+q+q^2+2q^3+2q^4+\cdots\end{aligned}$$

Then we have $p_d(1)=1$, $p_d(2)=1$, $p_d(3)=2$ and $p_d(4)=2$.

Example 1.4. *Let $p_o(n)$ be the number of partitions of n into odd parts. The generating function is*

$$\begin{aligned}\sum_{n\geqslant 0} p_o(n)q^n &= (1+q+q^2+\cdots)(1+q^{3\cdot 1}+q^{3\cdot 2}+\cdots)(1+q^{5\cdot 1}+q^{5\cdot 2}+\cdots)\cdots \\ &= \frac{1}{1-q^1}\frac{1}{1-q^3}\frac{1}{1-q^5}\cdots \\ &= \prod_{k\geqslant 1}\frac{1}{1-q^{2k-1}}.\end{aligned}$$

The following notations will be used throughout the text to get the expressions simplified.

$$\begin{aligned}
(a;q)_n &= \begin{cases} 1, & n=0 \\ (1-a)\cdot(1-aq)\cdots(1-aq^{n-1}), & n \geqslant 1 \end{cases} \\
(a;q)_\infty &= \lim_{n\to\infty}(a;q)_n = \prod_{n\geqslant 0}(1-aq^n).
\end{aligned}$$

The three generating functions we presented before can be translated in this new notation by:

$$\begin{aligned}
F(q) := \sum_{n\geqslant 0} p(n)q^n &= \frac{1}{(q;q)_\infty} \\
F_d(q) := \sum_{n\geqslant 0} p_d(n)q^n &= (-q;q)_\infty \\
F_o(q) := \sum_{n\geqslant 0} p_o(n)q^n &= \frac{1}{(q;q^2)_\infty}.
\end{aligned}$$

Theorem 1.1 (Euler's Identity)**.** *The number of partitions of n into odd parts is equal to the partitions of n into distinct parts, that is, $p_d(n) = p_o(n)$. In terms of Generating Functions, it is equivalent to*

$$(-q;q)_\infty = \frac{1}{(q;q^2)_\infty}.$$

Proof. The proof we present here is about showing the equality between both generating functions. A bijective proof can be found in [4] or [13]. Consider the following identity

$$(q^2;q^2)_\infty = (-q;q)_\infty(q;q)_\infty.$$

By doing operations on both sides, we have

$$\begin{aligned}
\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} &= (-q;q)_\infty \cdot \frac{(q;q)_\infty}{(q;q^2)_\infty} \\
\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} &= (-q;q)_\infty(q^2;q^2)_\infty \\
\frac{1}{(q;q^2)_\infty} &= (-q;q)_\infty \\
F_o(q) &= F_d(q).
\end{aligned}$$

□

1.3 The Rogers-Ramanujan Identities

The Rogers-Ramanujan Identities deals with parts congruent to ± 1 and ± 2 modulo 5. The first one establishes an equality, in number, between sets of partitions into parts congruent to ± 1 modulo 5 and those ones in which the difference between consecutive parts are, at least 2. The second one is similar to the first. The changes are on the congruence, now into parts congruent to $\pm 2 \pmod 5$, and joining a restriction that all parts must be greater than 1 in the second set.

Theorem 1.2 (First Rogers-Ramanujan Identity)**.** *The number of partitions into parts congruent to* 1 *or* -1 *modulo* 5 *is equal to the number of partitions whose parts are* 2*-distinct, that is, the difference between consecutive parts is, at least* 2*. Namely,*

$$p_{\pm 1(5)}(n) = p(n,\ 2\text{-}\textit{distincts parts}).$$

In terms of generating functions, the identity becomes

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-1})(1-q^{5n-4})}.$$

Theorem 1.3 (Second Rogers-Ramanujan Identity)**.** *The number of partitions into parts congruent to* 2 *or* -2 *modulo* 5 *is equal to the number of partitions whose parts are* 2*-distinct and greater than* 1*. Namely*

$$p_{\pm 2(5)}(n) = p(n,\ 2\text{-}\textit{distincts parts greater than } 1).$$

In terms of generating functions, the identity becomes

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-2})(1-q^{5n-3})}.$$

They were first discovered by Leonard James Rogers in 1894. They were subsequently rediscovered (without a proof) by Srinivasa Ramanujan some time before 1913. Ramanujan had no proof, but rediscovered Rogers's paper in 1917, and they then published a joint new proof [14]. Even though they are about partitions, a one-to-one correspondence was only settled in the year of 1981, by Garsia and Milne [12].

1.4 Matrix Representation

In this section, we intend to give a brief description of a new representation for partition in terms of two-lines matrices that was introduced in [15]. This idea will join us throughout the text as starting point for nice results that it implies.

That paper presented two different interpretations for unrestricted partitions as two-lines matrices. They are

Theorem 1.4 (Theorem 4.1, [15])**.** *The number of unrestricted partitions of* n *is equal to the number of two-line matrices of the form*

$$\begin{pmatrix} c_1 & c_2 & \cdots & c_s \\ d_1 & d_2 & \cdots & d_s \end{pmatrix}, \tag{1.1}$$

where $c_s = 0$*,* $c_t = c_{t+1} + d_{t+1}$*, and the sum of all entries is equal to* n*.*

Theorem 1.5 (Theorem 4.3, [15])**.** *The number of unrestricted partitions of* n *is equal to the number of two-line matrices of the form*

$$\begin{pmatrix} c_1 & c_2 & \cdots & c_s \\ d_1 & d_2 & \cdots & d_s \end{pmatrix}, \tag{1.2}$$

where $d_t \neq 0$*,* $c_t \geqslant 1 + c_{t+1} + d_{t+1}$*, and the sum of all entries is equal to* n*.*

They proved both identities as a corollary from other identity. In [8], the first one was proved in two different ways by exhibiting two different bijections between unrestricted partitions and two-line matrices. One of them is built in such a way that the sums of the entries in each column of the matrix are the parts of the partition. As an example, it is like

$$(8,5,5,3,2,2,1) \leftrightarrow \begin{pmatrix} 5 & 5 & 3 & 2 & 2 & 1 & 0 \\ 3 & 0 & 2 & 1 & 0 & 1 & 1 \end{pmatrix}. \tag{1.3}$$

One can observe that the elements on the second row gives a complete description of the conjugate partition. The entry d_t represents the multiplicity of the part t in the conjugate partition,

The second bijection is built by making a connection between the number of columns and the greatest part of the partition. See the next example.

$$(6,5,2,2) \leftrightarrow \begin{pmatrix} 4 & 2 & 2 & 2 & 1 & 0 \\ 0 & 2 & 0 & 0 & 1 & 1 \end{pmatrix}. \tag{1.4}$$

Now, one can note that the elements on the second row give a complete description of the partition itself, where the entry d_t is how many times t appears as a part.

The second theorem was also proved in [8] by a bijection. Next we show an example of a partition and a matrix that are straight related by the map they constructed. In the form it was settled, the matrix would represent the parts below the Durfee square in the partition and in its conjugate.

$$\begin{aligned} (5,4,4,2,2,1) \quad &\leftrightarrow \quad \begin{pmatrix} 9 & 4 & 2 \\ 1 & 2 & 0 \end{pmatrix} \\ &= \quad \begin{pmatrix} 5 & 3 & 1 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

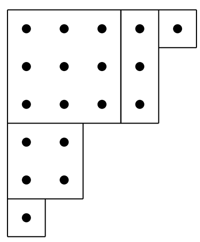


Figure 1 – Ferrers Graph of $(5,4,4,2,2,1)$

The decomposition into sum of other matrices represents the Durfee Square, parts 2 and 1 below the Durfee square and parts 3 and 1 below the Durfee square in the conjugate partition, respectively.

A similar representation was also useful to describe the coefficients of some mock theta functions, first considering their unsigned version and setting a weight to the partitions generated by their general terms. For many mock theta functions, in [9] we find a characterization of those matrices, which helps us to evaluate the coefficients of each function.

For example, the next theorem characterizes the expansion of the Mock Theta Function $\phi(q)$ into power series by describing how to count each coefficient according to the number of two-lines matrices that satisfy some relations.

Definition 1.4. *The Mock Theta Function of order* 3*,* $\phi(q)$ *is defined by*

$$\phi(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q^2;q^2)_n}.$$

If we not consider the signal, we can define the following unsigned version of $\phi(q)$:

$$\phi^*(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^2;q^2)_n}.$$

In [9], the following combinatorial interpretation for this function in terms of two-line matrices is given.

Theorem 1.6. *The coefficient of* q^n *in the expansion of the unsigned version of* $\phi(q)$ *is equal to the number of elements in the set of matrices of the form*

$$A = \begin{pmatrix} c_1 & c_2 & \cdots & c_s \\ d_1 & d_2 & \cdots & d_s \end{pmatrix}, \tag{1.5}$$

with non-negative integer entries whose sum is n*, satisfying*

$$\begin{aligned} c_s &= 1; \quad d_t \geqslant 0; \quad d_t \equiv 0 \pmod 2; \\ c_t &= 2 + c_{t+1} + d_{t+1}, \ \forall t < s. \end{aligned}$$

2 Matrix representation for partitions subjected to some restrictions

Finding a closed formula for unrestricted partitions is still a great open problem in number theory. But, setting restrictions on the parts has been useful to find nice identities that relate them, such as Euler's and Shur's identities. If we set up some conditions on the parts, we can establish a combinatorial relation to two-lines matrices, subject to some rules. The entries of such matrices might tell us properties of the respective partition.

In this chapter, we present some kind of partitions and its respective matrix representation. By summing the elements of the second line, this value represents one property for the related partition. If we classify them according to theses sums and organize them in a table, we discover identities till then unknown that it clearly reveals. In each section, we focus on partitions into distinct, odd and even parts, besides parts which are Fibonacci Numbers.

In the case of distinct parts, we show two characterizations, whereas we present a new one for partitions into odd parts which, taking advantage of Euler's Identity, in number, are the same as those ones subjected to the first restriction.

2.1 Partitions into distinct parts and rank

If we start by considering the number of parts of partitions whose parts are distinct, we can associate them to one kind of matrices. The following theorem is a combinatorial interpretation about these partitions based on how many parts they have.

Theorem 2.1. *The number of partitions of n into distinct parts is equal to the number of two-line matrices*

$$A = \begin{pmatrix} c_1 & c_2 & \cdots & c_s \\ d_1 & d_2 & \cdots & d_s \end{pmatrix}, \tag{2.1}$$

whose entries are non-negative numbers whose sum is n and satisfying the following relations

$$c_s = 1;\ d_t \geqslant 1, \tag{2.2}$$

$$c_t = 1 + c_{t+1} + d_{t+1}, \tag{2.3}$$

Proof. We are going to construct a bijection between the considered sets. First, take a partition into distinct parts,

$$n = \lambda_1 + \lambda_2 + \cdots + \lambda_s.$$

We can relate it to a matrix, whose entries sum n, in the following way:

$$A = \begin{pmatrix} \lambda_2 + 1 & \lambda_3 + 1 & \cdots & \lambda_s + 1 & 1 \\ \lambda_1 - \lambda_2 - 1 & \lambda_2 - \lambda_3 - 1 & \cdots & \lambda_{s-1} - \lambda_s - 1 & \lambda_s - 1 \end{pmatrix}, \tag{2.4}$$

It's easy to see the matrix above satisfies the conditions (2.2) and (2.3). The inverse map is described just adding up the columns and getting a partition into distinct parts, where s is the number of parts. □

Example 2.1. *By the generating function of partitions into distinct parts , we have*

$$\sum_{n\geqslant 0} p_d(n)q^n = \prod_{i\geqslant 1}(1+q^i) = 1+q+q^2+2q^3+2q^4+3q^5+4q^6+5q^7+6q^8+8q^9+\cdots.$$

If we consider $n = 8$, we have the following partitions and their respective matrix representations:

$p_d(8)$	*Matrix representation*	*Sum on the 2^{nd} line*
(8)	$\begin{pmatrix} 1 \\ 7 \end{pmatrix}$	7
$(7,1)$	$\begin{pmatrix} 2 & 1 \\ 5 & 0 \end{pmatrix}$	5
$(6,2)$	$\begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix}$	4
$(5,3)$	$\begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix}$	3
$(5,2,1)$	$\begin{pmatrix} 3 & 2 & 1 \\ 2 & 0 & 0 \end{pmatrix}$	2
$(4,3,1)$	$\begin{pmatrix} 4 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	1

By summing the entries of the second line, this number describes the rank of the related partition. Indeed,

$$\begin{aligned} d_1+d_2+\cdots+d_s &= (\lambda_s-\lambda_{s-1}-1)+(\lambda_{s-1}-\lambda_{s-2}-1)+\cdots+(\lambda_2-\lambda_1-1) \\ &= \lambda_s - s, \end{aligned}$$

which is the largest part minus the number of parts.

Remark 2.1. *It is not possible to have partitions into distinct parts whose rank is a negative number. Indeed, if m is the largest part, we have, at most, $m-1$ distinct parts besides it.*

Definition 2.1. *Let $p_r(n,k)$ be the number of partitions of n into distinct parts whose rank is k. We have that $\sum_{k=0}^{n-1} p_r(n,k) = p_d(n)$.*

Example 2.2. *We have four partitions of 18 into distinct parts and rank equals to 7, then $p_r(18,7) = 4$. They are*

$$(9,8), (10,7,1), (10,6,2) \text{ and } (10,5,3).$$

For a fixed n we classified its partitions according to their sums on the second line. By counting the appearance of sums, we can organize the data in a table, which is presented next. The entry in line n and column $n-j$ is the number of times j appears as sum of second line in type (2.1) matrices. In this case, the entry $(n, n-j)$ is $p_r(n,j)$.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
1	1																																							
2	1	0																																						
3	1	0	1																																					
4	1	0	1	0																																				
5	1	0	1	1	0																																			
6	1	0	1	1	0	1																																		
7	1	0	1	1	1	1	0																																	
8	1	0	1	1	1	1	1	0																																
9	1	0	1	1	1	2	1	1	0																															
10	1	0	1	1	1	2	1	2	0	1																														
11	1	0	1	1	1	2	2	2	1	1	0																													
12	1	0	1	1	1	2	2	2	2	2	1	0																												
13	1	0	1	1	1	2	2	3	2	3	1	1	0																											
14	1	0	1	1	1	2	2	3	2	4	2	2	1	0																										
15	1	0	1	1	1	2	2	3	3	4	3	3	2	0	1																									
16	1	0	1	1	1	2	2	3	3	4	4	4	3	2	1	0																								
17	1	0	1	1	1	2	2	3	3	5	4	5	4	3	2	1	0																							
18	1	0	1	1	1	2	2	3	3	5	4	6	5	5	4	2	1	0																						
19	1	0	1	1	1	2	2	3	3	5	5	7	6	6	5	4	2	1	0																					
20	1	0	1	1	1	2	2	3	3	5	5	7	7	7	7	6	4	2	1	0																				
21	1	0	1	1	1	2	2	3	3	5	5	7	7	8	8	8	6	4	3	0	1																			
22	1	0	1	1	1	2	2	3	3	5	5	7	7	9	9	10	8	7	5	2	1	0																		
23	1	0	1	1	1	2	2	3	3	5	5	7	8	9	10	11	10	9	8	4	3	1	0																	
24	1	0	1	1	1	2	2	3	3	5	5	7	8	9	11	12	12	12	11	7	6	2	1	0																
25	1	0	1	1	1	2	2	3	3	5	5	7	8	10	11	13	13	14	13	11	9	5	3	1	0															
26	1	0	1	1	1	2	2	3	3	5	5	7	8	10	11	14	14	16	16	14	13	9	6	2	1	0														
27	1	0	1	1	1	2	2	3	3	5	5	7	8	10	12	14	15	17	18	17	17	13	10	6	3	1	0													
28	1	0	1	1	1	2	2	3	3	5	5	7	8	10	12	14	16	18	20	20	20	18	15	10	6	3	0	1												
29	1	0	1	1	1	2	2	3	3	5	5	7	8	10	12	15	16	19	21	22	23	22	20	15	12	6	3	1	0											
30	1	0	1	1	1	2	2	3	3	5	5	7	8	10	12	15	16	20	22	24	26	26	25	21	17	11	7	3	1	0										
31	1	0	1	1	1	2	2	3	3	5	5	7	8	10	12	15	16	20	23	25	28	29	29	27	24	18	13	7	3	1	0									
32	1	0	1	1	1	2	2	3	3	5	5	7	8	10	12	15	16	20	24	26	30	32	33	32	31	25	20	13	7	3	1	0								
33	1	0	1	1	1	2	2	3	3	5	5	7	8	10	12	15	16	20	24	27	31	34	36	37	37	32	29	21	14	8	3	1	0							
34	1	0	1	1	1	2	2	3	3	5	5	7	8	10	12	15	16	20	24	28	32	36	39	41	42	40	37	30	23	14	8	3	1	0						
35	1	0	1	1	1	2	2	3	3	5	5	7	8	10	12	15	16	20	25	28	33	37	41	44	47	46	45	40	33	24	16	8	3	1	0					
36	1	0	1	1	1	2	2	3	3	5	5	7	8	10	12	15	16	20	25	28	34	38	43	47	51	52	53	49	44	35	26	16	8	4	0	1				
37	1	0	1	1	1	2	2	3	3	5	5	7	8	10	12	15	16	20	25	29	34	39	44	49	54	57	59	58	55	47	39	27	17	9	3	1	0			
38	1	0	1	1	1	2	2	3	3	5	5	7	8	10	12	15	16	20	25	29	34	40	45	51	57	61	65	66	65	59	53	40	29	18	8	4	1	0		
39	1	0	1	1	1	2	2	3	3	5	5	7	8	10	12	15	16	20	25	29	35	40	46	52	59	64	70	73	74	71	67	55	45	31	18	10	3	1	0	
40	1	0	1	1	1	2	2	3	3	5	5	7	8	10	12	15	16	20	25	29	35	40	47	53	61	67	74	79	82	82	80	71	61	47	32	20	9	4	1	0

Table 1 – Table of the characterization given by Theorem (2.1)

Looking at the 9^{th} line, from the right to the left, we have:

- no partition into distinct parts with rank 0.
- 1 partition into distinct parts with rank 1: $(4, 3, 2)$.
- 1 partition into distinct parts with rank 2: $(5, 3, 1)$.
- 2 partitions into distinct parts with rank 3: $(6, 2, 1), (5, 4)$.
- 1 partition into distinct parts with rank 4: $(6, 3)$.
- 1 partition into distinct parts with rank 5: $(7, 2)$.
- 1 partition into distinct parts with rank 6: $(8, 1)$.
- no partition into distinct parts with rank 7.
- 1 partition into distinct parts with rank 8: (9).

By observing the table, one can see below certain entry, the column becomes constant. This property is

Theorem 2.2. *For all $n, i \geqslant 0$, we have*

$$p_r(2n+3, n) = p_r(2n+3+i, n+i).$$

Proof. By establishing a bijection between the sets of partitions counted by both sides of the equality, the statement would be true. Taking a partition of $2n+3$ whose rank is n, adding up i to the largest part, we obtain a partition of $2n+i+3$ where the rank has increased the same amount, then belonging to the set counted by $r(2n+3+i, n+i)$.

The inverse map is defined just decreasing i from the largest part of a partition of $2n+3+i$ with rank equals to $n+i$. We must show that it is always possible to decrease it and still have a partition into distinct parts. Besides, we must assure that the first part minus i is also larger than the second one. By considering

$$\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_k,$$

it remains to prove that $\lambda_1 - i > \lambda_2$.

As the rank of λ is $n+i$, we have $\lambda_1 = n+k+i$. From this, we note that the partition must have more than one part, otherwise

$$|\lambda| = |\lambda_1| = n+i+1 \neq 2n+3+i.$$

If $\lambda_2 \geqslant n+k$, then $\lambda_2 = n+k+l$ for some $l \geqslant 0$. Hence the partitioned integer would be, at least,

$$\lambda_1 + \lambda_2 = (n+k+i) + (n+k+l) = 2n+2k+l.$$

As $k \geqslant 2$, it is a contradiction. So, the inverse map is well defined.

□

Theorem 2.3. *For all $n \geqslant 0$, we have*

$$p_r(2n+3, n) = p_d(n+2, \text{ parts} \geqslant 2),$$

where $p_d(n, \text{ parts} \geqslant 2)$ is the number of partitions of n into distinct parts with no part 1.

Proof. Let $P_r(2n+3, n)$ be the set of partitions of $2n+3$ into distinct parts with rank n and $P_d(n+2,, \text{ parts} \geqslant 2)$ the set of partitions of $n+2$ into distinct parts larger than or equal to 2. We shall establish a bijection between both sets, in order to have

$$|P_r(2n+3, n)| = |P_d(n+2,, \text{ parts} \geqslant 2)|.$$

So, given $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ a partition lying in $P_r(2n+3, n)$, we remove the largest part λ_1 and add 1 to the remaining parts, getting a partition $\mu = (\mu_1, \mu_2, \ldots, \mu_{k-1})$ of $n+2$ into distinct parts larger than or equal to 2, such that $\mu_1 = \lambda_2 + 1, \mu_2 = \lambda_3 + 1, \ldots, \mu_{k-1} = \lambda_k + 1$.

Conversely, given $\mu = (\mu_1, \mu_2, \ldots, \mu_t)$ a partition of $P_d(n+2, \text{ parts} \geqslant 2)$, we subtract 1 from each part and add a part of size $n+1+t$. Thus we get a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{t+1})$ of $2n+3$ into distinct parts. Note that, as

$$(\mu_1 - 1) + (\mu_2 - 1) + \ldots + (\mu_t - 1) = n + 2 - t,$$

then

$$\lambda_2 = |\mu_1 - 1| \leqslant n + 2 - t - 1 < n + 1 + t.$$

So, $|\lambda_1| = n + 1 + t$ is the greatest part of λ. Moreover, $r(\lambda) = n + 1 + t - (t+1) = n$, and the bijection is well defined. ☐

Example 2.3. *We illustrate the previous bijection by an example. Let us consider $n = 10$ and its 8 partitions that make up the set $P_r(23, 10)$.*

$P_r(23, 10)$		$p_d(12, \text{ parts} \geqslant 2)$
$(14, 6, 2, 1)$	$(6, 2, 1)$	$(7, 3, 2)$
$(14, 5, 3, 1)$	$(5, 3, 1)$	$(6, 4, 2)$
$(14, 4, 3, 2)$	$(4, 3, 2)$	$(5, 4, 3)$
$(13, 9, 1)$	$(9, 1)$	$(10, 2)$
$(13, 8, 2)$	$(8, 2)$	$(9, 3)$
$(13, 7, 3)$	$(7, 3)$	$(8, 4)$
$(13, 6, 4)$	$(6, 4)$	$(7, 5)$
$(12, 11)$	(11)	(12)

2.2 Partitions into distinct parts and number of parts

Instead of focusing on the parts, now we take as basis the greatest part of such partitions. Here we present another combinatorial way to see partitions into distinct parts. First we prove an identity for the generating function, which helps us set a straight relation to another kind of two-lines matrices.

Lemma 2.1. *The generating function for partitions into distinct parts is*

$$F_d(q) = 1 + q + \sum_{k=2}^{\infty} (1+q)(1+q^2)\cdots(1+q^{k-1})q^k.$$

Proof. Note that

$$\prod_{k=1}^{n}(1+q^k) = \prod_{k=1}^{n-1}(1+q^k) + \prod_{k=1}^{n-1}(1+q^k)q^n,$$

and, by induction, we get

$$\prod_{k=1}^{n}(1+q^k) = 1 + q + \sum_{k=2}^{n}(1+q)(1+q^2)\cdots(1+q^{k-1})q^k.$$

Once we have the generating function for partitions into distinct parts equals to

$$F_d(q) = \prod_{k=1}^{\infty}(1+q^k),$$

letting $n \to \infty$, the identity is proven. □

Theorem 2.4. *The number of partitions of n into distinct parts is equal to the number of two-line matrices*

$$A = \begin{pmatrix} c_1 & c_2 & \cdots & c_s \\ d_1 & d_2 & \cdots & d_s \end{pmatrix}, \tag{2.5}$$

whose entries are non-negative integers that sum n and satisfy the following relations:

$$c_s = 0, d_s = 1 \tag{2.6}$$

$$d_t \in \{0,1\}, \tag{2.7}$$

$$c_t = c_{t+1} + d_{t+1}, \tag{2.8}$$

Proof. By Lemma 2.1, we have that

$$f_d(q) = 1 + q + \sum_{s=2}^{\infty}(1+q)(1+q^2)\cdots(1+q^{s-1})q^s.$$

Each term in the sum of the previous representation of $f_d(q)$ generates partitions into distinct parts whose largest part is s. Consider $j_i \in \{0,1\}$ the number of times i appears as part and write this partition as

$$\lambda = j_s \cdot s + j_{s-1} \cdot (s-1) + \cdots + j_2 \cdot 2 + j_1 \cdot 1.$$

The following matrix corresponds to one that satisfies the conditions (2.6) to (2.8),

$$A = \begin{pmatrix} 1 + j_2 + \cdots + j_{s-1} & \cdots & 1 + j_{s-2} + j_{s-1} & 1 + j_{s-1} & 1 & 0 \\ j_1 & \cdots & j_{s-3} & j_{s-2} & j_{s-1} & 1 \end{pmatrix}.$$

In a inverse way, for a fixed number s of columns, we can easily relate each matrix to a partition into distinct parts with largest part s. The second row of those matrices describes the number of parts of the related partition, as we can see by summing its entries. □

Example 2.4. *In Example* 2.1*, we presented all six partitions of* 8 *into distinct parts and their respective representations into matrices whose sum of the second line represents their ranks. Now, in this new representation, the second line gives the number of parts on them. They are listed below.*

$p_d(8)$	*Matrix representation*	*Sum on the* 2^{nd} *line*
(8)	$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	1
$(7,1)$	$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	2
$(6,2)$	$\begin{pmatrix} 2 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$	2
$(5,3)$	$\begin{pmatrix} 2 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$	2
$(5,2,1)$	$\begin{pmatrix} 2 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix}$	3
$(4,3,1)$	$\begin{pmatrix} 2 & 2 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix}$	3

We classify the partitions according to the sum of the second line of its respective matrix like we did in Section 1. Now, the entry $(n, n-j)$ represents the number of partitions of n that have j distinct parts. We denote it as $p_d(n, j)$.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
1	1																																							
2	0	1																																						
3	0	1	1																																					
4	0	0	1	1																																				
5	0	0	0	2	1																																			
6	0	0	0	1	2	1																																		
7	0	0	0	0	1	3	1																																	
8	0	0	0	0	0	2	3	1																																
9	0	0	0	0	0	0	3	4	1																															
10	0	0	0	0	0	0	1	4	4	1																														
11	0	0	0	0	0	0	0	1	5	5	1																													
12	0	0	0	0	0	0	0	0	2	7	5	1																												
13	0	0	0	0	0	0	0	0	0	3	8	6	1																											
14	0	0	0	0	0	0	0	0	0	0	5	10	6	1																										
15	0	0	0	0	0	0	0	0	0	0	1	6	12	7	1																									
16	0	0	0	0	0	0	0	0	0	0	0	1	9	14	7	1																								
17	0	0	0	0	0	0	0	0	0	0	0	0	2	11	16	8	1																							
18	0	0	0	0	0	0	0	0	0	0	0	0	0	3	15	19	8	1																						
19	0	0	0	0	0	0	0	0	0	0	0	0	0	0	5	18	21	9	1																					
20	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	7	23	24	9	1																				
21	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	10	27	27	10	1																			
22	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	13	34	30	10	1																		
23	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	18	39	33	11	1																	
24	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	3	23	47	37	11	1																
25	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	5	30	54	40	12	1															
26	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	7	37	64	44	12	1														
27	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	11	47	72	48	13	1													
28	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	14	57	84	52	13	1												
29	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	20	70	94	56	14	1											
30	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	26	84	108	61	14	1										
31	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	3	35	101	120	65	15	1									
32	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	5	44	119	136	70	15	1								
33	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	7	58	141	150	75	16	1							
34	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	11	71	164	169	80	16	1						
35	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	15	90	192	185	85	17	1					
36	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	21	110	221	206	91	17	1				
37	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	28	136	255	225	96	18	1			
38	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	38	163	291	249	102	18	1		
39	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	3	49	199	333	270	108	19	1	
40	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	5	65	235	377	297	114	19	1

Table 2 – Table from the characterization given by Theorem (2.4)

Looking at the 15^{th} line, from right to left, we have:

- 1 partition into 1 distinct parts;
- 7 partitions into 2 distinct parts;
- 12 partitions into 3 distinct parts;
- 6 partitions into 4 distinct parts;
- 1 partition into 5 distinct parts;
- no partitions into more than 5 distinct parts.

The pattern of this table allows us to see some properties, listed as follows. The first one justifies the entries 0 on the table and the second one characterizes the sequences lying just above the first null entries in each column.

Theorem 2.5. *Let* $T_n = \dfrac{n(n+1)}{2}$ *be the* n^{th} *triangular number. If* $T_n \leqslant k < T_{n+1}$*, then*

$$p_d(k, n+i) = 0, \text{ for all } i \geqslant 1.$$

Proof. Let us suppose there is a partition of k into $n+i$ distinct parts, with $T_n \leqslant k < T_{n+1}$. In other words,

$$k = \lambda_1 + \lambda_2 + \cdots + \lambda_{n+i},$$

with $\lambda_t \neq \lambda_j, \forall t \neq j$. As we have $\lambda_{n-t+i+1} \geqslant t$, then

$$k = \lambda_1 + \lambda_2 + \cdots + \lambda_{n+i} \geqslant (n+i) + \cdots + 2 + 1 = T_{n+i},$$

which is an absurd. □

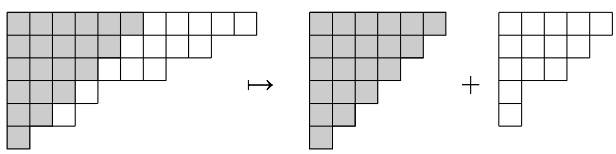
Theorem 2.6. *For* $n, k \geqslant 1$,

$$p_d(T_n + k, n) = p(k, \text{ with at most } n \text{ parts }).$$

Proof. Given a partition of $T_n + k$ into n distinct parts and considering its Young Diagram, it is possible to remove i squares from the $i-th$ smallest part, for $i = 1, 2, ..., n$. By doing this, what is left is a partition of n with no restrictions.

Conversely, by adding the partition $(n, n-1, ..., 2, 1)$ on any partition of k, we obtain a new one of $T_n + k$ into n distinct parts. Therefore, the map is well defined and it is a bijection. □

Example 2.5. *If we consider* $k = 14$ *and* $n = 6$*, we illustrate the previous bijection.*



2.3 Partitions into odd parts considering the largest part

In the two previous sections we exhibited two different forms to look at partitions into distinct parts. From Euler's Identity, the generating function for partitions into odd parts is the same as partitions whose parts are distinct. We have that

$$F_o(q) = \prod_{i\geqslant 1} \frac{1}{1-q^{2i-1}} = \prod_{i\geqslant 1}(1+q^i) = F_d(q).$$

Here we prove an equality for $F_o(q)$ which allows us to set a new relation between partitions into odd parts and two-lines matrices. As a consequence, these three representations are, in number, the same.

Lemma 2.2. *We have*

$$F_o(q) = \sum_{n=1}^{\infty} \frac{q^n}{(q^2;q^2)_n}.$$

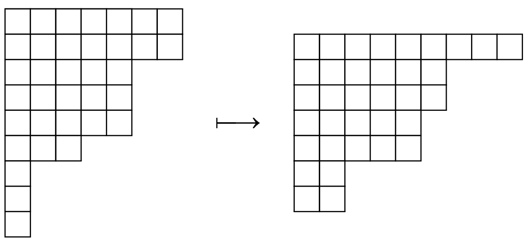
Proof. The general term

$$\frac{q^s}{(q^2;q^2)_s} = \frac{q^s}{(1-q^2)(1-q^4)\cdots(1-q^{2s})}$$

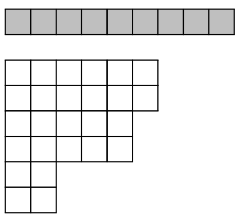
generates partitions into two colors (dark and light gray) in which there is only one light gray part s and any number of dark even parts less than or equal to $2s$.

We prove the statement by building a bijection between partitions into odd parts and those ones into two colors that satisfy the conditions above. In order to make the bijection easier to understand, as we explain the steps, we use an example.

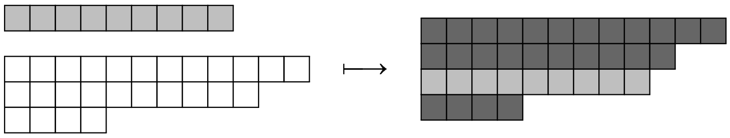
First, take a partition λ into odd parts and consider its conjugated λ'.



Separate the first part λ'_1 and turn it into the unique light gray part.



As the original partition had only odd parts, the remaining parts, after this process, appear in pairs. So, merge each pair of parts and paint them with dark gray.



It is clear that the resulting parts are, at most, 2 times λ'_1. The reverse process is easily constructible. ☐

Theorem 2.7. *The number of partitions of n into odd parts (equivalently, distinct parts) is equal to the number of two-line matrices in the form* 2.1*, with non-negative integers as entries whose sum is n and satisfy the relations:*

$$c_s = 1; \tag{2.9}$$
$$d_i \equiv 0 \pmod 2; \tag{2.10}$$
$$c_t = c_{t+1} + d_{t+1}, t < s. \tag{2.11}$$

Proof. Instead of taking a partition into odd parts, we are going to work with partitions into two colors generated by

$$\sum_{n=1}^{\infty} \frac{q^n}{(q^2;q^2)_n}.$$

Both sets have the same cardinality as settled by Lemma 2.2. Write this partition as

$$n = s + j_1 \cdot 2 + j_2 \cdot 4 + \cdots + j_s \cdot 2s,$$

where s is the unique light gray part and j_i counts the appearance of the even dark gray part $2i$. Rearrange it as

$$n = (2j_1 + 2j_2 + \cdots + 2j_s) + 1 + (1 + 2j_s) + (1 + 2j_s + 2j_{s-1}) + \cdots + (1 + 2j_s + \cdots + 2j_2).$$

Organizing them in a matrix in the form

$$A = \begin{pmatrix} 1 + 2j_s + 2j_{s-1} + \cdots + 2j_2 & \cdots & 1 + 2j_s + 2j_{s-1} & 1 + 2j_s & 1 \\ 2j_1 & \cdots & 2jd_{s-2} & 2j_{s-1} & 2j_s \end{pmatrix},$$

it satisfies the conditions (2.9) to (2.11). The inverse map can be easily built based on the entries on the second line that a half of d_i is how many times $2i$ appears as a part. ☐

Remark 2.2. *Again we want to sum all entries in the second line to seek an information for the related partition. Now, following the inverse map of the bijection given as proof for Lemma* 2.2*, this value plus* 1 *represents the largest part of the partition into odd parts.*

Example 2.6. *In order to distinguish light and dark gray parts, we bold the dark ones. We consider $n = 8$, that we know, from Example* 2.1*, there are six partitions whose parts are odd numbers. Next we show them as well as the respective partitions into two colors and this new matrix representation. Remember that the sum of second line plus* 1 *gives the largest part for the original partition.*

$p_o(8)$		*Matrix representation*	*Sum on the 2^{nd} line plus 1*
$(7,1)$	$(2,\mathbf{2},\mathbf{2},\mathbf{2})$	$\begin{pmatrix} 1 & 1 \\ 6 & 0 \end{pmatrix}$	7
$(5,3)$	$(\mathbf{4},2,\mathbf{2})$	$\begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}$	5
$(5,1,1,1)$	$(4,\mathbf{2},\mathbf{2})$	$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 4 & 0 & 0 & 0 \end{pmatrix}$	5
$(3,3,1,1)$	$(4,\mathbf{4})$	$\begin{pmatrix} 3 & 1 & 1 & 1 \\ 0 & 2 & 0 & 0 \end{pmatrix}$	3
$(3,1,1,1,1,1)$	$(6,\mathbf{2})$	$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	3
$(1,1,1,1,1,1,1,1)$	(8)	$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	1

2.4 Partitions into even parts

If we set that the parts must be always even, the generating function for this kind of partitions is

$$F_e(q) = \prod_{i\geqslant 1} \frac{1}{1-q^{2i}}.$$

As before, we can associate each one of those partitions of n to two-lines matrix whose entries sum n and satisfy some conditions. This theorem is proved next.

Theorem 2.8. *The number of partitions of n into even parts is equal to the number of two-line matrices*

$$A = \begin{pmatrix} c_1 & c_2 & \cdots & c_s \\ d_1 & d_2 & \cdots & d_s \end{pmatrix},$$

whose entries are non-negative integers summing n and satisfy the following relations:

$$c_s = d_s; \tag{2.12}$$

$$c_t = d_t + c_{t+1} + d_{t+1}, t < s. \tag{2.13}$$

Proof. We prove this statement by showing a bijection between both sets considered. We start considering a partition into even parts of n, like

$$n = j_s \cdot 2 \cdot s + \cdots + j_2 \cdot 2 \cdot 2 + j_1 \cdot 2 \cdot 1.$$

Once the parts must be even numbers, it is clear that we only have partitions for even values of n. Then, write the previous partitions as

$$n = (j_1 + \cdots + j_s) + j_s + (j_{s-1} + 2j_s) + (j_{s-2} + 2j_{s-1} + 2j_s) + \cdots + (j_1 + 2j_2 + \cdots + 2j_s).$$

Consider the following matrix

$$A = \begin{pmatrix} j_1 + 2j_2 + \cdots + 2j_s & \cdots & j_{s-2} + 2j_{s-1} + 2j_s & j_{s-1} + 2j_s & j_s \\ j_1 & \cdots & j_{s-2} & j_{s-1} & j_s \end{pmatrix}.$$

It corresponds to one that satisfies the conditions (2.12) and (2.13).

In a inverse way, for a fixed number s of columns, we can easily relate each matrix to a partition into even parts by determining how many times each number appear in the partition. These quantities are expressed on the second line. When we add up these entries, we obtain the number of parts of such partitions. □

Example 2.7. *By calling $p_e(n)$ the number of partitions of n into even parts and considering its generating function, we have*

$$F_e(q) = \sum_{n\geqslant 0} p_e(n)q^n = \prod_{i\geqslant 1} \frac{1}{1-q^{2i}} = 1 + q^2 + 2q^4 + 3q^6 + 5q^8 + 7q^{10} + 11q^{12} + 15q^{14} + \cdots .$$

For $n = 10$, next we have the seven following partitions and their respective matrix representations:

$p_e(10)$	*Matrix representation*	*Sum on the second line*
(10)	$\begin{pmatrix} 2 & 2 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	1
$(8,2)$	$\begin{pmatrix} 3 & 2 & 2 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$	2
$(6,4)$	$\begin{pmatrix} 4 & 3 & 1 \\ 0 & 1 & 1 \end{pmatrix}$	2
$(6,2,2)$	$\begin{pmatrix} 4 & 2 & 1 \\ 2 & 0 & 1 \end{pmatrix}$	3
$(4,4,2)$	$\begin{pmatrix} 5 & 2 \\ 1 & 2 \end{pmatrix}$	3
$(4,2,2,2)$	$\begin{pmatrix} 5 & 1 \\ 3 & 1 \end{pmatrix}$	4
$(2,2,2,2,2)$	$\begin{pmatrix} 5 \\ 5 \end{pmatrix}$	5

This representation of partitions provides information about the parts, as we could see. As the sum of the second line gives us the number of parts, we can classify all partitions according to this sum, which is the same as classifying them according to the number of parts. The next definition will be useful for all results we are going to present.

Definition 2.2. *For non-negative integers n and k, let $p_e(n, k)$ be the number of partitions of n into k even parts. We have $\sum_k p_e(n, k) = p_e(n)$. This number is the same as the number of matrices we have considered whose sum on the second line is equal to k.*

Remark 2.3. *Besides the fact we can only split even numbers into even parts, the maximum numbers of parts is its half. So, we will only consider partitions counted by $p_e(2n, k)$, where $k \leqslant n$.*

Classifying those partitions according to the number of parts (equivalently to the sum of the second line of its matrix representation), we organize them in a table. Next we present this table, created by *Maple*, where the entry $(2n, j)$ expresses the number of partitions of $2n$ into $n - j + 1$ even parts.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37
2	1																																				
4	1	1																																			
6	1	1	1																																		
8	1	1	2	1																																	
10	1	1	2	2	1																																
12	1	1	2	3	3	1																															
14	1	1	2	3	4	3	1																														
16	1	1	2	3	5	5	4	1																													
18	1	1	2	3	5	6	7	4	1																												
20	1	1	2	3	5	7	9	8	5	1																											
22	1	1	2	3	5	7	10	11	10	5	1																										
24	1	1	2	3	5	7	11	13	15	12	6	1																									
26	1	1	2	3	5	7	11	14	18	18	14	6	1																								
28	1	1	2	3	5	7	11	15	20	23	23	16	7	1																							
30	1	1	2	3	5	7	11	15	21	26	30	27	19	7	1																						
32	1	1	2	3	5	7	11	15	22	28	35	37	34	21	8	1																					
34	1	1	2	3	5	7	11	15	22	29	38	44	47	39	24	8	1																				
36	1	1	2	3	5	7	11	15	22	30	40	49	58	57	47	27	9	1																			
38	1	1	2	3	5	7	11	15	22	30	41	52	65	71	70	54	30	9	1																		
40	1	1	2	3	5	7	11	15	22	30	42	54	70	82	90	84	64	33	10	1																	
42	1	1	2	3	5	7	11	15	22	30	42	55	73	89	105	110	101	72	37	10	1																
44	1	1	2	3	5	7	11	15	22	30	42	56	75	94	116	131	136	119	84	40	11	1															
46	1	1	2	3	5	7	11	15	22	30	42	56	76	97	123	146	164	163	141	94	44	11	1														
48	1	1	2	3	5	7	11	15	22	30	42	56	77	99	128	157	186	201	199	164	108	48	12	1													
50	1	1	2	3	5	7	11	15	22	30	42	56	77	100	131	164	201	230	248	235	192	120	52	12	1												
52	1	1	2	3	5	7	11	15	22	30	42	56	77	101	133	169	212	252	288	300	282	221	136	56	13	1											
54	1	1	2	3	5	7	11	15	22	30	42	56	77	101	134	172	219	267	318	352	364	331	255	150	61	13	1										
56	1	1	2	3	5	7	11	15	22	30	42	56	77	101	135	174	224	278	340	393	434	436	391	291	169	65	14	1									
58	1	1	2	3	5	7	11	15	22	30	42	56	77	101	135	175	227	285	355	423	488	525	522	454	333	185	70	14	1								
60	1	1	2	3	5	7	11	15	22	30	42	56	77	101	135	176	229	290	366	445	530	598	638	618	532	377	206	75	15	1							
62	1	1	2	3	5	7	11	15	22	30	42	56	77	101	135	176	230	293	373	460	560	653	732	764	733	612	427	225	80	15	1						
64	1	1	2	3	5	7	11	15	22	30	42	56	77	101	135	176	231	295	378	471	582	695	807	887	919	860	709	480	249	85	16	1					
66	1	1	2	3	5	7	11	15	22	30	42	56	77	101	135	176	231	296	381	478	597	725	863	984	1076	1090	1009	811	540	270	91	16	1				
68	1	1	2	3	5	7	11	15	22	30	42	56	77	101	135	176	231	297	383	483	608	747	905	1060	1204	1291	1297	1175	931	603	297	96	17	1			
70	1	1	2	3	5	7	11	15	22	30	42	56	77	101	135	176	231	297	384	486	615	762	935	1116	1303	1455	1549	1527	1367	1057	674	321	102	17	1		
72	1	1	2	3	5	7	11	15	22	30	42	56	77	101	135	176	231	297	385	488	620	773	957	1158	1380	1586	1761	1845	1801	1579	1206	748	351	108	18	1	
74	1	1	2	3	5	7	11	15	22	30	42	56	77	101	135	176	231	297	385	489	623	780	972	1188	1436	1686	1930	2112	2194	2104	1824	1360	831	378	114	18	1

Table 3 – Table of the characterization given by Theorem (2.8)

Looking at the line number 14, from the right to the left, we have

- 1 partition into 1 even part: (14).
- 3 partitions into 2 even parts: $(12, 2), (10, 4), (8, 6)$.
- 4 partitions into 3 even parts: $(10, 2, 2), (8, 4, 2), (6, 4, 4), (6, 6, 2)$.
- 3 partitions into 4 even parts: $(8, 2, 2, 2), (6, 4, 2, 2), (4, 4, 4, 2)$.
- 2 partitions into 5 even parts: $(6, 2, 2, 2, 2), (4, 4, 2, 2, 2)$.
- 1 partition into 6 even parts: $(4, 2, 2, 2, 2, 2)$.
- 1 partition into 7 even parts: $(2, 2, 2, 2, 2, 2, 2)$.

The table provides that some quantities remain constant in columns beneath the entries $p_e(4n, n)$. This information can be translated as

$$p_e(4n + 2i, n + i) = p_e(4n, n); i \geqslant 0.$$

If we prove that they keep constant in columns, the sequence made of these numbers is the same as the sequence of unrestricted partitions. The following theorem presents the properties we have mentioned.

Theorem 2.9. *For all $n \geqslant 1$ and $i \geqslant 0$ we have*

$$p_e(4n + 2i, n + i) = p(n)$$

Proof. The statement we are going to prove is $p_e(4n, n) = p(n)$. For the case $i \geqslant 1$, there is a similar bijection to the one we did in Theorem 2.2, which proves that $p_e(4n+2i, n+i) = p_e(4n, n)$. The only difference here is that we remove i parts 2 instead of i parts 1. Again, it is possible to assert there are, at least, i parts 2 to remove. We do it by supposing there are less than i and it becomes an absurd.

For the case $i = 0$, we start with a partition which lies in $P_e(4n, n)$ and has the form

$$4n = \lambda_1 + \lambda_2 + \cdots + \lambda_n.$$

As each part is even, we write it as $\lambda_i = 2\mu_i$. Then

$$(2\mu_1, 2\mu_2, ..., 2\mu_n)$$

is a partition of $4n$. We decrease 2 from each part, then divide the result by 2. We obtain

$$\Big(\frac{\mu_1 - 2}{2}, \frac{\mu_2 - 2}{2}, \cdots, \frac{\mu_n - 2}{2}\Big),$$

a partition of n, where some parts might possible be zero. $\square$

Example 2.8. *We describe both bijections with an example. Consider $n = 5$, $i = 3$ and the three sets $P_e(26, 8)$, $P_e(20, 5)$ and $P(5)$, whose cardinalities are 7 for all.*

$P_e(26,8)$	$P_e(20,5)$				$P(5)$
$(12,2,2,2,2,2,2,2)$	$(12,2,2,2,2)$	$\longmapsto$	(10)	$\longmapsto$	(5)
$(10,4,2,2,2,2,2,2)$	$(10,4,2,2,2)$	$\longmapsto$	$(8,2)$	$\longmapsto$	$(4,1)$
$(8,6,2,2,2,2,2,2)$	$(8,6,2,2,2)$	$\longmapsto$	$(6,4)$	$\longmapsto$	$(3,2)$
$(8,4,4,2,2,2,2,2)$	$(8,4,4,2,2)$	$\longmapsto$	$(6,2,2)$	$\longmapsto$	$(3,1,1)$
$(6,6,4,2,2,2,2,2)$	$(6,6,4,2,2)$	$\longmapsto$	$(4,4,2)$	$\longmapsto$	$(2,2,1)$
$(6,4,4,4,2,2,2,2)$	$(6,4,4,4,2)$	$\longmapsto$	$(4,2,2,2)$	$\longmapsto$	$(2,1,1,1)$
$(4,4,4,4,4,2,2,2)$	$(4,4,4,4,4)$	$\longmapsto$	$(2,2,2,2,2)$	$\longmapsto$	$(1,1,1,1,1)$

2.5 Partitions into Fibonacci Numbers

The Fibonacci sequence is a sequence of integers numbers, starting with 0 and 1, in which each subsequent term is the sum of the two previous ones. This sequence received the Italian mathematician Leonardo of Pisa's name, better known as Fibonacci, which describes, the growth of a rabbit population.

In mathematical terms, the sequence is recursively defined by the following formula

$$F_n = F_{n-1} + F_{n-2}, n \geqslant 2$$

whose initial conditions are $F_0 = 0$ and $F_1 = 1$. This sequence has as first terms

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, ...$$

Analysed as a sequence number, it is only a simple organization numerals receiving a touch of mathematical logic. But what makes this order numbers, a special discovery, is its connection with the phenomena of nature and the approximate value of the constant 1.6, the ratio between a number and its predecessor in the sequence.

In this section, we consider partitions whose parts are in the set of Fibonacci Numbers. Although we are not able to find nice information for this kind of partitions, we could get a matrix representation to them.

Theorem 2.10. *The number of partitions of n into parts that are Fibonacci numbers is equal to the number of two-lines matrices in the form given by Theorem* **??**, *with non-negative integers entries whose sum is n and satisfy the following relations:*

$$c_s = 0, \quad c_{s-1} = d_s \tag{2.14}$$

$$c_t = c_{t+2} + c_{t+1} + d_{t+1}, \quad t < s - 1. \tag{2.15}$$

Proof. We will not consider here the duplicity of the part 1 in the sequence. So, taking a partition into Fibonacci numbers, we write it as

$$n = d_1 F_2 + d_2 F_3 + \cdots + d_s F_{s+1},$$

where d_i are the multiplicities of each F_{i+1}. Each Fibonacci number can be rewritten as

$$\begin{aligned} F_2 &= 1 \\ F_3 &= 1 + F_1 \\ F_4 &= 1 + F_1 + F_2 \\ &\vdots \\ F_{s+1} &= 1 + F_1 + F_2 + F_3 + \cdots + F_{s-1}. \end{aligned}$$

Then, the partition can be read as

$$n = (d_1 + d_2 + \cdots + d_s) \cdot 1 + (d_2 + \cdots + d_s)F_1 + \cdots + d_s F_{s-1},$$

and we put on a matrix like

$$A = \begin{pmatrix} d_s F_{s-1} + d_{s-1} F_{s-2} + \cdots + d_2 F_1 & \cdots & d_s F_2 + d_{s-1} F_1 & d_s F_1 & 0 \\ d_1 & \cdots & d_{s-2} & d_{s-1} & d_s \end{pmatrix}.$$

Observe that the entries satisfy the conditions (2.14) and (2.15) and those ones in the second line express how many parts of $F_2, ..., F_{s+1}$ the correspondent partition has. The inverse map can be defined just knowing the multiplicity of each part, set on the second line. □

Example 2.9. *By calling $p_{fib}(n)$ the number of partitions into Fibonacci numbers and considering its generating function, we have*

$$F_{fib}(q) = \prod_{i \geqslant 2} \frac{1}{(1 - q^{F_i})} = 1 + q + 2q^2 + 3q^3 + 4q^4 + 6q^5 + 8q^6 + 10q^7 + \cdots.$$

For $n = 5$, next we list its six considered partitions and their respective matrix representations:

$p_{fib}(5)$	*Matrix representation*	*Sum on the* 2^{nd} *line*
(5)	$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	1
$(3,2)$	$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$	2
$(3,1,1)$	$\begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$	3
$(2,2,1)$	$\begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$	3
$(2,1,1,1)$	$\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$	4
$(1,1,1,1,1)$	$\begin{pmatrix} 0 \\ 5 \end{pmatrix}$	5

3 Some Unsigned Mock Theta Functions

If we consider some Mock Theta Functions without sign in their general term, we can read them as generating function for some kind of partitions. Setting a weight for each partition, we can turn back to evaluate the coefficients of their expansion into power series. This work was done in [9], where the authors established a straight relation between those coefficients and a set of two-line matrices, whose integer entries are subject to some rules. To do so, first they considered the unsigned version then, after setting the structure of the related matrix, they saw that the weight normally depends on the sum of the second line of this, when it is required.

Our work is based on the representation for the coefficients by a two-line matrix given in [9]. Even though most of them are presented in that paper, we prove the ones we are going to use. From this and for each function, we classify all generated partitions of an integer n which have the same sum for the second row of its associated matrix and build a table to organize it. Some patterns are clearly suggested by the table. Probably, there are many more than the ones we have already found.

In this chapter we study four Mock Theta Functions of order 5 as generating functions for partitions. Two of them will be consider without the signal and properties about its partitions will be enunciated and proved. Some results are known and, where they appear, we are going to refer where they can be found. When it is not mentioned, the identities are new.

Some identities we present next are in the article [5] - *Identities for partitions generated by the unsigned versions of some mock theta functions* - accepted for publication (Bulletin of the brazilian Mathematical Society). It is composed by a selection of the following results. They come from a combinatorial interpretation for the coefficientes of some unsigned Mock Theta Functions.

3.1 Mock Theta Function $f_1(q)$

We start by considering the Mock Theta Function of order 5

$$f_1(q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(-q;q)_n} = 1 + q^2 - q^3 + q^4 - q^5 + 2q^6 - 2q^7 + q^8 - q^9 + \cdots .$$

If we consider its expansion, we get negative coefficients brought fourth by the denominator of its general term. By removing this sign we get

$$f_1^*(q) = \sum_{s=0}^{\infty} \frac{q^{s^2+s}}{(q;q)_s} = \sum_{s=0}^{\infty} \frac{q^{2(1+2+3+\cdots+s)}}{(1-q)(1-q^2)\cdots(1-q^s)}, \tag{3.1}$$

whose general term generates the partitions of n with no gaps containing at least two parts equal to each one of the numbers $1, 2, 3, \ldots, s$. By conjugation, another interpretation can be seen as generator of partitions of n into exactly s parts such that the smallest part $\lambda_s \geqslant 2$ and the difference between consecutive parts is $\lambda_t - \lambda_{t+1} \geqslant 2$.

Definition 3.1. *Let $p_{f_1}(n)$ be the number of partitions of n where, if s is the greatest part, there are, at least, two copies of each number smaller than or equal to s. Then, we have*

$$f_1^*(q) = \sum_{n=0}^{\infty} p_{f_1}(n) q^n.$$

Example 3.1. $p_{f_1}(9) = 3$, *and the three considered partitions are*

$$(\mathbf{2}, 2, 2, \mathbf{1}, 1, 1);$$

$$(2, 2, \mathbf{1}, \mathbf{1}, \mathbf{1}, 1, 1);$$

$$(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, 1, 1).$$

The following combinatorial interpretation for the function $f_1^*(q)$ is given in [9].

Theorem 3.1. *The coefficient of q^n in the expansion of* (3.1) *is equal to the number of elements in the set of matrices of the form*

$$A = \begin{pmatrix} c_1 & c_2 & \cdots & c_s \\ d_1 & d_2 & \cdots & d_s \end{pmatrix}, \tag{3.2}$$

with non-negative integer entries whose sum is n and satisfying

$$c_s = 2;\ d_t \geqslant 0; \tag{3.3}$$

$$c_t = 2 + c_{t+1} + d_{t+1},\ \forall t < s. \tag{3.4}$$

Proof. According to the general term of (3.1), we can decompose n as

$$n = 2 \cdot 1 + 2 \cdot 2 + 2 \cdot 3 + \cdots + 2 \cdot s + (1 \cdot d_1 + 2 \cdot d_2 + \cdots + s \cdot d_s)$$

or, equivalently, as the sum of the entries of the matrix

$$A = \begin{pmatrix} 2s + d_2 + \cdots + d_s & \cdots & 6 + d_{s-1} + d_s & 4 + d_s & 2 \\ d_1 & \cdots & d_{s-2} & d_{s-1} & d_s \end{pmatrix},$$

Noting that the entries satisfy conditions (3.3) to (3.4), the theorem is proved. ☐

The second row of the matrices mentioned before describes how many parts $1, 2, ..., s$ the related partition has beyond the two copies that necessarily appear. To know how many of these parts the partition has, we have to sum the d_i, for $i = 1, 2, \ldots, s$.

Definition 3.2. *Let $p_{f_1}(n, k)$ be the number of partitions counted by $p_{f_1}(n)$ having k other parts beyond the two copies that must appear. When we consider an example, those k parts will be bold.*

Example 3.2. *Considering $n = 24$ and $k = 6$, we have 4 partitions that satisfy the conditions having 6 parts beyond those that must appear twice. They are*

$$(3, 3, \mathbf{3}, \mathbf{3}, \mathbf{3}, 2, 2, 1, 1, \mathbf{1}, \mathbf{1}, \mathbf{1});$$

$$(3, 3, \mathbf{3}, \mathbf{3}, 2, 2, \mathbf{2}, \mathbf{2}, 1, 1, \mathbf{1}, \mathbf{1});$$

$$(3, 3, \mathbf{3}, 2, 2, \mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{2}, 1, 1, \mathbf{1});$$

$$(3, 3, 2, 2, \mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{2}, 1, 1).$$

The author defined the **weight** for partitions generated by this function as

$$\omega_{f_1}(\lambda) = (-1)^{\sum d_i},$$

where the elements d_i are the entries of the second row. So, the coefficient of q^n in the expansion of f_1 can be rewritten as

$$f_1(q) = \sum_{n=0}^{\infty} \Big(\sum_{\textit{even } k} p_{f_1}(n, k) - \sum_{\textit{odd } k} p_{f_1}(n, k) \Big) q^n.$$

For a fixed n, we classify its partitions of the type described in Definition 3.2 according to the sum on the second row of the matrix associated to it. By counting the appearance of each number in these sums, we can organize the data on a table, which is presented next. The entry in line n and column $n - j$ is the number of times j appears as sum of the entries of the second row in type (3.2) matrices.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
1	0																																							
2	0	1																																						
3	0	1	0																																					
4	0	1	0	0																																				
5	0	1	0	0	0																																			
6	0	1	0	0	0	1																																		
7	0	1	0	0	0	1	0																																	
8	0	1	0	0	0	1	1	0																																
9	0	1	0	0	0	1	1	0	0																															
10	0	1	0	0	0	1	1	1	0	0																														
11	0	1	0	0	0	1	1	1	0	0	0																													
12	0	1	0	0	0	1	1	1	1	0	0	1																												
13	0	1	0	0	0	1	1	1	1	0	0	1	0																											
14	0	1	0	0	0	1	1	1	1	1	0	1	1	0																										
15	0	1	0	0	0	1	1	1	1	1	0	1	1	1	0																									
16	0	1	0	0	0	1	1	1	1	1	1	1	1	2	0	0																								
17	0	1	0	0	0	1	1	1	1	1	1	1	1	2	1	0	0																							
18	0	1	0	0	0	1	1	1	1	1	1	2	1	2	2	1	0	0																						
19	0	1	0	0	0	1	1	1	1	1	1	2	1	2	2	2	0	0	0																					
20	0	1	0	0	0	1	1	1	1	1	1	2	2	2	2	3	1	0	0	1																				
21	0	1	0	0	0	1	1	1	1	1	1	2	2	2	2	3	2	1	0	1	0																			
22	0	1	0	0	0	1	1	1	1	1	1	2	2	3	2	3	3	2	0	1	1	0																		
23	0	1	0	0	0	1	1	1	1	1	1	2	2	3	2	3	3	3	1	1	1	1	0																	
24	0	1	0	0	0	1	1	1	1	1	1	2	2	3	3	3	3	4	2	2	1	2	1	0																
25	0	1	0	0	0	1	1	1	1	1	1	2	2	3	3	3	3	4	3	3	1	2	2	0	0															
26	0	1	0	0	0	1	1	1	1	1	1	2	2	3	3	4	3	4	4	4	2	2	3	2	0	0														
27	0	1	0	0	0	1	1	1	1	1	1	2	2	3	3	4	3	4	4	5	3	3	3	3	1	0	0													
28	0	1	0	0	0	1	1	1	1	1	1	2	2	3	3	4	4	4	4	6	4	4	3	4	3	1	0	0												
29	0	1	0	0	0	1	1	1	1	1	1	2	2	3	3	4	4	4	4	6	5	5	4	4	4	3	0	0	0											
30	0	1	0	0	0	1	1	1	1	1	1	2	2	3	3	4	4	5	4	6	6	6	5	5	5	5	2	0	0	1										
31	0	1	0	0	0	1	1	1	1	1	1	2	2	3	3	4	4	5	4	6	6	7	6	6	5	6	4	1	0	1	0									
32	0	1	0	0	0	1	1	1	1	1	1	2	2	3	3	4	4	5	5	6	6	8	7	7	6	7	6	4	1	1	1	0								
33	0	1	0	0	0	1	1	1	1	1	1	2	2	3	3	4	4	5	5	6	6	8	8	8	7	8	7	6	3	1	1	1	0							
34	0	1	0	0	0	1	1	1	1	1	1	2	2	3	3	4	4	5	5	7	6	8	9	9	8	9	8	8	6	3	1	2	1	0						
35	0	1	0	0	0	1	1	1	1	1	1	2	2	3	3	4	4	5	5	7	6	8	9	10	9	10	9	9	8	6	2	2	2	1	0					
36	0	1	0	0	0	1	1	1	1	1	1	2	2	3	3	4	4	5	5	7	7	8	9	11	10	11	10	11	10	9	5	3	3	3	0	0				
37	0	1	0	0	0	1	1	1	1	1	1	2	2	3	3	4	4	5	5	7	7	8	9	11	11	12	11	12	11	11	8	5	3	4	2	0	0			
38	0	1	0	0	0	1	1	1	1	1	1	2	2	3	3	4	4	5	5	7	7	9	9	11	12	13	12	13	13	13	11	9	5	5	4	2	0	0		
39	0	1	0	0	0	1	1	1	1	1	1	2	2	3	3	4	4	5	5	7	7	9	9	11	12	14	13	14	14	15	13	12	8	6	5	5	1	0	0	
40	0	1	0	0	0	1	1	1	1	1	1	2	2	3	3	4	4	5	5	7	7	9	10	11	12	15	14	15	15	17	15	15	12	9	7	7	4	1	0	0

Table 4 – Table from the characterization given by Theorem 3.1

Looking at the 10^{th} line, from the right to the left, there are:

- no partitions with no parts beyond those ones which must appear twice;
- no partitions with 1 part beyond those ones which must appear twice;
- 1 partition with $2, 3$ and 4 parts beyond those ones which must appear twice;
- no partitions with $5, 6$ and 7 parts beyond those ones which must appear twice;
- 1 partition with 8 parts beyond those ones which must appear twice;
- no partitions with 9 parts beyond those ones which must appear twice.

By observing the table above, looking at the diagonals, we get some results.

Proposition 3.1. *For all $n \geqslant 1$ and $0 \leqslant i \leqslant n-1$ we have*

$$p_{f_1}(n^2 + n + 1 + i, 1) = 1.$$

Proof. First of all, note that the largest part of any partition that belongs to $P_{f_1}(n^2+n+1+i,1)$ must be n. Indeed, if $n+1$ were the greatest part, we would have, at least,

$$(n+1) + (n+1) + n + n + \cdots + 1 + 1 = n^2 + 3n + 2,$$

which is greater than $n^2 + n + 1 + i$, for $0 \leqslant i \leqslant n-1$. On the other hand, if $n-1$ were the greatest part, we would have

$$(n-1) + (n-1) + \cdots + 1 + 1 = n^2 - n$$

and, as a part is at most $n-1$, the number been partitioned would be at most $n^2 - n + (n-1)$, which is smaller than $n^2 + n + 1 + i$, if $0 \leqslant i \leqslant n-1$.

Then, we have to write

$$n^2 + n + 1 + i = n + n + \cdots + 1 + 1 + k = (n+1)n + k = n^2 + n + k,$$

with $k \leqslant n$. It is clear that we can only have $k = i + 1$. □

Proposition 3.2. *For all $n \geqslant 1$ and $0 \leqslant i \leqslant n$, we have*

$$p_{f_1}(n^2 \pm i, 2) = \begin{cases} \dfrac{n-1}{2} - \left\lfloor \dfrac{i}{2} \right\rfloor & \text{odd } n, \\ \dfrac{n}{2} - \left\lfloor \dfrac{i+1}{2} \right\rfloor & \text{even } n. \end{cases}$$

Proof. The greatest part of any partition counted by $p_{f_1}(n^2 \pm i, 2)$ must be $n-1$. So, we write

$$n^2 \pm i = (n-1) + (n-1) + \cdots + 1 + 1 + r + s,$$

with $1 \leqslant s \leqslant r \leqslant n-1$, which implies

$$r + s = n^2 \pm i - n(n-1) = n \pm i.$$

Then we need to determinate the number of solutions of equations $r+s=n+i$ and $r+s=n-i$, with $1 \leqslant s \leqslant r \leqslant n-1$. Let us solve the problem for $r+s=n+i$, the negative case being adaptable by changing signs. First of all, the number of solutions of equation $r+s=n+i$ with no restriction on r is $\left\lfloor \frac{n+i}{2} \right\rfloor$. Now we discard the solutions where $r > n-1$.

If $i=0$, note that $r > n-1$ implies $r=n$, and so $s=0$, which never occurs. So, there's no solution to discard and the number we're looking for is just

$$\left\lfloor \frac{n}{2} \right\rfloor = \begin{cases} \frac{n-1}{2}, & \text{odd } n, \\ \frac{n}{2}, & \text{even } n \end{cases}$$

For $i \geqslant 1$, as $s \geqslant 1$ and $r > n-1$, we can write $r = n-1+k \quad$ with $1 \leqslant k \leqslant i$, and so, for each value of k we get one value of s.

Then, the number of solutions we want is $\left\lfloor \frac{n+i}{2} \right\rfloor - i$. Next, we analyse possible parities of n and i and their combinations, in order to see that this number is equal to the one given by the proposition.

If n and i are odd, we write $n=2l+1$ and $i=2j+1$. So,

$$\begin{aligned} \left\lfloor \frac{n+i}{2} \right\rfloor - i &= \left\lfloor \frac{2l+2j+2}{2} \right\rfloor - (2j+1) \\ &= l-j \\ &= \frac{n-1}{2} - \frac{i-1}{2} \end{aligned}$$

If n is odd and i is even, we write $n=2l+1$ and $i=2j$. So,

$$\begin{aligned} \left\lfloor \frac{n+i}{2} \right\rfloor - i &= \left\lfloor \frac{2l+2j+1}{2} \right\rfloor - 2j \\ &= l-j \\ &= \frac{n-1}{2} - \frac{i}{2} \end{aligned}$$

If n and i are even, we write $n=2l$ and $i=2j$. So,

$$\begin{aligned} \left\lfloor \frac{n+i}{2} \right\rfloor - i &= \left\lfloor \frac{2l+2j}{2} \right\rfloor - 2j \\ &= l-j \\ &= \frac{n}{2} - \frac{i}{2} \end{aligned}$$

If n is even and i is odd, we write $n=2l$ and $i=2j+1$. So,

$$\begin{aligned} \left\lfloor \frac{n+i}{2} \right\rfloor - i &= \left\lfloor \frac{2l+2j+1}{2} \right\rfloor - (2j+1) \\ &= l-j-1 \\ &= \frac{n}{2} - \frac{i+1}{2} \end{aligned}$$

Thus, the proposition is proven. □

Proposition 3.3. *For all $n \geqslant 3$ we have*

$$p_{f_1}(n^2 - 3, 3) = p(n - 3, 3).$$

Proof. First of all, it is easy to prove that the largest part of any partition counted by $p_{f_1}(n^2-3,3)$ must be $n-1$, by considering possible parts larger or smaller than it. We are able to write

$$\begin{aligned} n^2 - 3 &= (n-1) + (n-1) + \cdots + 1 + 1 + r + s + t \\ &= n(n-1) + r + s + t \\ &= n^2 - n + r + s + t, \end{aligned}$$

which is the same as

$$r + s + t = n - 3,$$

with $1 \leqslant t \leqslant s \leqslant r \leqslant n - 1$. The number of solution of this equation clearly is $p(n - 3, 3)$. □

Proposition 3.4. *For all $n \geqslant 1$ we have*

(*i*) $p_{f_1}(4n^2 + n + i, 3) = p_{f_1}(4n^2 + n - i, 3)$, *for* $0 \leqslant i \leqslant n - 2$;

(*ii*) $p_{f_1}(4n^2 + 5n + 2 + i, 3) = p_{f_1}(4n^2 + 5n + 2 - i, 3)$, *for* $0 \leqslant i \leqslant n - 2$;

(*iii*) $p_{f_1}(4n^2 + 5n + i, 3) = T_n$, *for* $i = 0, 1, 2, 3$.

(*iv*) $p_{f_1}(4n^2 + n, 3) = \left\lfloor \dfrac{n^2+1}{2} \right\rfloor$;

Proof. (*i*) First of all, note that the largest part of any partition counted by $p_{f_1}(4n^2 + n \pm i, 3)$ is $2n - 1$. This can be proved in the same way as we did before. So, we have to build a bijection between sets $P_{f_1}(4n^2 + n + i, 3)$ and $P_{f_1}(4n^2 + n - i, 3)$.

Let λ be a partition of $4n^2 + n + i$ in the form

$$\lambda = (1, 1, 2, 2, \ldots, 2n - 1, 2n - 1, r, s, t),$$

with $1 \leqslant r \leqslant s \leqslant t \leqslant 2n - 1$. Then, r, s and t satisfy

$$r + s + t = 3n + i.$$

Consider now the partition $4n^2 + n - i$, whose parts probably are not ordered,

$$\mu = (1, 1, 2, 2, \ldots, 2n - 1, 2n - 1, 2n - r, 2n - s, 2n - t).$$

Conversely, let μ be a partition of $4n^2 + n - i$ in the form

$$\lambda = (1, 1, 2, 2, \ldots, 2n - 1, 2n - 1, r, s, t),$$

with $1 \leqslant r \leqslant s \leqslant t \leqslant 2n - 1$. Then

$$r + s + t = 3n - i.$$

As before, taking $\lambda = (1, 1, 2, 2, \ldots, 2n - 1, 2n - 1, 2n - r, 2n - s, 2n - t)$, we get a partition of $4n^2 + n + i$.

(ii) In this case, the largest part of any partition of $p_{f_1}(4n^2 + 5n + 2 + i, 3)$ is $2n$. The bijection between $P_{f_1}(4n^2 + 5n + 2 + i, 3)$ and $P_{f_1}(4n^2 + 5n + 2 - i, 3)$ is similar to the one we gave in item *(i)*.

Given $\lambda = (1, 1, 2, 2, \ldots, 2n, 2n, r, s, t)$, a partition counted by $p_{f_1}(4n^2 + 5n + 2 - i, 3)$, then r, s and t, must satisfy $1 \leqslant r \leqslant s \leqslant t \leqslant 2n$ and $r + s + t = 3n + 1 - i$. So, consider

$$\mu = (1, 1, 2, 2, \ldots, 2n, 2n, 2n + 1 - r, 2n + 1 - s, 2n + 1 - t)$$

the correspondent partition lying in $P_{f_1}(4n^2 + 5n + 2 + i, 3)$.

The reverse map is analogous.

(iii) The largest part of any partition counted by $p_{f_1}(4n^2 + 5n + i, 3)$ is $2n$. So, we have

$$\begin{aligned} 4n^2 + 5n + i &= 1 + 1 + 2 + 2 + \cdots + 2n + 2n + r + s + t \\ &= 4n^2 + 2n + r + s + t, \end{aligned}$$

with $0 \leqslant r \leqslant s \leqslant t \leqslant 2n$, which implies

$$r + s + t = 3n + i, 0 \leqslant r \leqslant s \leqslant t \leqslant 2n \tag{3.5}$$

So, to prove statement *(iii)*, we need to prove that the number of solutions of equation (3.5) is equal to T_n.

Let us begin with equation (3.5) with no restriction in parts r, s and t, other than $0 \leqslant r \leqslant s \leqslant t$. The number of solutions is the same as $p(3n + i, 3)$, which, as we already know, is

$$\left\{\frac{(3n + i + 3)^2}{12}\right\} - \left\lfloor \frac{3n + i}{2} \right\rfloor - 1.$$

Now we have to eliminate the solutions we do not want. For this we will split the rest of the proof into two parts, proving for $i = 0$ and $i = 1$, and, after that, setting bijections between $P_{f_1}(4n^2 + 5n, 3)$ and $P_{f_1}(4n^2 + 5n + 3, 3)$ and between $P_{f_1}(4n^2 + 5n + 1, 3)$ and $P_{f_1}(4n^2 + 5n + 2, 3)$.

- If $i = 0$, equation (3.5) turns to $r + s + t = 3n$, and the number of its solutions with no restriction is
$$\left\{\frac{3(n + 1)^2}{4}\right\} - \left\lfloor \frac{3n}{2} \right\rfloor - 1.$$

  Claim. For all $n \geqslant 1$,
$$\left\{\frac{3(n + 1)^2}{4}\right\} - \left\lfloor \frac{3n}{2} \right\rfloor - 1 = \frac{n(n + 1)}{2} + \left\lfloor \frac{(n - 1)^2}{4} \right\rfloor. \tag{3.6}$$

  * Indeed, if n is even, say $n = 2k$,
$$\left\{\frac{3(2k + 1)^2}{4}\right\} - \left\lfloor \frac{3(2k)}{2} \right\rfloor - 1 = \left\{\frac{3(4k^2 + 4k + 1)}{4}\right\} - 3k - 1 = 3k^2,$$
    and
$$\frac{2k(2k + 1)}{2} + \left\lfloor \frac{(2k - 1)^2}{4} \right\rfloor = \frac{4k^2 + 2k}{2} + k^2 - k = 3k^2.$$

* If n is odd, saying $n = 2k+1$,

$$\left\{\frac{3(2k+2)^2}{4}\right\} - \left\lfloor\frac{3(2k+1)}{2}\right\rfloor - 1 = \left\{\frac{3(4k^2+8k+4)}{4}\right\} - (3k+1) - 1 = 3k^2+3k+1,$$

and

$$\frac{(2k+1)(2k+2)}{2} + \left\lfloor\frac{(2k+1-1)^2}{4}\right\rfloor = \frac{4k^2+6k+2}{2} + k^2 = 3k^2+3k+1.$$

Next we eliminate the solutions that do not satisfy $1 \leqslant r \leqslant s \leqslant t \leqslant 2n$.

Note that only t can be greater than $2n$, otherwise $r+s+t > 3n$. So, if $t > 2n$ we can write $t = 2n+i$ with $1 \leqslant i \leqslant n-2$ and the equation becomes $r+s = n-i$. For each value of i the number of solutions is $\left\lfloor\frac{n-i}{2}\right\rfloor$.

Claim. For all $n \geqslant 3$,

$$\sum_{i=1}^{n-2}\left\lfloor\frac{n-i}{2}\right\rfloor = \left\lfloor\frac{(n-1)^2}{4}\right\rfloor. \tag{3.7}$$

We prove this claim by induction on n. First of all, for $n = 3$ the equality holds. Supposing that for n the equality holds, for $n+1$, we have

$$\begin{aligned}\sum_{i=1}^{n-1}\left\lfloor\frac{n+1-i}{2}\right\rfloor &= \sum_{i=0}^{n-2}\left\lfloor\frac{n-i}{2}\right\rfloor\\ &= \sum_{i=1}^{n-2}\left\lfloor\frac{n-i}{2}\right\rfloor + \left\lfloor\frac{n}{2}\right\rfloor\\ &= \left\lfloor\frac{(n-1)^2}{4}\right\rfloor + \left\lfloor\frac{n}{2}\right\rfloor.\end{aligned}$$

* For odd n, say $n = 2j+1$,

$$\left\lfloor\frac{(2j+1-1)^2}{4}\right\rfloor + \left\lfloor\frac{2j+1}{2}\right\rfloor = \left\lfloor\frac{4j^2}{4}\right\rfloor + j = j^2+j = \left\lfloor\frac{(2j+1)^2}{4}\right\rfloor.$$

* For even n, say $n = 2j$,

$$\left\lfloor\frac{(2j-1)^2}{4}\right\rfloor + \left\lfloor\frac{2j}{2}\right\rfloor = \left\lfloor\frac{4j^2-4j+1}{4}\right\rfloor + j = j^2-j+j = j^2 = \left\lfloor\frac{(2j)^2}{4}\right\rfloor.$$

Then

$$\sum_{i=1}^{n-1}\left\lfloor\frac{n+1-i}{2}\right\rfloor = \left\lfloor\frac{n^2}{4}\right\rfloor,$$

and the claim is proved.

Now, the number of solutions of equation (3.5) with $1 \leqslant r \leqslant s \leqslant t \leqslant 2n$ is

$$\frac{n(n+1)}{2} + \left\lfloor\frac{(n-1)^2}{4}\right\rfloor - \left\lfloor\frac{(n-1)^2}{4}\right\rfloor = \frac{n(n+1)}{2} = T_n.$$

- If $i = 1$ equation (3.5) turns to $r + s + t = 3n + 1$, and the number of its solutions with no restriction is $\left\{\dfrac{(3n+4)^2}{12}\right\} - \left\lfloor \dfrac{3n+1}{2} \right\rfloor - 1$.
  Like before, considering odd n and even n, we have $n \geqslant 1$,
  $$\left\{\frac{(3n+4)^2}{12}\right\} - \left\lfloor \frac{3n+1}{2} \right\rfloor - 1 = \frac{n(n+1)}{2} + \left\lfloor \frac{n^2}{4} \right\rfloor.$$
  Now, having the number of solutions of equation $r + s + t = 3n + 1$ with no restriction, other than $0 \leqslant r \leqslant s \leqslant t$, we eliminate those ones that do not satisfy $1 \leqslant r \leqslant s \leqslant t \leqslant 2n$.
  Note that only t can be greater than $2n$, otherwise $r + s + t > 3n + 1$. If $t > 2n$, we can write $t = 2n + i$ with $1 \leqslant i \leqslant n - 1$ and the equation becomes $r + s = n + 1 - i$. For each value of i the number of solutions is $\left\lfloor \dfrac{n+1-i}{2} \right\rfloor$.
  In the same way we did for equality (3.7), we have: For all $n \geqslant 2$,
  $$\sum_{i=1}^{n-1} \left\lfloor \frac{n+1-i}{2} \right\rfloor = \left\lfloor \frac{n^2}{4} \right\rfloor.$$
  So, the number of solutions of equation (3.5) with $1 \leqslant r \leqslant s \leqslant t \leqslant 2n$ is for $i = 1$ is
  $$\frac{n(n+1)}{2} + \left\lfloor \frac{n^2}{4} \right\rfloor - \left\lfloor \frac{n^2}{4} \right\rfloor = \frac{n(n+1)}{2} = T_n.$$

Now, to prove the case when $i = 3$ we set a bijection between $P_{f_1}(4n^2 + 5n, 3)$ and $P_{f_1}(4n^2 + 5n + 3, 3)$.

Given a partition counted by $p_{f_1}(4n^2+5n, 3)$, it is of the form $\lambda = (1, 1, 2, 2, 3, 3, \ldots, 2n, 2n, r, s, t)$, with $1 \leqslant r \leqslant s \leqslant t \leqslant 2n$ and $r + s + t = 3n$.

Writing $\mu = (1, 1, 2, 2, \ldots, 2n + 1 - r, 2n + 1 - s, 2n + 1 - t)$, μ is a partition lying in $P_{f_1}(4n^2 + 5n + 3, 3)$ because

$$1 + 1 + 2 + 2 + \cdots + 2n + 2n + (2n + 1 - r) + (2n + 1 - s) + (2n + 1 - t) =$$

$$\begin{aligned} &= 4n^2 + 2n + 6n + 3 - (r + s + t) \\ &= 4n^2 + 6n + 3 - 3n \\ &= 4^2 + 5n + 3. \end{aligned}$$

Conversely, given a partition counted by $P_{f_1}(4n^2 + 5n + 3, 3)$, it is like

$$\mu = (1, 1, 2, 2, 3, 3, \ldots, 2n, 2n, a, b, c),$$

with $1 \leqslant a \leqslant b \leqslant c \leqslant 2n$ and $a + b + c = 3n + 3$. So,

$$\lambda = (1, 1, 2, 2, 3, 3, \ldots, 2n, 2n, 2n + 1 - a, 2n + 1 - b, 2n + 1 - c)$$

is a partition counted by $P_{f_1}(4n^2 + 5n, 3)$ because

$$1 + 1 + 2 + 2 + \cdots + 2n + 2n + (2n + 1 - a) + (2n + 1 - b) + (2n + 1 - t) =$$

$$\begin{aligned} &= 4n^2 + 2n + 6n + 3 - (a + b + c) \\ &= 4n^2 + 8n + 3 - 3n - 3 \\ &= 4^2 + 5n. \end{aligned}$$

The bijection between sets $P_{f_1}(4n^2 + 5n + 1, 3)$ and $P_{f_1}(4n^2 + 5n + 2, 3)$ is analogous.

(*iv*) By item (*i*), taking $i = 0$, we have that the largest part of any partition counted by $p_{f_1}(4n^2 + n, 3)$ is $2n - 1$. Thus, $p_{f_1}(4n^2 + n, 3)$ is the number of solutions of the following equation

$$\begin{aligned} 4n^2 + n &= 1 + 1 + 2 + 2 + \cdots + 2n - 1 + 2n - 1 + r + s + t \\ &= 4n^2 - 2n + r + s + t, \end{aligned}$$

which is equivalent to

$$r + s + t = 3n, 1 \leqslant r \leqslant s \leqslant t \leqslant 2n - 1 \tag{3.8}$$

To prove statement (*iv*), we need to prove that the number of solutions of equation (3.8) is equal to $\left\lfloor \frac{n^2+1}{2} \right\rfloor$.

Let us begin with equation (3.8) with no restriction in parts r, s and t. The number of solutions is the same as $p(3n, 3)$, which we already know

$$\left\{ \frac{(3n+3)^2}{12} \right\} - \left\lfloor \frac{3n}{2} \right\rfloor - 1.$$

By considering odd n and even n, as we did in the previous item, we have that: for all $n \geqslant 1$,

$$\left\{ \frac{(3n+3)^2}{12} \right\} - \left\lfloor \frac{3n}{2} \right\rfloor - 1 = \left\lfloor \frac{n^2+1}{2} \right\rfloor + \left\lfloor \frac{n^2}{4} \right\rfloor. \tag{3.9}$$

Now we have to eliminate the solutions that do not satisfy $1 \leqslant r \leqslant s \leqslant t \leqslant 2n - 1$. As before, note that only t can be greater than $2n-1$, otherwise $r+s+t > 3n+1$. If $t > 2n-1$, we can write $t = 2n + 1 + i$ with $1 \leqslant i \leqslant n - 1$ and the equation becomes $r + s = n + 1 - i$. For each value of i the number of solutions is $\left\lfloor \frac{n+1-i}{2} \right\rfloor$.

In the same way we did for equality (3.7), we have: For all $n \geqslant 2$,

$$\sum_{i=1}^{n-1} \left\lfloor \frac{n+1-i}{2} \right\rfloor = \left\lfloor \frac{n^2}{4} \right\rfloor. \tag{3.10}$$

And the identity holds.

□

Proposition 3.5. *For all $n \geqslant 2$ and $0 \leqslant i \leqslant 3$, we have*

$$p_{f_1}(2T_n + i, 4) = p_{f_1}(2T_n - i, 4).$$

Proof. Any partition counted by $P_{f_1}(2T_n \pm i, 4)$ has largest part $n-1$. This can be proved in the same way we did many times before.

Given λ a partition counted by $P_{f_1}(2T_n + i, 4)$, it is of the form

$$\lambda = (n-1, n-1, \ldots, 2, 2, 1, 1, k_1, k_2, k_3, k_4),$$

with $1 \leqslant k_j \leqslant n-1$. So, we have $k_1 + k_2 + k_3 + k_4 = n^2 + n + i - n(n-1) = 2n + i$.

If we take $\mu = (n-1, n-1, \ldots, 2, 2, 1, 1, n-k_1, n-k_2, n-k_3, n-k_4)$, we get a partition of

$$\begin{aligned} n-1+n-1+ \quad \ldots \quad & +1+1+n-k_1+n-k_2+n-k_3+n-k_4 \\ = \quad & n(n-1) + 4n - (k_1+k_2+k_3+k_4) \\ = \quad & n^2 + 3n - (2n+i) \\ = \quad & n^2 + n - i \\ = \quad & 2T_n - i, \end{aligned}$$

with $1 \leqslant n - k_j \leqslant n-1$. So, μ is a partition counted by $P_{f_1}(2T_n - i, 4)$.

The reverse map is clear.

□

3.2 Mock Theta Function $F_1(q)$

Consider the mock theta function of order 5

$$F_1(q) = \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(q;q^2)_{n+1}}. \tag{3.11}$$

Its general term

$$\frac{q^{4(1+2+3+\cdots+s)}}{(1-q)(1-q^3)\cdots(1-q^{2s+1})},$$

generates the partitions of n with:
- the even parts ranging from 2 to $2s$ with no gaps, and each one of them with multiplicity 2;
- any odd part is less than or equal to $2s+1$.

The next Theorem can be found in [9]. It also relates the coefficients of the Mock Theta Function $F_1(q)$ to a set composed by two-line matrices. Differently from what we did for $f_1(q)$, here we do not need to set a weight for partitions the general term of $F_1(q)$ provides, just because its expansion does not have any negative term.

Theorem 3.2. *The coefficient of q^n in the expansion of* (3.11) *is equal to the number of elements in the set of matrices of the form*

$$A = \begin{pmatrix} c_1 & c_2 & \cdots & c_{s+1} \\ d_1 & d_2 & \cdots & d_{s+1} \end{pmatrix}, \tag{3.12}$$

with non-negative integer entries satisfying

$$c_{s+1} = 0; d_t \geqslant 0; \tag{3.13}$$

$$c_t = 4 + c_{t+1} + 2d_{t+1},\ \forall t < s+1; \tag{3.14}$$

$$n = \sum c_t + \sum d_t. \tag{3.15}$$

Proof. According to the general term of (3.11), we can decompose n as

$$n = (4 + 8 + 12 + \cdots + 4s) + (1 \cdot d_1 + 3 \cdot d_3 + \cdots + (2s+1) \cdot d_{s+1})$$

or, equivalently, as the sum of the entries of the matrix

$$A = \begin{pmatrix} 4s + 2d_2 + \cdots + 2d_{s+1} & \cdots & 8 + 2d_s + 2d_{s+1} & 4 + 2d_{s+1} & 0 \\ d_1 & \cdots & d_{s-1} & d_s & d_{s+1} \end{pmatrix},$$

Noting that the entries satisfy conditions (3.14) to (3.15), the theorem is proved. □

Definition 3.3. *Let $p_{F_1}(n)$ be the number of partitions of n where, if the largest even part is $2s$, then all even number smaller than or equal to $2s$ must appear twice and the odd parts must be smaller than or equal to $2s+1$. Hence,*

$$F_1(q) = \sum_{n=0}^{\infty} p_{F_1}(n) q^n.$$

The second row of the matrices of type (3.12) describes the odd parts from 1 to $2s+1$, of the partition associated to each matrix. To know how many odd parts the partition has, we have to sum the d_i, for $i = 1, 2, \ldots, s+1$.

Definition 3.4. *Let $p_{F_1}(n,k)$ be the number of partitions of n counted by $p_{F_1}(n)$, where there are k odd parts. So*

$$p_{F_1}(n) = \sum_k p_{F_1}(n,k).$$

Example 3.3. *We have $p_{F_1}(28,8) = 4$ and $P_{F_1}(28,8)$ is composed of by the following four partitions:*

$$(\mathbf{5}, \mathbf{5}, 4, 4, 2, 2, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1});$$

$$(\mathbf{5}, 4, 4, \mathbf{3}, \mathbf{3}, 2, 2, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1});$$

$$(4, 4, \mathbf{3}, \mathbf{3}, \mathbf{3}, \mathbf{3}, 2, 2, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1});$$

$$(\mathbf{3}, \mathbf{3}, \mathbf{3}, \mathbf{3}, \mathbf{3}, \mathbf{3}, \mathbf{3}, \mathbf{3}, 2, 2).$$

For a fixed n, we classify its partitions like in Definition 3.3 according to the sum on the second row of the matrix associated to it, consequently, according to the number of odd parts. By counting the appearance of each number in these sums, we can organize the data on a table, which is presented next. The entry in line n and column $n-j$ is the number of times j appears as sum of the entries of the second row in type (3.12) matrices.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
1	0																																						
2	0	0																																					
3	0	0	0																																				
4	0	0	0	1																																			
5	0	0	0	1	0																																		
6	0	0	0	1	0	0																																	
7	0	0	0	1	0	1	0																																
8	0	0	0	1	0	1	0	0																															
9	0	0	0	1	0	1	0	0	0																														
10	0	0	0	1	0	1	0	1	0	0																													
11	0	0	0	1	0	1	0	1	0	0	0																												
12	0	0	0	1	0	1	0	1	0	0	0	1																											
13	0	0	0	1	0	1	0	1	0	1	0	1	0																										
14	0	0	0	1	0	1	0	1	0	1	0	1	0	0																									
15	0	0	0	1	0	1	0	1	0	1	0	1	0	1	0																								
16	0	0	0	1	0	1	0	1	0	1	0	2	0	1	0	0																							
17	0	0	0	1	0	1	0	1	0	1	0	2	0	1	0	1	0																						
18	0	0	0	1	0	1	0	1	0	1	0	2	0	1	0	2	0	0																					
19	0	0	0	1	0	1	0	1	0	1	0	2	0	2	0	2	0	0	0																				
20	0	0	0	1	0	1	0	1	0	1	0	2	0	2	0	2	0	1	0	0																			
21	0	0	0	1	0	1	0	1	0	1	0	2	0	2	0	2	0	2	0	0	0																		
22	0	0	0	1	0	1	0	1	0	1	0	2	0	2	0	3	0	2	0	1	0	0																	
23	0	0	0	1	0	1	0	1	0	1	0	2	0	2	0	3	0	2	0	2	0	0	0																
24	0	0	0	1	0	1	0	1	0	1	0	2	0	2	0	3	0	2	0	3	0	0	0	1															
25	0	0	0	1	0	1	0	1	0	1	0	2	0	2	0	3	0	3	0	3	0	1	0	1	0														
26	0	0	0	1	0	1	0	1	0	1	0	2	0	2	0	3	0	3	0	3	0	2	0	1	0	0													
27	0	0	0	1	0	1	0	1	0	1	0	2	0	2	0	3	0	3	0	3	0	3	0	2	0	1	0												
28	0	0	0	1	0	1	0	1	0	1	0	2	0	2	0	3	0	3	0	4	0	3	0	3	0	1	0	0											
29	0	0	0	1	0	1	0	1	0	1	0	2	0	2	0	3	0	3	0	4	0	3	0	4	0	1	0	1	0										
30	0	0	0	1	0	1	0	1	0	1	0	2	0	2	0	3	0	3	0	4	0	3	0	5	0	2	0	2	0	0									
31	0	0	0	1	0	1	0	1	0	1	0	2	0	2	0	3	0	3	0	4	0	4	0	5	0	3	0	2	0	1	0								
32	0	0	0	1	0	1	0	1	0	1	0	2	0	2	0	3	0	3	0	4	0	4	0	5	0	4	0	3	0	2	0	0							
33	0	0	0	1	0	1	0	1	0	1	0	2	0	2	0	3	0	3	0	4	0	4	0	5	0	5	0	4	0	3	0	0	0						
34	0	0	0	1	0	1	0	1	0	1	0	2	0	2	0	3	0	3	0	4	0	4	0	6	0	5	0	5	0	3	0	2	0	0					
35	0	0	0	1	0	1	0	1	0	1	0	2	0	2	0	3	0	3	0	4	0	4	0	6	0	5	0	6	0	4	0	3	0	0	0				
36	0	0	0	1	0	1	0	1	0	1	0	2	0	2	0	3	0	3	0	4	0	4	0	6	0	5	0	7	0	5	0	4	0	1	0	0			
37	0	0	0	1	0	1	0	1	0	1	0	2	0	2	0	3	0	3	0	4	0	4	0	6	0	6	0	7	0	6	0	5	0	3	0	0	0		
38	0	0	0	1	0	1	0	1	0	1	0	2	0	2	0	3	0	3	0	4	0	4	0	6	0	6	0	7	0	7	0	6	0	4	0	1	0	0	
39	0	0	0	1	0	1	0	1	0	1	0	2	0	2	0	3	0	3	0	4	0	4	0	6	0	6	0	7	0	8	0	7	0	5	0	3	0	0	0	
40	0	0	0	1	0	1	0	1	0	1	0	2	0	2	0	3	0	3	0	4	0	4	0	6	0	6	0	8	0	8	0	8	0	6	0	5	0	0	0	1

Table 5 – Table from the characterization given by Theorem (3.2)

Example 3.4. *Looking at some numbers that appear in the* 23^{th} *line, we have:*

- *In the* 20^{th} *column, we have* $p_{F_1}(23, 23-20) = p_{F_1}(23, 3) = 2$;
- *In the* 18^{th} *column, we have* $p_{F_1}(23, 23-18) = p_{F_1}(23, 5) = 2$;
- *In the* 16^{th} *column, we have* $p_{F_1}(23, 23-16) = p_{F_1}(23, 7) = 3$;
- *In the* 14^{th} *column, we have* $p_{F_1}(23, 23-14) = p_{F_1}(23, 9) = 2$;
- *In the* 13^{th} *column, we have* $p_{F_1}(23, 23-13) = p_{F_1}(23, 10) = 0$.

By observing the table, one note that the columns become constant below certain entry. Beyond this, the columns in odd posititions are null columns. In fact, this occurs when n and the number k of odd parts have different parity.

This sequence made of those constant numbers in columns represents the same sequence of the number of partitions into parts congruent to $\pm 2 \pmod 5$, used in the Second Rogers-Ramanujan Identity. The informations we described before are set out and proved next into two theorems.

Theorem 3.3. *For all* $n \geqslant 1$ *and* $i \geqslant 0$*, we have*

$$p_{F_1}(3n+1, n-1) = p_{F_1}(3n+1+i, n-1+i).$$

Proof. We present a bijective proof that simply maps a partition $p_{F_1}(3n+1, n-1)$ onto a new one that belongs to $p_{F_1}(3n+1+i, n-1+i)$, by adding i parts of size 1. Now, the goal is to ensure there are always at least i parts 1 in a partition that lies in $P_{F_1}(3n+1+i, n-1+i)$. Let us suppose, by contradiction, we have less than i parts 1. To simplify the notation, call the multiplicity of each part λ_k by $x(\lambda_k)$.

As we have supposed, $x(1) \leqslant i-1$. Then the largest s from the definition that can appear as a part is 2 and the smallest is 1, otherwise

$$\begin{aligned}
3n+1+i &= 2s(s+1) + \sum_{i=1}^{s+1}(2i-1)\cdot x(2i-1) \\
&> 4(2+1) + \sum_{i=1}^{3}(2i-1)\cdot x(2i-1) \\
&\geqslant 12 + x(1) + 3\underbrace{(x(3)+x(5))}_{n-1+i-x(1)} \\
&= 12 + x(1) + 3n - 3 + 3i - 3x(1).
\end{aligned}$$

It gives us that

$$\begin{aligned}
3n+1+i &> 9 - 2x(1) + 3n + 3i \\
x(1) &> i+4,
\end{aligned}$$

which is a contradiction.

Now, still supposing that $x(1) < i$, we can write

$$\begin{aligned} 3n+1+i &= 4+8+x(1)+3x(3)+5x(5) \\ 3n+1+i &= 12+3\underbrace{(x(1)+x(3)+x(5))}_{n-1+i}-2x(1)+2x(5) \\ 3n+1+i &= 9+3n+3i-2x(1)+2x(5) \\ -i-4 &= -x(1)+x(5) \end{aligned}$$

Since $x(1) \leqslant i-1$, it follows that $x(5) \leqslant -5$. This is an absurd and then, the theorem is proved. □

Theorem 3.4. *For all $n \geqslant 1$, we have*

$$p_{F_1}(3n+1, n-1) = p(n+1|\textit{parts congruent to} \pm 2 \pmod 5).$$

Proof. We denote by $P^*(n+1)$ the set of partitions of $n+1$ in which, if s is the largest part, every part from 1 to s appears at least twice. So, we can build a bijection between sets $P_{F_1}(3n+1, n-1)$ and $P^*(n+1)$ by decreasing 1 from every odd part of a partition counted by $p_{F_1}(3n+1, n-1)$ and then dividing all parts by 2. Clearly, the reverse map is possible.

As an example, take the partition $(5,4,4,3,3,3,3,3,2,2,1,1,1,1,1)$ that belongs to $P_{F_1}(37, 11)$. The resulting partition is $(2,2,2,1,1,1,1,1,1,1)$, in $P^*(13)$.

Looking at the 2^{nd} Rogers-Ramanujan Identity, we are going to prove that $p^*(n+1)$ is equal to the number of partitions of $n+1$ into 2-distinct parts, greater than or equal to 2. Consider the following steps.

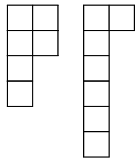
- Given a partition in $P^*(n+1)$, split it into two new ones, the first one made of two copies of each part from 1 to s, and the other one with the remaining parts.
- In the first partition, merge equal parts $1, 2, ..., s$, getting $(2, 4, ..., 2s)$. From the second one, take its conjugate. Once that its parts are smaller than or equal to s, now it has at most s parts.
- Get both partitions together side-by-side.

□

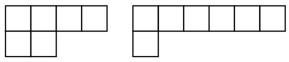
Example 3.5. *For $n = 12$, the partition $(2,2,4,4,5,3,3,3,3,3,1,1,1,1,1) \in P_{F_1}(37, 11)$ leads to $(10, 3)$, a partition of 13 into 2-distinct parts greater than or equal to 2.*

$$(2,2,4,4,5,3,3,3,3,3,1,1,1,1,1) \longrightarrow (2,2,4,4,4,2,2,2,2,2)$$

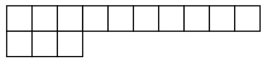
$$\longrightarrow (1,1,2,2,2,1,1,1,1,1) \longrightarrow$$



$$\longrightarrow$$



$$\longrightarrow$$



Proposition 3.6. *For all* $n \geqslant 1$ *we have*

$$p_{F_1}(n,0) = \begin{cases} 1, & \text{if } n = 2s(s+1) \\ 0, & \text{otherwise.} \end{cases}$$

Proof. As any partition counted by $p_{F_1}(n,0)$ has its even parts repeated 2 times from 2 to $2s$ and no other part appears, we can only write

$$\begin{aligned} n &= 2(2+4+\cdots+2s) \\ n &= 2s(s+1), \end{aligned}$$

and we get what we need. □

Proposition 3.7. *For all* $n \geqslant 1$ *we have*

$$p_{F_1}(2n^2+2n+i,1) = \begin{cases} 0, & \text{for even } i,\ 0 \leqslant i \leqslant 2n, \\ 1, & \text{for odd } i,\ 1 \leqslant i \leqslant 2n+1. \end{cases}$$

Proof. Note that the largest even part of multiplicity 2 of any partition counted by $p_{F_1}(2n^2 + 2n + i, 1)$ must be $2n$. Indeed, if it were larger than $2n$, it would be at least $2n+2$ and the number we would have to partition would be greater than or equal to

$$2(2+4+\cdots+2n+2n+2) = 2(n+1)(n+2) = 2n^2+6n+4.$$

This number is also greater than $2n^2+2n+i$.

By an analogous argument, we can prove that the largest even part cannot be smaller than $2n-2$.

So, $2n^2+2n+i$ must be

$$2n^2+2n+i = 2(2+4+\cdots+2n)+t, \quad \text{with odd } t \text{ and } 1 \leqslant t \leqslant 2n+1.$$

This implies

$$2n^2+2n+i = 2n^2+2n+t \Longrightarrow t = i.$$

So, for each value of i, we have one solution for the equation above, in case i is odd. Otherwise, there is no possible solution. □

Proposition 3.8. *For all* $n \geqslant 1$ *and* $1 \leqslant i \leqslant 2n$ *we have* *(i)* $p_{F_1}(8n^2 \pm (2i-1), 2) = 0$;

(ii) $p_{F_1}(8n^2 \pm 2i, 2) = n - \left\lfloor \frac{i}{2} \right\rfloor$;

(iii) $p_{F_1}(8n^2 - 8n + 1 \pm (2i-1), 2) = 0$;

(iv) $p_{F_1}(8n^2 - 8n + 1 \pm 2i, 2) = n - \left\lfloor \frac{i+1}{2} \right\rfloor$.

Proof. We prove just items (i) and (ii). Items (iii) and (iv) have analogous proofs of (i) and (ii), respectively.

(i) The largest even part of multiplicity 2 of any partition counted by $p_{F_1}(8n^2 \pm (2i-1), 2)$ must be $4n-2$. So, we write

$$8n^2 \pm (2i-1) = 2(2+4+\cdots+(4n-2)) + r + t, \quad \text{with odd } r, t \text{ and } 1 \leqslant r \leqslant t \leqslant 4n-1,$$

that gives us the equation

$$r + t = 4n \pm (2i-1).$$

Hence, as $4n \pm (2i-1)$ is an odd number, we cannot write it as a sum of two odd numbers r and t. Then $p_{F_1}(8n^2 \pm (2i-1), 2) = 0$.

(ii) Again, the largest even part of multiplicity 2 of any partition counted by $p_{F_1}(8n^2 \pm 2i, 2)$ is $4n-2$. So,

$$8n^2 \pm 2i = 2(2+4+\cdots+4n-2) + r + t$$

implies

$$r + t = 4n \pm 2i, \quad \text{with odd } r, t \text{ and } 1 \leqslant r \leqslant t \leqslant 4n-1.$$

So, in order to prove the statement, we need to find the number of solutions of the last equation. This can be rewritten as

$$2k - 1 + 2m - 1 = 4n \pm 2i, \quad \text{with } 1 \leqslant k \leqslant m \leqslant 2n,$$

which is the same as

$$k + m = 2n \pm i + 1, \quad \text{with } 1 \leqslant k \leqslant m \leqslant 2n. \tag{3.16}$$

First we analyse the case with $-i$. The number of solutions of equation

$$k + m = 2n - i + 1, \quad \text{with } 1 \leqslant k \leqslant m \leqslant 2n,$$

without counting the order, is $\left\lfloor \frac{2n-i+1}{2} \right\rfloor = n + \left\lfloor \frac{-i+1}{2} \right\rfloor = n - \left\lfloor \frac{i}{2} \right\rfloor$.

For the case with $+i$, we evaluate the number of solutions, without counting the order, of

$$k + m = 2n + i + 1, \quad \text{with } 1 \leqslant k \leqslant m \leqslant 2n.$$

Considering the previous equation with no restrictions, the number of its solutions is $\left\lfloor \dfrac{2n+i+1}{2} \right\rfloor$. We just need to discard the solutions where $m > 2n$. Writing $m = 2n + t$ with $1 \leqslant t \leqslant i$, the solutions we eliminate are the same, in number, as the solutions of equation $k + t = i + 1$ with $1 \leqslant t \leqslant i$, which are i.

Then, the number of solutions of (3.16) is $\left\lfloor \dfrac{2n+i+1}{2} \right\rfloor - i$, that is equal to $n - \left\lfloor \dfrac{i}{2} \right\rfloor$.

□

Proposition 3.9. *For all $n \geqslant 1$ and $i = 0, 2, 4, 6$, we have*

$$p_{F_1}(8n^2 + 2n - 3 + i, 3) = T_n.$$

Proof. The largest even part of multiplicity 2 of any partition counted by $p_{F_1}(8n^2 + 2n - 3 + i, 3)$ is $4n - 2$. So, we have

$$8n^2 + 2n - 3 + i = 2(2 + 4 + \cdots + 4n - 2) + r + s + t,$$

with odd r, s, t and $1 \leqslant r \leqslant s \leqslant t \leqslant 4n - 1$. The equation implies

$$r + s + t = 6n - 3 + i.$$

By noting that r, s, t are odd numbers, we can write it as

$$2R - 1 + 2S - 1 + 2T - 1 = 6n - 3 + i,$$

or

$$R + S + T = 3n + j \quad \text{with } j = 0, 1, 2, 3 \text{ and } 1 \leqslant R \leqslant S \leqslant T \leqslant 2n.$$

The number of solutions with no restrictions, without counting the order of parts, is equal to the number of partitions of $3n + j$ into 3 parts. This formula is already known and it can be found in [4], that is

$$\left\{ \frac{(3n+j+3)^2}{12} \right\} - \left\lfloor \frac{3n+j}{2} \right\rfloor - 1.$$

Now we have to eliminate the solutions where $T > 2n$. As R and S are, at least 1, writing $T = 2n + k$, with $k \geqslant 1$, k may range from 1 to $n + j - 2$. So, for each value of k, we have to exclude the solutions of $R + S = n + j - k$, that, in number, are $\left\lfloor \dfrac{n+j-k}{2} \right\rfloor$. This formula can be also found in [4]. The amount of solutions we need to eliminate is $\displaystyle\sum_{k=1}^{n+j-2} \left\lfloor \frac{n+j-k}{2} \right\rfloor$. So,

$$p_{F_1}(8n^2 + 2n - 3 + i, 3) = \left\{ \frac{(3n+j+3)^2}{12} \right\} - \left\lfloor \frac{3n+j}{2} \right\rfloor - 1 - \sum_{k=1}^{n+j-2} \left\lfloor \frac{n+j-k}{2} \right\rfloor.$$

Finally, to see that it is equal to T_n, we follow the same steps we did in item (iii) of Proposition (3.4).

□

Proposition 3.10. *For all $n \geqslant 2$ we have*

(i) $p_{F_1}(8n^2-6n+1\pm(2i-1),3)=0$*, if* $0\leqslant i\leqslant n$*;*

(ii) $p_{F_1}(8n^2-6n+1,3)=\left\lfloor\dfrac{n^2+1}{2}\right\rfloor$*;*

(iii) $p_{F_1}(8n^2-6n+1+2i,3)=p_{F_1}(8n^2-6n+1-2i,3)$*, if* $0\leqslant i\leqslant n$*.*

Proof.

(i) The largest even part of any partition counted by $p_{F_1}(8n^2-6n+1\pm(2i-1),3)$ is $4n-4$. Then, we write

$$8n^2-6n+1\pm(2i-1)=2(2+4+\cdots+4n-4))+r+s+t$$

with odd r,s,t and $1\leqslant r\leqslant s\leqslant t\leqslant 4n-3$, which implies

$$r+s+t=6n-3\pm(2i-1).$$

Noting that $6n-3\pm(2i-1)$ is an even number, clearly it cannot be written as sum of three odd numbers.

(ii) Again, the largest even part of any partition counted by $p_{F_1}(8n^2-6n+1,3)$ is $4(n-1)$. Then, we write

$$8n^2-6n+1=2(2+4+\cdots+4(n-1))+r+s+t$$

with odd r,s,t and $1\leqslant r\leqslant s\leqslant t\leqslant 4n-3$, which implies

$$r+s+t=6n-3.$$

What we need is evaluate the number of solutions of

$$r+s+t=6n-3\quad\text{with odd } r,s,t \text{ and } 1\leqslant r\leqslant s\leqslant t\leqslant 4n-3.$$

As r,s,t are odd, we can write

$$2R-1+2S-1+2T-1=6n-3\quad\text{with } 1\leqslant R\leqslant S\leqslant T\leqslant 2(n-1),$$

or

$$R+S+T=3n\quad\text{with } 1\leqslant R\leqslant S\leqslant T\leqslant 2n-1.$$

Note that this equation is exactly the same as the one stablished in (3.8), whose number of solution has already been calculated and is equal to $\left\lfloor\dfrac{n^2+1}{2}\right\rfloor$.

(iii) We will build a bijection between sets $P_{F_1}(8n^2-6n+1+2i,3)$ and $P_{F_1}(8n^2-6n+1-2i,3)$. The largest even part of any partition counted by $p_{F_1}(8n^2-6n+1\pm 2i,3)$ is $4(n-1)$.

So, given a partition counted by $p_{F_1}(8n^2-6n+1+2i,3)$, we have

$$2(2+4+\cdots+4(n-1))+r+s+t=8n^2-6n+1+2i,$$

with odd r, s, t and $1 \leqslant r \leqslant s \leqslant t \leqslant 4n-3$, which implies

$$r+s+t = 6n-3+2i.$$

If we take the partition

$$\lambda = (2,2,4,4,\ldots,4(n-1),4(n-1),4n-2-t,4n-2-s,4n-2-r),$$

it belongs to $p_{F_1}(8n^2-6n+1-2i,3)$, because $4n-2-t, 4n-2-s, 4n-2-r$ are odd, with $1 \leqslant 4n-2-t \leqslant 4n-2-s \leqslant 4n-2-r \leqslant 4n-3$, and

$$\begin{aligned} 2(2+4+\cdots 4(n-1))+3(4n-2)-(t+s+r) &= 8n^2-12n+4+12n-6-(r+s+t) \\ &= 8n^2-2-(6n-3+2i) \\ &= 8n^2-6n+1-2i. \end{aligned}$$

The way back is easy to set. □

3.3 Mock Theta Function $f_0(q)$

We start by considering the Mock Theta Function of order 5

$$f_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n} = 1+q-q^2+q^3-q^6+q^7+q^9-2q^{10}+q^{11}+\cdots.$$

If we consider its expansion, we get negative coefficients brought fourth by the denominator of its general term. By removing this signal we get

$$f_0^*(q) = \sum_{s=0}^{\infty} \frac{q^{s^2}}{(q;q)_s} = \sum_{s=0}^{\infty} \frac{q^{1+3+5+\cdots+(2s-1)}}{(1-q)(1-q^2)\cdots(1-q^s)}, \tag{3.17}$$

whose general term generates the partitions of n containing each one of the odd numbers $1,3,5,...,2s-1$ as part of multiplicity 1 and any number of parts less than or equal to s. Also, it is the generating function for superdistinct partitions (partitions where $\lambda_i - \lambda_{i+1} \geqslant 2$) of n into exactly s parts. To see this, just add an unrestricted partition into at most s parts to the right of the triangle $1+3+\ldots+(2s-1)$.

Definition 3.5. *Let $p_{f_0}(n)$ be the number of partitions of n where, if $2s-1$ is the greatest odd part, each one of odd numbers smaller than it must appear once. Besides those parts, any quantity of parts smaller than or equal to s. When we show examples for this partitions, we bold the parts beyond the sequence with no gaps of odd parts that are part of them. So,*

$$f_0^*(q) = \sum_{n=0}^{\infty} p_{f_0}(n) q^n.$$

Example 3.6. $p_{f_0}(9) = 5$, *because the five considered partitions are*

$$(5,3,1);$$

$$(3, \mathbf{2}, \mathbf{2}, \mathbf{1}, 1);$$

$$(3, \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, 1);$$

$$(3, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, 1);$$

$$(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, 1);$$

As we can see, the coefficients of the unsigned version are the number of partitions that satisfy the Definition 3.5. We explore this identity to prove a combinatorial interpretation for the function $f_0^*(q)$, given in [9]. This sets a bijection between those partitions and two-line matrices, subject to some restrictions on the entries.

Theorem 3.5. *The coefficient of q^n in the expansion of* (3.17) *is equal to the number of elements in the set of matrices of the form*

$$A = \begin{pmatrix} c_1 & c_2 & \cdots & c_s \\ d_1 & d_2 & \cdots & d_s \end{pmatrix}, \tag{3.18}$$

with non-negative integer entries satisfying

$$c_s = 1; d_t \geqslant 0; \tag{3.19}$$

$$c_t = 2 + c_{t+1} + d_{t+1}, \ \forall t < s; \tag{3.20}$$

$$n = \sum c_t + \sum d_t. \tag{3.21}$$

Proof. According to the general term of (3.17), we can decompose n as

$$n = 1 + 3 + 5 + \cdots + (2s - 1) + (1 \cdot d_1 + 2 \cdot d_2 + \cdots + s \cdot d_s)$$

or, equivalently, as the sum of the entries of the matrix

$$A = \begin{pmatrix} 2s - 1 + d_2 + \cdots + d_s & \cdots & 5 + d_{s-1} + d_s & 3 + d_s & 1 \\ d_1 & \cdots & d_{s-2} & d_{s-1} & d_s \end{pmatrix},$$

with $d_i \geqslant 0$ for all i.

Noting that the entries satisfy conditions (3.19) to (3.21), the theorem is proved. □

The second row of the matrices of type (3.18) describes the parts from 1 to s, besides the odd parts from 1 to $2s - 1$, of the partition associated to it. To know how many parts from 1 to s the partition has, we have to sum the d_i, for $i = 1, 2, \ldots, s$.

Definition 3.6. *Let $p_{f_0}(n, k)$ be the number of partitions counted by $p_{f_0}(n)$ having k parts beyond the sequence of consecutive odd numbers starting from* 1*, in accordance with Definition 3.5.*

Example 3.7. *We have that* $p_{f_0}(25,9) = 5$. *Its five partitions are listed below.*

$$(7,5,3,1,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1});$$

$$(5,3,\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1},1);$$

$$(5,\mathbf{3},3,\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1},1);$$

$$(5,\mathbf{3},\mathbf{3},3,\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},1);$$

$$(5,\mathbf{3},\mathbf{3},\mathbf{3},3,\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},1).$$

As the authors did for $f_1(q)$ in [9], they defined the same **weight** for partitions generated by this function. It is

$$\omega_{f_0}(\lambda) = (-1)^{\sum d_i},$$

where the elements d_i are the entries of the second row. Equivalently, $\sum d_i$ counts the number of parts beyond the odd sequence. So, the coefficient of q^n the expansion of f_1 can be rewritten as

$$f_0(q) = \sum_{n=0}^{\infty} \Big(\sum_{even\ k} p_{f_0}(n,k) - \sum_{odd\ k} p_{f_0}(n,k) \Big) q^n.$$

For a fixed n, we classify its partitions of type described in Definition 3.6 according to the sum on the second row of the matrix associated to it, which is the same as the number of parts beyond the odd sequence. By counting the appearance of each number in these sums, we can organize the data on a table, which is presented next. The entry in line n and column $n-j$ is the number of times j appears as sum of the entries of the second row in type (3.18) matrices.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
1	1																																							
2	1	0																																						
3	1	0	0																																					
4	1	0	0	1																																				
5	1	0	0	1	0																																			
6	1	0	0	1	1	0																																		
7	1	0	0	1	1	0	0																																	
8	1	0	0	1	1	1	0	0																																
9	1	0	0	1	1	1	0	0	1																															
10	1	0	0	1	1	1	1	0	1	0																														
11	1	0	0	1	1	1	1	0	1	1	0																													
12	1	0	0	1	1	1	1	1	1	1	1	0																												
13	1	0	0	1	1	1	1	1	1	1	2	0	0																											
14	1	0	0	1	1	1	1	1	2	1	2	1	0	0																										
15	1	0	0	1	1	1	1	1	2	1	2	2	1	0	0																									
16	1	0	0	1	1	1	1	1	2	2	2	2	2	0	0	1																								
17	1	0	0	1	1	1	1	1	2	2	2	2	3	1	0	1	0																							
18	1	0	0	1	1	1	1	1	2	2	3	2	3	2	1	1	1	0																						
19	1	0	0	1	1	1	1	1	2	2	3	2	3	3	2	1	1	1	0																					
20	1	0	0	1	1	1	1	1	2	2	3	3	3	3	3	2	1	2	1	0																				
21	1	0	0	1	1	1	1	1	2	2	3	3	3	3	4	3	2	2	2	0	0																			
22	1	0	0	1	1	1	1	1	2	2	3	3	4	3	4	4	3	2	3	2	0	0																		
23	1	0	0	1	1	1	1	1	2	2	3	3	4	3	4	5	4	3	3	3	1	0	0																	
24	1	0	0	1	1	1	1	1	2	2	3	3	4	4	4	5	5	4	4	4	3	1	0	0																
25	1	0	0	1	1	1	1	1	2	2	3	3	4	4	4	5	6	5	5	4	4	3	0	0	1															
26	1	0	0	1	1	1	1	1	2	2	3	3	4	4	5	5	6	6	6	5	5	5	2	0	1	0														
27	1	0	0	1	1	1	1	1	2	2	3	3	4	4	5	5	6	7	7	6	6	6	4	1	1	1	0													
28	1	0	0	1	1	1	1	1	2	2	3	3	4	4	5	6	6	7	8	7	7	7	6	4	2	1	1	0												
29	1	0	0	1	1	1	1	1	2	2	3	3	4	4	5	6	6	7	9	8	8	8	7	6	4	1	2	1	0											
30	1	0	0	1	1	1	1	1	2	2	3	3	4	4	5	6	7	7	9	9	9	9	9	8	7	3	2	2	1	0										
31	1	0	0	1	1	1	1	1	2	2	3	3	4	4	5	6	7	7	9	10	10	10	10	9	9	6	3	3	3	0	0									
32	1	0	0	1	1	1	1	1	2	2	3	3	4	4	5	6	7	8	9	10	11	11	11	11	11	9	6	4	4	2	0	0								
33	1	0	0	1	1	1	1	1	2	2	3	3	4	4	5	6	7	8	9	10	12	12	12	12	13	11	9	6	5	4	2	0	0							
34	1	0	0	1	1	1	1	1	2	2	3	3	4	4	5	6	7	8	10	10	12	13	13	13	15	13	12	10	7	5	5	1	0	0						
35	1	0	0	1	1	1	1	1	2	2	3	3	4	4	5	6	7	8	10	10	12	14	14	14	16	15	14	13	10	7	7	4	1	0	0					
36	1	0	0	1	1	1	1	1	2	2	3	3	4	4	5	6	7	8	10	11	12	14	15	15	17	17	17	16	14	10	9	7	4	0	0	1				
37	1	0	0	1	1	1	1	1	2	2	3	3	4	4	5	6	7	8	10	11	12	14	16	16	18	18	19	18	17	14	12	9	8	3	0	1	0			
38	1	0	0	1	1	1	1	1	2	2	3	3	4	4	5	6	7	8	10	11	13	14	16	17	19	19	21	21	20	18	16	12	11	7	2	1	1	0		
39	1	0	0	1	1	1	1	1	2	2	3	3	4	4	5	6	7	8	10	11	13	14	16	18	20	20	22	23	23	21	20	16	14	11	7	2	1	1	0	
40	1	0	0	1	1	1	1	1	2	2	3	3	4	4	5	6	7	8	10	11	13	15	16	18	21	21	23	25	26	24	24	21	18	15	12	6	2	2	1	0

Table 6 – Table of the characterization given by Theorem (3.5)

Example 3.8. *Looking at some numbers that appear in the* 16^{th} *line, from right to left, we have:*

- 1 *partition with* $0, 8, 9, 10, 11, 12$ *and* 15 *other parts beyond the ones that must appear.*
- 2 *partitions with* $3, 4, 5, 6$ *and* 7 *other parts beyond the ones that must appear.*
- *no partitions with* $1, 2, 13$ *and* 14 *other parts beyond the ones that must appear.*

By observing the table above we get some interesting results.

Remark 3.1. *By observing the first diagonal, corresponding to partitions with no other part besides* $1, 3, \ldots, 2s-1$*, we can easily note that only squares are able to be partitioned into distinct odd parts with no gaps. That is,* $n^2 = 1 + 3 + \cdots + (2n-1)$.

The next theorem describes the numbers that are in the second diagonal of Table 6. These numbers correspond to the set of partitions with only one part beyond the sequence of consecutive odd numbers that must appear.

Proposition 3.11. *For all* $n \geqslant 0$ *and* $0 \leqslant i \leqslant n$*, we have*

(i) $p_{f_0}(2T_n + 1 + i, 1) = 0$;

(ii) $p_{f_0}(2T_n + n + 2 + i, 1) = 1$.

Proof. (i) Suppose we can write $2T_n + 1 + i$ as a sum of s consecutive odd parts plus one part k, with $1 \leqslant k \leqslant s$. In this case, the largest part is $2n - 1$.

Let us suppose $\lambda = (1, 3, \ldots, 2n-1, k)$, with $1 \leqslant k \leqslant n$, is a partition of $2T_n + 1 + i$. Then,

$$\begin{aligned} 1 + 3 + \cdots + 2n - 1 + k &= 2T_n + 1 + i \\ n^2 + k &= n^2 + n + 1 + i \\ k &= n + 1 + i, \end{aligned}$$

which is an absurd.

(ii) The largest part of any partition of $2T_n + n + 2 + i$ must be $2n + 1$.

Then, let $\lambda = (1, 3, \ldots, 2n+1, k)$, with $1 \leqslant k \leqslant n+1$, be a partition of $2T_n + n + 2 + i$.

$$\begin{aligned} 1 + 3 + \cdots + 2n + 1 + k &= 2T_n + n + 2 + i \\ (n+1)^2 + k &= n^2 + 2n + 2 + i \\ k &= 1 + i. \end{aligned}$$

As $i \in \{0, 1, 2, \ldots, n\}$ and $k \in \{1, 2, 3, \ldots, n+1\}$, for each value of i we get only one value of k.

□

In order to prove the next Proposition, first we need the following lemmas.

Lemma 3.1. *For all odd $n \geqslant 1$ and $0 \leqslant i \leqslant n$, any solution (y_1, y_2) of the equation $y_1 + y_2 = n + 1 + i$, with $1 \leqslant y_1 \leqslant y_2 \leqslant n$, can be written as*

(i) $(x_1 + j, x_2 + j)$, *if* $i = 2j$, *or*

(ii) $(x_1 + j - 1, x_2 + j)$, *if* $i = 2j - 1$,

where (x_1, x_2) is solution of equation $x_1 + x_2 = n + 1$, with $1 + j \leqslant x_1 \leqslant x_2 \leqslant n - j$.

Proof. (i) Suppose $i = 2j$ and $y_1 + y_2 = n + 1 + i$. First of all, note that as $y_1 + y_2 = n + 1 + i$ and $y_2 \leqslant n$, we have $y_1 \geqslant i + 1$. So, we can write $y_1 = i + l = 2j + l$ with $l \geqslant 1$. Writing $x_1 = j + l$ we get $y_1 = x_1 + j$.

As $y_2 \geqslant y_1$, we can write $y_2 = y_1 + t$ with $t \geqslant 0$. So, $y_2 = x_1 + j + t$ and we call $x_2 = x_1 + t$. Note that $x_1 + x_2 = (y_1 - j) + (y_2 - j) = y_1 + y_2 - 2j = n + 1 + i - i = n + 1$ and that $x_1 + x_2 = n + 1$ implies $x_2 = n + 1 - x_1 = n + 1 - (j + l) \leqslant n - j$.

(ii) If $i = 2j - 1$, then n and i are both odd, which means $y_1 < y_2$. Again, as $y_2 \leqslant n$, $y_1 \geqslant i + 1$ or $y_1 = i + l = l + j + (j - 1)$, with $l \geqslant 1$. We take $x_1 = l + j$.

As $y_1 < y_2$, then $y_2 = y_1 + t = x_1 + (j - 1) + t$, with $t \geqslant 0$. We take $x_2 = x_1 - 1 + t$. Noting that $x_1 + x_2 = y_1 - (j - 1) + y_2 - j = y_1 + y_2 - i = n + 1 + i - i = n + 1$ and that $x_2 = n + 1 - x_1 = n + 1 - (l + j) \leqslant n - j$, we get what we need.

□

Lemma 3.2. *For all even $n \geqslant 1$ and $0 \leqslant i \leqslant n$, any solution (y_1, y_2) of the equation $y_1 + y_2 = n + 1 + i$, with $1 \leqslant y_1 \leqslant y_2 \leqslant n$, can be written as*

(i) $(x_1 + j, x_2 + j)$, *if* $i = 2j$, *where (x_1, x_2) is solution of equation $x_1 + x_2 = n + 1$ with $1 + j \leqslant x_1 \leqslant x_2 \leqslant n - j$, or*

(ii) $(x_1 + j, x_2 + j - 1)$, *if* $i = 2j - 1$, *where (x_1, x_2) is solution of equation $x_1 + x_2 = n + 1$, with $j \leqslant x_1 \leqslant x_2 \leqslant n - j + 1$.*

Proof. The proof of item (i) is analogous to the proof of item (ii) in Lemma 3.1.

(ii) Again $y_2 \leqslant n$ implies $y_1 \geqslant i + 1$ and we write $y_1 = i + l = l + (j - 1) + j$, taking $x_1 = l + (j - 1)$.

As $y_2 \geqslant y_1$, we write $y_2 = y_1 + t = x_1 + j + t$ with $t \geqslant 0$. Note that $x_1 + x_2 = n + 1$ with even n implies x_1 and x_2 have different parities. So, as $x_2 \geqslant x_1$, we have $x_2 > x_1$. Therefore we take $x_2 = x_1 + t + 1$.

Now, $x_1 + x_2 = y_1 - j + y_2 - (j - 1) = y_1 + y_2 - (2j - 1) = n + 1 + i - i = n + 1$, with $x_2 = n + 1 - x_1 = n + 1 - (l + j - 1) \leqslant n + 1 - j$.

□

Proposition 3.12. *For all $n \geqslant 1$ and $0 \leqslant i \leqslant n$ we have*

$$p_{f0}(2T_n + 1 \pm i, 2) = \begin{cases} \lfloor \frac{n+1}{2} \rfloor - \lfloor \frac{i+1}{2} \rfloor, & \text{for odd } n \\ \lfloor \frac{n+1}{2} \rfloor - \lfloor \frac{i}{2} \rfloor, & \text{for even } n. \end{cases}$$

Proof. Let n be an odd positive integer and suppose we can write $2T_n + 1 \pm i$ as a sum of s consecutive odd parts plus two parts y_1 and y_2, with $1 \leqslant y_1 \leqslant y_2 \leqslant s$. As done many times before, we can prove that the greatest part of this partition has to be $2n - 1$. Therefore,

$$\begin{aligned} 1 + 3 + \cdots + 2n - 1 + y_1 + y_2 &= 2T_n + 1 + \pm i \\ n^2 + y_1 + y_2 &= n^2 + n + 1 \pm i \\ y_1 + y_2 &= n + 1 \pm i \end{aligned} \tag{3.22}$$

Now we want to know the number of positive integer solutions of equation (3.22). Let us solve the problem for $y_1 + y_2 = n + 1 + i$, the negative case being adaptable by changing signs.

First, if n is odd, by Lemma 3.1 the the number of solutions we are looking for is the same as the number of solutions of equation $x_1 + x_2 = n + 1$, with $1 + j \leqslant x_1 \leqslant x_2 \leqslant n - j$ and $j = \left\lfloor \frac{i+1}{2} \right\rfloor$. As there are $\left\lfloor \frac{n+1}{2} \right\rfloor$ solutions for equation $x_1 + x_2 = n + 1$ with no restriction, other than $x_1 \leqslant x_2$, we have to exclude those where $x_1 < j + 1$ or $x_2 > n - j$. But observe that, if we have $x_1 < j + 1$, automatically we get $x_2 > n - j$ and *vice versa*. So we only need to analyse the case where $x_1 < j + 1$. Clearly, those solutions are in number of $\left\lfloor \frac{j}{2} \right\rfloor = \left\lfloor \frac{i+1}{2} \right\rfloor$.

So, the number of solutions we were looking for is $\left\lfloor \frac{n+1}{2} \right\rfloor - \left\lfloor \frac{i+1}{2} \right\rfloor$.

If n is even we use Lemma 3.2. Again there are $\left\lfloor \frac{n+1}{2} \right\rfloor$ solutions for equation $x_1 + x_2 = n + 1$ with no restriction. If $i = 2j - 1$ we have to exclude the solutions where $x_1 < j$ or $x_2 > n + 1 - j$, which are equivalent. If $i = 2j$ we have to exclude the solutions where $x_1 < j + 1$ or $x_2 > n + j$. In both cases this number is $\left\lfloor \frac{i}{2} \right\rfloor$.

So, the number of solutions we were looking for is $\left\lfloor \frac{n+1}{2} \right\rfloor - \left\lfloor \frac{i}{2} \right\rfloor$. □

Proposition 3.13. *For all $n \geqslant 1$ we have*

$$p_{f0}(n^2, 3) = p(n - 2, \leqslant 3).$$

Proof. The largest part of any partition counted by $p_{f0}(n^2, 3)$ must be $2n - 3$. As

$$n^2 - (1 + 3 + \cdots + ()2n - 3)) = 2n - 1,$$

it remains that $2n - 1$ need to be partitioned into three parts smaller than or equal to $n - 1$. That is,

$$\lambda_1 + \lambda_2 + \lambda_3 = 2n - 1, \quad \text{with } \lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant n - 1.$$

Let us consider $(\lambda_1, \lambda_2, \lambda_3)$, whose parts are solutions for the equation above. By writing $\mu = (\mu_1, \mu_2, \mu_3) = (n-1-\lambda_3, n-1-\lambda_2, n-1-\lambda_1)$, we get a partition of $n-2$. Some μ_i might be zero and we omit them. □

Example 3.9. *Considering $n = 7$, we get the partitions below:*

$P_{f_0}(49,3)$	$(\lambda_1, \lambda_2, \lambda_3)$	(μ_1, μ_2, μ_3)	$P(5, \leqslant 3)$
$(11, 9, 7, \mathbf{6}, \mathbf{6}, 5, 3, 1, \mathbf{1})$	$(6, 6, 1)$	$(5, 0, 0)$	(5)
$(11, 9, 7, \mathbf{6}, 5, \mathbf{5}, 3, \mathbf{2}, 1)$	$(6, 5, 2)$	$(4, 1, 0)$	$(4, 1)$
$(11, 9, 7, \mathbf{6}, 5, \mathbf{4}, 3, \mathbf{3}, 1)$	$(6, 4, 3)$	$(3, 2, 0)$	$(3, 2)$
$(11, 9, 7, 5, \mathbf{5}, \mathbf{5}, 3, \mathbf{3}, 1)$	$(5, 5, 3)$	$(3, 1, 1)$	$(3, 1, 1)$
$(11, 9, 7, 5, \mathbf{5}, \mathbf{4}, \mathbf{4}, 3, 1)$	$(5, 4, 4)$	$(2, 2, 1)$	$(2, 2, 1)$

In order to prove the next proposition, we will need the following lemma.

Lemma 3.3. *For all $n \geqslant 1$, if we define*

$$A := \{(r, s, t) \in \mathbb{Z}^3 \; ; \; r + s + t = 3n + 3, 1 \leqslant t \leqslant s \leqslant r \leqslant 2n \text{ and } (r = 2n \text{ or } t = 1)\}, \qquad (3.23)$$

its cardinality is n, that is $|A| = n$.

Proof. We separate the proof in two cases: (i) $r = 2n$ and (ii) $t = 1$.

(i) If $r = 2n$, then

$$\begin{aligned} r + s + t &= 3n + 3 \\ s + t &= 3n + 3 - 2n \\ s + t &= n + 3, \end{aligned}$$

which has $\lfloor \frac{n+3}{2} \rfloor$ solutions.

(ii) If $t = 1$, then

$$r + s = 3n + 2.$$

As $r \leqslant 2n$, then $s \geqslant n + 2$. So, note that r and s can assume $2n - (n+2) + 1 = n - 1$ different values. This gives us $\left\lfloor \frac{n-1+1}{2} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor$ different solutions, without counting the order, as usual.

Now, note that the solution of $r + s + t = 3n + 3$ where $r = 2n$ and $t = 1$ has been counted in both cases. So, the number of solutions of equation $r + s + t = 3n + 3$ satisfying the desired conditions, i.e., the number of elements of the set (3.23) is equal to

$$\left\lfloor \frac{n+3}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor - 1.$$

By considering both parities of n, we can prove that $\left\lfloor \frac{n+3}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor - 1 = n$ and the result follows.

□

Proposition 3.14. *For all $n \geqslant 1$ we have*

$$p_{f_0}(4n^2 + 3n + 3, 3) = T_n$$

Proof. The largest part of any partition counted by $p_{f_0}(4n^2 + 3n + 3, 3)$ must be $4n - 1$. The three parts less than or equal to $2n$ must satisfy $r + s + t = 3n + 3$.

This means that

$$p_{f_0}(4n^2 + 3n + 3, 3) = |\{(r, s, t)\ ;\ r + s + t = 3n + 3, 1 \leqslant t \leqslant s \leqslant r \leqslant 2n\}|$$

which is the same as

$$p_{f_0}(4n^2 + 3n + 3, 3) = |\{(r, s, t)\ ;\ r + s + t = 3n + 3, 2 \leqslant t \leqslant s \leqslant r \leqslant 2n - 1\}|$$

$$+ \quad |\{(r, s, t)\ ;\ r + s + t = 3n + 3, 1 \leqslant t \leqslant s \leqslant r \leqslant 2n \textit{ and } (r = 2n \textit{ or } t = 1)\}|$$

By Lemma 3.3, it turns to

$$p_{f_0}(4n^2 + 3n + 3, 3) = |\{(r, s, t)\ ;\ r + s + t = 3n + 3, 2 \leqslant t \leqslant s \leqslant r \leqslant 2n - 1\}| + n,$$

and we only need to prove that

$$|\{(r, s, t)\ ;\ r + s + t = 3n + 3, 2 \leqslant t \leqslant s \leqslant r \leqslant 2n - 1\}| = p_{f_0}(4n^2 - 5n + 4, 3),$$

that will be by showing a bijection.

Given a partition counted by $p_{f_0}(4n^2 - 5n + 4, 3)$, consider the parts r, s, t satisfying $r + s + t = 3n$, with $1 \leqslant t \leqslant s \leqslant r \leqslant 2n - 2$. Set the partition $(r + 1, s + 1, t + 1)$ and note that $2 \leqslant t + 1 \leqslant s + 1 \leqslant r + 1 \leqslant 2n - 1$.

The result follows by induction. □

Example 3.10. *In the following table, we show the partitions of* $P_{f_0}(4n^2+3n+3,3)$*, for* $n=1,2,3,4$*, according to Theorem* (3.14).

$n=1$	$n=2$	$n=3$	$n=4$
$P_{f_0}(10,3)$	$P_{f_0}(25,3)$	$P_{f_0}(48,3)$	$P_{f_0}(79,3)$
$(3,\mathbf{2},\mathbf{2},\mathbf{2},1)$	$(7,5,3,\mathbf{3},\mathbf{3},\mathbf{3},1)$	$(11,9,7,5,\mathbf{4},\mathbf{4},\mathbf{4},3,1)$	$(15,13,11,9,7,5,\mathbf{5},\mathbf{5},\mathbf{5},3,1)$
	$(7,5,\mathbf{4},3,\mathbf{3},\mathbf{2},1)$	$(11,9,7,5,\mathbf{5},\mathbf{4},3,\mathbf{3},1)$	$(15,13,11,9,7,\mathbf{6},5,\mathbf{5},\mathbf{4},3,1)$
	$(7,5,\mathbf{4},\mathbf{4},3,1,\mathbf{1})$	$(11,9,7,5,\mathbf{5},\mathbf{5},3,\mathbf{2},1)$	$(15,13,11,9,7,\mathbf{6},\mathbf{6},5,3,\mathbf{3},1)$
		$(11,9,7,\mathbf{6},5,\mathbf{5},3,1,\mathbf{1})$	$(15,13,11,9,7,\mathbf{7},\mathbf{6},5,3,\mathbf{2},1)$
		$(11,9,7,\mathbf{6},5,\mathbf{4},3,\mathbf{2},1)$	$(15,13,11,9,7,\mathbf{7},5,\mathbf{5},3,\mathbf{3},1)$
		$(11,9,7,\mathbf{6},5,3,\mathbf{3},\mathbf{3},1)$	$(15,13,11,9,7,\mathbf{7},5,\mathbf{4},\mathbf{4},3,1)$
			$(15,13,11,9,7,\mathbf{7},\mathbf{7},5,3,1,\mathbf{1})$
			$(15,13,11,9,\mathbf{8},7,\mathbf{6},5,3,1,\mathbf{1})$
			$(15,13,11,9,\mathbf{8},7,5,\mathbf{5},3,\mathbf{2},1)$
			$(15,13,11,9,\mathbf{8},7,5,\mathbf{4},3,\mathbf{3},1)$

Theorem 3.6. *For all* $n \geqslant 2$ *and* $i \geqslant 0$ *we have*

$$p_{f_0}(2n+i, n-2+i) = p_b(n+2),$$

where $p_b(n)$ *is the number of balanced partitions of* n*, i.e., the number of partitons of* n *where the smallest part equals the number of parts.*

Proof. We begin with $i=0$.

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a partition counted by $p_b(n+2)$. Note that $\lambda_k = k$ and the Ferrers graph of this partition has Durfee square of size k. Considering the k^2 points of the Durfee square, let's write them as the partition $(2k-1, 2k-3, \ldots, 3, 1)$. We denote by $\lambda^{(1)}$ the partition on the right side of the Durfee square. So,

$$n+2 = k^2 + |\lambda^{(1)}|.$$

Observe that, as $n \geqslant 2$, we have $k \geqslant 2$, and as $n+2 \geqslant k^2 + \lambda_1 - k$, then $\lambda_1 - k \leqslant n+2-k^2 \leqslant n-2$.

Let us consider $\overline{\lambda^{(1)}}$ the conjugate partition of $\lambda^{(1)}$ and add to its left side the partition $(\underbrace{1,1,\ldots,1}_{n-2})$. As the number of parts of $\lambda^{(1)}$ is less than k (because $\lambda_k = k$), each part of this new partition we built is less than k.

Observe that the number we have partitioned now is

$$n+2-k^2+(n-2) = 2n-k^2.$$

By joining this partition to the partition $(2k-1, 2k-3, \ldots, 3, 1)$, we get

$$2n-k^2+2k-1+2k-3+\cdots+3+1 = 2n-k^2+k^2 = 2n,$$

partitioned into k odd parts plus $n-2$ excesses less than k. In other words, it is a partition counted by $p_{f_0}(2n, n-2)$.

Conversely, given a partition counted by $p_{f_0}(2n, n-2)$, let us suppose its greatest odd part is $2k-1$. Then we transform the parts $2k-1, 2k-3, \ldots, 3, 1$ into the partition $(\underbrace{k, k \ldots, k}_{k})$. The remaining parts are the $n-2$ excesses less than or equal to k. We subtract one unit from each of these excesses and conjugate this new parts. By adding this conjugated partition to the right side of the partition $(k, k, \ldots, k)$, we obtain a balanced partition of $n+2$.

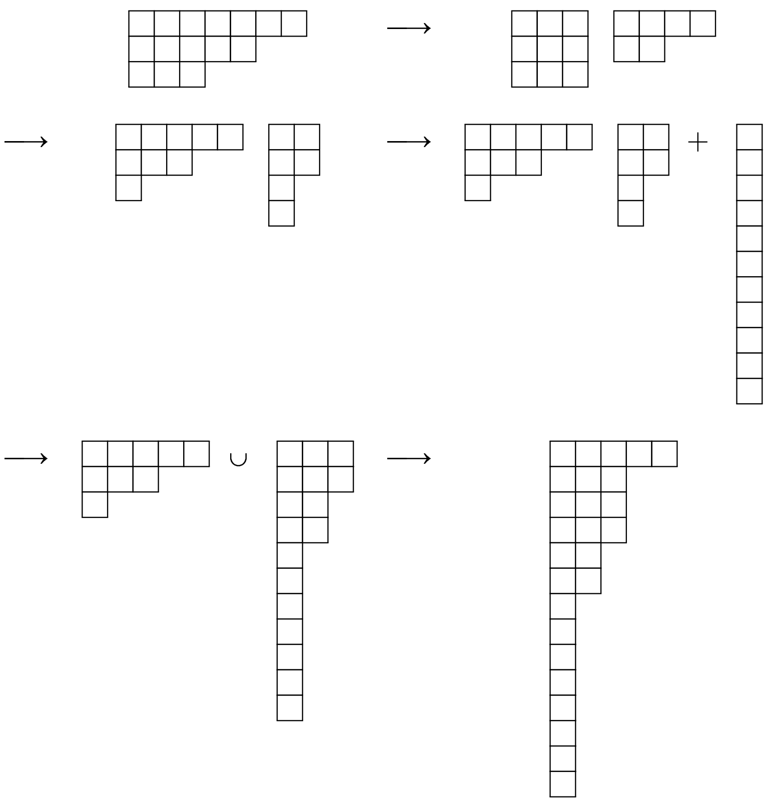
If $i \geqslant 1$, the only way to have a partition counted by $p_{f_0}(2n+i, n-2+i)$ is by having i parts of size one below the Durfee square of this partition. Clearly, a bijection between $P_{f_0}(2n, n-2)$ and $P_{f_0}(2n+i, n-2+i)$, in one way, adds parts of size one and, in the other way, removes them. $\square$

Example 3.11. *For* $n = 13$*, the partition* $(7, 5, 3) \in P_b(15)$ *leads to*

$$(5, \mathbf{3}, \mathbf{3}, 3, \mathbf{2}, \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, 1),$$

a partition of 26 *that lies in* $P_{f_0}(26, 11)$*.*

$$(7, 5, 3) \longrightarrow (3, 3, 3) + (4, 2)$$



3.4 Mock Theta Function $F_0(q)$

In this section, we consider the mock theta function of order 5

$$F_0(q) = \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q;q^2)_n}. \tag{3.24}$$

Now, the expansion in power series does not consider signal. So, the general term

$$\frac{q^{2(1+3+5+\cdots+(2s-1))}}{(1-q)(1-q^3)\cdots(1-q^{2s-1})},$$

can be interpreted as generating function for partitions of n into odd parts from 1 to $2s-1$, with no gaps, in a way that each part has multiplicity at least two.

Theorem 3.7. *The coefficient of q^n in the expansion of* (3.24) *is equal to the number of elements in the set of matrices of the form*

$$A = \begin{pmatrix} c_1 & c_2 & \cdots & c_s \\ d_1 & d_2 & \cdots & d_s \end{pmatrix}, \tag{3.25}$$

with non-negative integer entries satisfying

$$c_s = 2; d_t \geqslant 0; \tag{3.26}$$

$$c_t = 4 + c_{t+1} + 2d_{t+1}, \ \forall t < s; \tag{3.27}$$

$$n = \sum c_t + \sum d_t. \tag{3.28}$$

Proof. According to the general term of (3.24), we can decompose n as

$$n = 2\cdot 1 + 2\cdot 3 + 2\cdot 5 + \cdots + 2\cdot(2s-1) + (1\cdot d_1 + 3\cdot d_2 + \cdots + (2s-1)\cdot d_s)$$

or, equivalently, as the sum of the entries of the matrix

$$A = \begin{pmatrix} 2(2s-1) + 2d_2 + \cdots + 2d_s & \cdots & 10 + 2d_{s-1} + 2d_s & 6 + 2d_s & 2 \\ d_1 & \cdots & d_{s-2} & d_{s-1} & d_s \end{pmatrix},$$

Noting that the entries satisfy conditions (3.26) to (3.28), the theorem is proved. □

The entries on the second row describe how many parts from 1 to $2s-1$ appear more than twice. To know this quantity, we have to sum them.

Definition 3.7. *Let $p_{F_0}(n,k)$ be the number of partitions of n into odd parts ranging from 1 to $2s-1$, each one appearing at least twice, and k odd parts beyond these two copies.*

As before, we can organize the data we have obtained by summing the second line in a table. The structure is the same as we have done in all tables before.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
1	0																																							
2	0	1																																						
3	0	1	0																																					
4	0	1	0	0																																				
5	0	1	0	0	0																																			
6	0	1	0	0	0	0																																		
7	0	1	0	0	0	0	0																																	
8	0	1	0	0	0	0	0	1																																
9	0	1	0	0	0	0	0	1	0																															
10	0	1	0	0	0	0	0	1	0	0																														
11	0	1	0	0	0	0	0	1	0	1	0																													
12	0	1	0	0	0	0	0	1	0	1	0	0																												
13	0	1	0	0	0	0	0	1	0	1	0	0	0																											
14	0	1	0	0	0	0	0	1	0	1	0	1	0	0																										
15	0	1	0	0	0	0	0	1	0	1	0	1	0	0	0																									
16	0	1	0	0	0	0	0	1	0	1	0	1	0	0	0	0																								
17	0	1	0	0	0	0	0	1	0	1	0	1	0	1	0	0	0																							
18	0	1	0	0	0	0	0	1	0	1	0	1	0	1	0	0	0	1																						
19	0	1	0	0	0	0	0	1	0	1	0	1	0	1	0	0	0	1	0																					
20	0	1	0	0	0	0	0	1	0	1	0	1	0	1	0	1	0	1	0	0																				
21	0	1	0	0	0	0	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0																			
22	0	1	0	0	0	0	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	0																		
23	0	1	0	0	0	0	0	1	0	1	0	1	0	1	0	1	0	2	0	1	0	1	0																	
24	0	1	0	0	0	0	0	1	0	1	0	1	0	1	0	1	0	2	0	1	0	2	0	0																
25	0	1	0	0	0	0	0	1	0	1	0	1	0	1	0	1	0	2	0	1	0	2	0	0	0															
26	0	1	0	0	0	0	0	1	0	1	0	1	0	1	0	1	0	2	0	2	0	2	0	1	0	0														
27	0	1	0	0	0	0	0	1	0	1	0	1	0	1	0	1	0	2	0	2	0	2	0	2	0	0	0													
28	0	1	0	0	0	0	0	1	0	1	0	1	0	1	0	1	0	2	0	2	0	2	0	2	0	1	0	0												
29	0	1	0	0	0	0	0	1	0	1	0	1	0	1	0	1	0	2	0	2	0	3	0	2	0	2	0	0	0											
30	0	1	0	0	0	0	0	1	0	1	0	1	0	1	0	1	0	2	0	2	0	3	0	2	0	3	0	0	0	0										
31	0	1	0	0	0	0	0	1	0	1	0	1	0	1	0	1	0	2	0	2	0	3	0	2	0	3	0	1	0	0	0									
32	0	1	0	0	0	0	0	1	0	1	0	1	0	1	0	1	0	2	0	2	0	3	0	3	0	3	0	2	0	0	0	1								
33	0	1	0	0	0	0	0	1	0	1	0	1	0	1	0	1	0	2	0	2	0	3	0	3	0	3	0	3	0	1	0	1	0							
34	0	1	0	0	0	0	0	1	0	1	0	1	0	1	0	1	0	2	0	2	0	3	0	3	0	3	0	3	0	2	0	1	0	0						
35	0	1	0	0	0	0	0	1	0	1	0	1	0	1	0	1	0	2	0	2	0	3	0	3	0	4	0	3	0	3	0	1	0	1	0					
36	0	1	0	0	0	0	0	1	0	1	0	1	0	1	0	1	0	2	0	2	0	3	0	3	0	4	0	3	0	4	0	2	0	1	0	0				
37	0	1	0	0	0	0	0	1	0	1	0	1	0	1	0	1	0	2	0	2	0	3	0	3	0	4	0	3	0	4	0	3	0	1	0	1	0			
38	0	1	0	0	0	0	0	1	0	1	0	1	0	1	0	1	0	2	0	2	0	3	0	3	0	4	0	4	0	4	0	4	0	2	0	2	0	0		
39	0	1	0	0	0	0	0	1	0	1	0	1	0	1	0	1	0	2	0	2	0	3	0	3	0	4	0	4	0	4	0	5	0	3	0	2	0	1	0	
40	0	1	0	0	0	0	0	1	0	1	0	1	0	1	0	1	0	2	0	2	0	3	0	3	0	4	0	4	0	4	0	5	0	4	0	2	0	2	0	0

Table 7 – Table from the characterization given by Theorem (3.7)

Remark 3.2. *The entries in columns in odd positions are zeros. These entries represent $p_{f_0}(n, n-k)$, where k is an odd number. Indeed, if n is even (resp. odd), we cannot decompose it into an odd (resp. even) number of odd parts.*

The main property is verified by looking at the non-zero columns. They are exactly the same of those ones from the table we build for the Mock Theta Function $f_0(q)$ in the last section. So, the results we present here can be proven by showing a relation between them and obtain similar identities.

There is a simple identity that relates the partitions generated by Mock Theta Function $F_0(q)$ with $f_0(q)$ we have considered, and it summarizes a lot of information.

Theorem 3.8. *For all $n \geqslant 1$ and $0 \leqslant i \leqslant n$, we have*

$$p_{F_0}(2n+i, i) = p_{f_0}(n+i, i).$$

Proof. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a partition of $2n+i$ counted by $p_{F_0}(2n+i, i)$, with largest odd part $2s-1$. Note that the exceeding parts sum $2n+i-2s^2$. As there are at least two copies of each odd part in λ, we remove one copy of each one of them, getting $2n+i-s^2$. Each exceeding odd part λ_j, we replace by $\mu_j = \dfrac{\lambda_j+1}{2}$, getting a part between 1 and s.

The reverse map is easy to get and, clearly, we have a bijection. □

Example 3.12. *By considering $n = 15$ and $i = 6$, we have $p_{F_0}(36, 6) = 4$. We illustrate the previous bijection in this case.*

$P_{F_0}(36,6)$	$\mapsto$	$P_{f_0}(21,6)$
$(\mathbf{5},\mathbf{5},\mathbf{5},5,5,3,3,\mathbf{1},\mathbf{1},\mathbf{1},1,1)$	$(\mathbf{3},\mathbf{3},\mathbf{3},5,3,\mathbf{1},\mathbf{1},\mathbf{1},1)$	$(5,\mathbf{3},\mathbf{3},\mathbf{3},3,\mathbf{1},\mathbf{1},\mathbf{1},1)$
$(\mathbf{5},\mathbf{5},5,5,\mathbf{3},\mathbf{3},3,3,\mathbf{1},\mathbf{1},1,1)$	$(\mathbf{3},\mathbf{3},5,\mathbf{2},\mathbf{2},3,\mathbf{1},\mathbf{1},1)$	$(5,\mathbf{3},\mathbf{3},3,\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1},1)$
$(\mathbf{5},5,5,\mathbf{3},\mathbf{3},\mathbf{3},\mathbf{3},3,3,\mathbf{1},1,1)$	$(\mathbf{3},5,\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2},3,\mathbf{1},1)$	$(5,\mathbf{3},3,\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{1},1)$
$(5,5,\mathbf{3},\mathbf{3},\mathbf{3},\mathbf{3},\mathbf{3},\mathbf{3},3,3,1,1)$	$(5,\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2},3,1)$	$(5,3,\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2},1)$

As consequence of Theorem 3.8, we get the following corollary.

Corollary 3.1. *For $n \geqslant 1$, we have the following identities:*

(*i*) $p_{F_0}(2n^2, 0) = p_{f_0}(n^2, 0) = 1$;

(*ii*) $p_{F_0}(2n^2+4n+3+2i, 1) = p_{f_0}(n^2+2n+2+i, 1) = 1$, $0 \leqslant i \leqslant n$;

(*iii*) $p_{F_0}(2n^2+2n \pm 2i, 2) = p_{f_0}(n^2+n+1 \pm i, 2)$;

(*iv*) $p_{F_0}(2n^2-3, 3) = p_{f_0}(n^2, 3) = p(n-2, \leqslant 3)$;

(*v*) $p_{F_0}(8n^2+6n+3, 3) = p_{f_0}(4n^2+3n+3, 3) = T_n$;

(*vi*) $p_{F_0}(3n+5+i, n-1+i) = p_{f_0}(2n+2+i, n-1+i) = p_b(n+3)$, $i \geqslant 0$.

4 Mock Theta Functions and partitions into two colors

Considering the models given in [9], in this chapter we focus on Mock Theta functions $\rho(q)$, $\sigma(q)$, $\nu(q)$ and $\lambda(q)$ the first two and the last one related to partitions into two colors. For $\nu(q)$, there is a signal which "weighs" the coefficients in its expansion in power series, so we first ignore it and analyze its general term as a generating function for a kind of partitions into one color. Although it is not apparent, we can map each one of them to a specific partition into two colors brought forth by $\lambda(q)$. By proving that this map is a bijection, both considered sets have the same cardinality. As a consequence, it proves an identity for $\nu(q)$ in a combinatorial way while it is already proved analytically.

The second line of matrices representing partitions generated by those Mock Theta functions also describes properties of the related partition. By summing the elements of the second line, we can classify the partitions according to these sums and organize the data in a table, as done in [1] for two matrix representations of unrestricted partitions. It allows us to investigate and discover properties, till then unknown, that clearly are suggested by the table.

By analyzing the table, the columns become constant below certain entry as it has happened for all tables we have built. Coincidentally in the next three tables (for $\rho(q), \sigma(q)$ and $\nu(q)$) these constant numbers represent the same sequence as three kind of partitions into distinct parts. These three identities are proven by showing bijections between the sets. Some identities in this chapter were collected and submitted as a paper [7].

4.1 Mock Theta Function $\rho(q)$

In this section, we study the Mock Theta function of order 6

$$\rho(q) = \sum_{n=0}^{\infty} \frac{(-q;q)_n q^{\binom{n+1}{2}}}{(q;q^2)_{n+1}}, \tag{4.1}$$

and obtain results and patterns that are suggested by a table, whose construction is made according to the sum of the second line of its matrix representation, found in [9].

Its general term

$$\frac{(1+q)(1+q^2)\cdots(1+q^s)q^{1+2+3+\cdots+s}}{(1-q)(1-q^3)\cdots(1-q^{2s+1})},$$

is the generating function for partitions of n into parts of two different colors:

- dark gray parts, ranging from 1 to s, with no gaps, and each part having multiplicity 1 or 2;
- any number of light gray odd parts less than or equal to $2s+1$.

Theorem 4.1. *The coefficient of q^n in the expansion of (4.1) is equal to the number of elements in the set of matrices of the form*

$$A = \begin{pmatrix} c_1 & c_2 & \cdots & c_s \\ d_1 & d_2 & \cdots & d_s \end{pmatrix}, \tag{4.2}$$

with non-negative integer entries satisfying

$$\begin{aligned} c_s &= 0;\ d_t \geqslant 0; && (4.3)\\ c_t &= i_t + c_{t+1} + 2d_{t+1},\ with\, i_t \in \{1,2\},\ \forall t < s; && (4.4)\\ n &= \sum c_t + \sum d_t. && (4.5) \end{aligned}$$

Proof. According to the general term of (4.1), we can decompose n as

$$n = ((1+j_1)\cdot 1 + (1+j_2)\cdot 2 + \cdots + (1+j_s)\cdot s) + (d_1 + 3d_3 + \cdots + (2s+1)\cdot d_{s+1})$$

with $j_t \in \{0,1\}$ and $d_t \geqslant 0$,or, equivalently, as the sum of the entries of the matrix

$$A = \begin{pmatrix} s + j_1 + \cdots + j_s + 2d_2 + \cdots + 2d_{s+1} & \cdots & 1 + j_s + 2d_{s+1} & 0 \\ d_1 & \cdots & d_s & d_{s+1} \end{pmatrix},$$

Noting that the entries satisfy conditions (4.3) to (4.5), the theorem is proved.

□

Definition 4.1. *From now on, in order to distinguish dark and light gray parts, we indicate the light ones by writing them inside a box. So, a partition in which* $(5,3,1,1)$ *are the light parts and* $(2,1,1)$ *are the dark gray ones can be expressed by*

$$(\boxed{5}, \boxed{3}, 2, \boxed{1}, \boxed{1}, 1, 1).$$

Example 4.1. *Looking at the first few terms of the expansion*

$$\rho(q) = 1 + 2q + 3q^2 + 4q^3 + 6q^4 + 8q^5 + 11q^6 + 14q^7 + 18q^8 + 24q^9 + \cdots,$$

we can see that there are 11 *partitions of* 6 *into parts we described above. Consequently, there are* 11 *matrices of type* (4.2) *whose sum of the second line elements is equal to* 6*. They are shown below.*

Partitions from $\rho(q)$	*Matrices of type* (4.2)
$(3,2,1)$	$\begin{pmatrix} 3 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
$(2,2,1,1)$	$\begin{pmatrix} 4 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
$(2,2,\boxed{1},1)$	$\begin{pmatrix} 3 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}$
$(2,\boxed{1},\boxed{1},1,1)$	$\begin{pmatrix} 3 & 1 & 0 \\ 2 & 0 & 0 \end{pmatrix}$
$(2,\boxed{1},\boxed{1},\boxed{1},1)$	$\begin{pmatrix} 2 & 1 & 0 \\ 3 & 0 & 0 \end{pmatrix}$

$(\boxed{3}, 2, 1)$	$\begin{pmatrix} 4 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$
$(\boxed{3}, \boxed{1}, 1, 1)$	$\begin{pmatrix} 4 & 0 \\ 1 & 1 \end{pmatrix}$
$(\boxed{1}, \boxed{1}, \boxed{1}, \boxed{1}, 1, 1)$	$\begin{pmatrix} 2 & 0 \\ 4 & 0 \end{pmatrix}$
$(\boxed{3}, \boxed{1}, \boxed{1}, 1)$	$\begin{pmatrix} 3 & 0 \\ 2 & 1 \end{pmatrix}$
$(\boxed{1}, \boxed{1}, \boxed{1}, \boxed{1}, \boxed{1}, 1)$	$\begin{pmatrix} 1 & 0 \\ 5 & 0 \end{pmatrix}$
$(\boxed{1}, \boxed{1}, \boxed{1}, \boxed{1}, \boxed{1}, \boxed{1})$	$\begin{pmatrix} 0 \\ 6 \end{pmatrix}$

The entries in the second row of the matrices of type (4.2) describe the light gray odd parts of the partition associated to each matrix. To know how many of these parts the partition has, we have to sum the d_i, for $i = 1, 2, \ldots, s + 1$.

Definition 4.2. *Let $p_\rho(n, k)$ be the number of partitions of n into parts of two different colors, counted by the general term of (4.1), having k light gray odd parts. We denote by $P_\rho(n, k)$ the set of partitions counted by $p_\rho(n, k)$.*

As an example, we have $p_\rho(10, 3) = 5$, and the elements of $P_\rho(10, 3)$ are

$$(\boxed{3}, \boxed{3}, \boxed{3}, 1);$$
$$(\boxed{3}, \boxed{3}, 2, \boxed{1}, 1);$$
$$(\boxed{5}, 2, \boxed{1}, \boxed{1}, 1);$$
$$(\boxed{3}, 2, 2, \boxed{1}, \boxed{1}, 1);$$
$$(3, 2, \boxed{1}, \boxed{1}, \boxed{1}, 1, 1).$$

For a fixed n, we classify its partitions of type described in definition 4.2 according to the sum on the second row of the matrix associated to each partition. By counting the appearance of each number in these sums, we can organize the data on a table, which is presented next. The entry in line n and column $n - j$ is the number of times j appears as sum of the entries of the second row in type (4.2) matrices.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
1	1	1																																							
2	1	1	1																																						
3	1	1	1	1																																					
4	1	1	1	2	1																																				
5	1	1	1	2	2	1																																			
6	1	1	1	2	2	2	2																																		
7	1	1	1	2	2	3	3	1																																	
8	1	1	1	2	2	3	4	3	1																																
9	1	1	1	2	2	3	4	4	4	2																															
10	1	1	1	2	2	3	4	5	5	4	2																														
11	1	1	1	2	2	3	4	5	6	6	5	2																													
12	1	1	1	2	2	3	4	5	6	7	8	5	2																												
13	1	1	1	2	2	3	4	5	6	8	9	8	6	2																											
14	1	1	1	2	2	3	4	5	6	8	10	10	10	7	2																										
15	1	1	1	2	2	3	4	5	6	8	10	11	13	11	7	3																									
16	1	1	1	2	2	3	4	5	6	8	10	12	14	14	13	9	3																								
17	1	1	1	2	2	3	4	5	6	8	10	12	15	16	17	15	9	3																							
18	1	1	1	2	2	3	4	5	6	8	10	12	15	17	20	20	16	10	3																						
19	1	1	1	2	2	3	4	5	6	8	10	12	15	18	21	23	23	19	11	3																					
20	1	1	1	2	2	3	4	5	6	8	10	12	15	18	22	25	27	26	21	12	4																				
21	1	1	1	2	2	3	4	5	6	8	10	12	15	18	22	26	30	31	30	24	13	4																			
22	1	1	1	2	2	3	4	5	6	8	10	12	15	18	22	27	31	34	37	34	26	15	4																		
23	1	1	1	2	2	3	4	5	6	8	10	12	15	18	22	27	32	36	41	42	38	30	16	4																	
24	1	1	1	2	2	3	4	5	6	8	10	12	15	18	22	27	32	37	44	47	48	45	32	17	5																
25	1	1	1	2	2	3	4	5	6	8	10	12	15	18	22	27	32	38	45	50	55	56	49	36	19	5															
26	1	1	1	2	2	3	4	5	6	8	10	12	15	18	22	27	32	38	46	52	59	64	63	56	40	20	5														
27	1	1	1	2	2	3	4	5	6	8	10	12	15	18	22	27	32	38	46	53	62	69	73	72	62	44	22	6													
28	1	1	1	2	2	3	4	5	6	8	10	12	15	18	22	27	32	38	46	54	63	72	80	84	81	7 0	48	23	6												
29	1	1	1	2	2	3	4	5	6	8	10	12	15	18	22	27	32	38	46	54	64	74	84	92	96	93	77	53	26	6											
30	1	1	1	2	2	3	4	5	6	8	10	12	15	18	22	27	32	38	46	54	64	75	87	97	106	110	103	87	59	27	7										
31	1	1	1	2	2	3	4	5	6	8	10	12	15	18	22	27	32	38	46	54	64	76	88	100	113	122	124	118	96	63	30	7									
32	1	1	1	2	2	3	4	5	6	8	10	12	15	18	22	27	32	38	46	54	64	76	89	102	117	130	139	142	131	106	70	32	7								
33	1	1	1	2	2	3	4	5	6	8	10	12	15	18	22	27	32	38	46	54	64	76	89	103	120	135	149	160	159	147	118	75	34	8							
34	1	1	1	2	2	3	4	5	6	8	10	12	15	18	22	27	32	38	46	54	64	76	89	104	121	138	156	172	181	181	164	129	83	37	8						
35	1	1	1	2	2	3	4	5	6	8	10	12	15	18	22	27	32	38	46	54	64	76	89	104	122	140	160	180	196	206	203	183	143	89	39	9					
36	1	1	1	2	2	3	4	5	6	8	10	12	15	18	22	27	32	38	46	54	64	76	89	104	122	141	163	185	206	224	233	229	203	157	97	42	10				
37	1	1	1	2	2	3	4	5	6	8	10	12	15	18	22	27	32	38	46	54	64	76	89	104	122	142	164	188	213	236	255	264	255	226	173	105	45	10			
38	1	1	1	2	2	3	4	5	6	8	10	12	15	18	22	27	32	38	46	54	64	76	89	104	122	142	165	190	217	244	270	290	296	287	250	188	115	48	10		
39	1	1	1	2	2	3	4	5	6	8	10	12	15	18	22	27	32	38	46	54	64	76	89	104	122	142	165	191	220	249	280	308	327	336	319	276	208	123	51	11	
40	1	1	1	2	2	3	4	5	6	8	10	12	15	18	22	27	32	38	46	54	64	76	89	104	122	142	165	192	221	252	287	320	349	372	375	356	307	225	133	55	11

Table 8 – Table from the characterization given by Theorem (4.1)

For example, looking at the 5^{th} line, from the right to the left, there are:

- 1 partition with no light gray odd parts: $(2,2,1)$.
- 2 partitions with 1 light gray odd part: $(2,\boxed{1},1,1)$ and $(\boxed{3},1,1)$
- 2 partitions with 2 light gray odd parts: $(2,\boxed{1},\boxed{1},1)$ and $(\boxed{3},\boxed{1},1)$.
- 1 partition with 3 light gray odd parts: $(\boxed{1},\boxed{1},\boxed{1},1,1)$
- 1 partition with 4 light gray odd parts: $(\boxed{1},\boxed{1},\boxed{1},\boxed{1},1)$
- 1 partition with 5 light gray odd parts: $(\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1})$.

By observing the table above we see that the columns become constant below certain entries. This result is described as follows:

Theorem 4.2. *For $n, i \geqslant 0$, we have*

(i) $p_\rho(3n+4+i, n+1+i) = p_\rho(3n+4, n+1)$;

(ii) $p_\rho(3n+5+i, n+1+i) = p_\rho(3n+5, n+1)$.

Proof. We prove only the first item by exhibiting a bijection between both sets of partitions. The second one has a similar proof.

A map we build from $P_\rho(3n+4, n+1)$ to $P_\rho(3n+4+i, n+1+i)$ is simply described by adding i light gray parts 1 to partitions lying in the first set. Clearly, the resulting partition lies in the second one.

In order to check that this map is in fact a bijection, we must assure that a partition counted by $p_\rho(3n+4+i, n+1+i)$ always has i light gray parts of size 1. So, let us suppose that there are $x(2j-1)$ light gray parts $2j-1$, with $x(1) < i$. As each dark gray part j has multiplicity $1+l_j$, with $l_j = 0$ or $l_j = 1$, then

$$\begin{aligned}
3n+4+i &= (1+l_1)\cdot 1 + (1+l_2)\cdot 2 + \cdots + (1+l_s)\cdot s + \sum_{j=1}^{s+1}(2j-1)\cdot x(2j-1) \\
&= \sum_{j=1}^{s}(1+l_j)\cdot j + \sum_{j=2}^{s+1}(2j-1)\cdot x(2j-1) + x(1) \\
&\geqslant \sum_{j=1}^{s}(1+l_j)\cdot j + \sum_{j=2}^{s+1} 3\cdot x(2j-1) + x(1) \\
&\geqslant \sum_{j=1}^{s}(1+l_j)\cdot j + 3\cdot(n+1+i-x(1)) + x(1)
\end{aligned}$$

So,

$$\begin{aligned}1+i &\geqslant \sum_{j=1}^{s}(1+l_j)\cdot j+3i-2x(1)\\ 1+i &> \sum_{j=1}^{s}(1+l_j)\cdot j+3i-2i\\ 1 &> \sum_{j=1}^{s}(1+l_j)\cdot j,\end{aligned}$$

which is a contradiction. □

Since the columns become constant, one can see that these fixed values represent the same sequence as the one for the number of partitions of certain integers into distinct parts. This result is described next.

Theorem 4.3. *For $n \geqslant 0$, we have*

(i) $p_\rho(3n+1,n) = p_d(2n+1)$,

(ii) $p_\rho(3n+2,n) = p_d(2n+2)$.

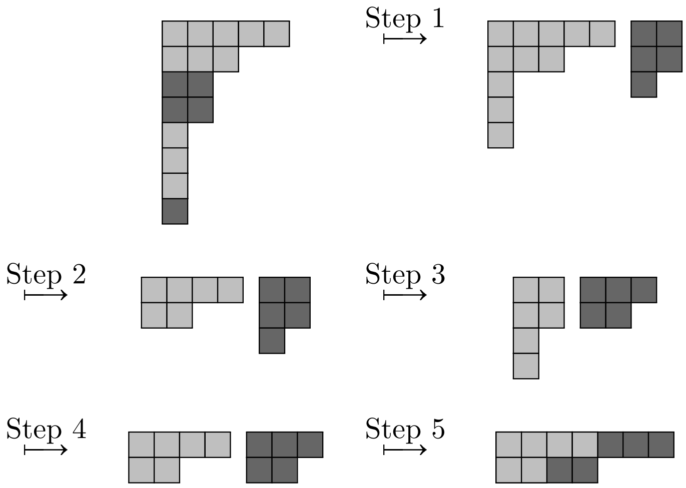
Proof. We prove the first item by exhibiting a bijection between sets $P_\rho(3n+1,n)$ and $P_d(2n+1)$. The same map also holds for item *(ii)*.

We classify each partition counted by $p_\rho(3n+1,n)$ according to its largest dark gray part, which is s from the definition of Mock Theta $\rho(q)$. As we also know, every positive part smaller than or equal to s must appear and there are n light gray odd parts smaller than or equal to $2s+1$.

The following steps describe the bijection between both sets, each one of them illustrated by an example.

- Step 1: Given a partition in $P_\rho(3n+1,n)$, consider its young diagram. Split it in two parts: the light gray and the dark gray one.
- Step 2: Decrease each light gray part by 1. Thus we get at most n light gray even parts smaller than or equal to $2s$.
- Step 3: Conjugate both partitions. In the dark gray one, as the largest part was s, now we have exactly s parts and in the light gray one, at most $2s$ parts.
- Step 4: Note that all dark gray parts are now distinct, because all numbers from 1 to s must appear. Since we have an even number of parts in the light gray partition, merge pairs of equal parts.
- Step 5: Finally, add both partitions side by side.

Example 4.2. *Taking $n = 5$, we start with $(\boxed{5}, \boxed{3}, 2, 2, \boxed{1}, \boxed{1}, \boxed{1}, 1)$, which lies in $P_\rho(16, 5)$.*



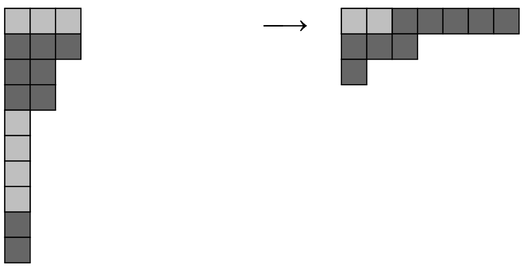
As image of $(\boxed{5}, \boxed{3}, 2, 2, \boxed{1}, \boxed{1}, \boxed{1}, 1)$ we get $(7, 4)$, which lies in $P_d(11)$.

Note that we always merge light gray parts of even size. Hence, if in the original dark gray partition the part s appears twice, the resulting smallest part will be even. In case it appears once, it will be odd.

Moreover, if the dark gray part k from the original partition has multiplicity 2, the resulting λ_k and λ_{k+1} parts have the same parity. In case of multiplicity 1, they have distinct parity. We can see this in the following two cases for $n = 5$.

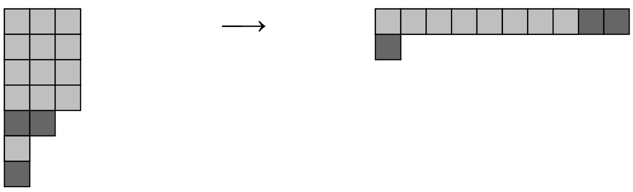
Example 4.3. *Case with dark gray parts with multiplicity 2:*

$$(\boxed{3}, 3, 2, 2, \boxed{1}, \boxed{1}, \boxed{1}, \boxed{1}, 1, 1) \quad \longrightarrow \quad (\boxed{2} + 5, 3, 1)$$



Case with dark gray parts with multiplicity 1:

$$(\boxed{3}, \boxed{3}, \boxed{3}, \boxed{3}, 2, \boxed{1}, 1) \quad \longrightarrow \quad (\boxed{8} + 2, 1)$$



Furthermore, the number of parts in the resulting partition is the same as the largest dark gray parts of size s in the original partition.

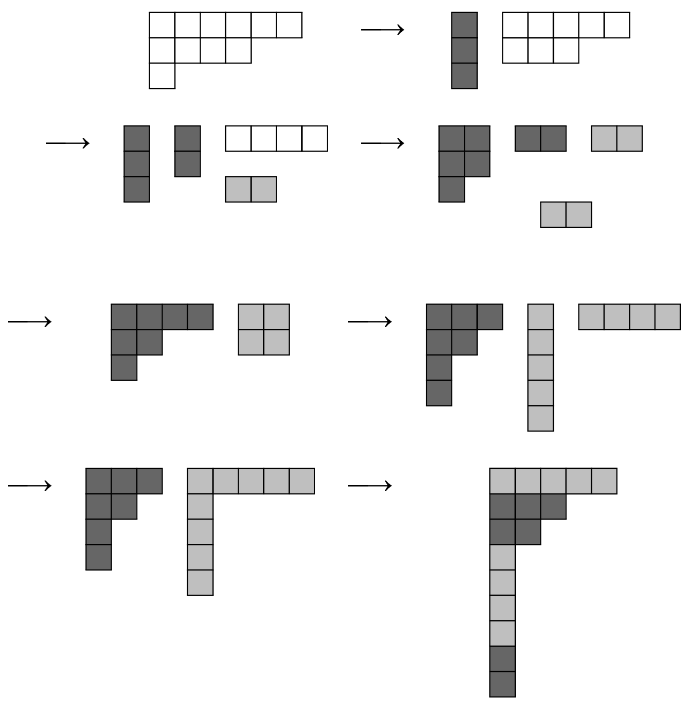
Based on the previous remarks, we are able to describe the inverse map. Taking a partition $\lambda = (\lambda_1, \lambda_2, ..., \lambda_s)$ of $2n+1$ into distinct parts, we need to know how many times each dark gray part from 1 to s will appear in the resulting partition.

We start by observing the smallest part λ_s. If it is odd, the dark part s in the resulting partition will have multiplicity 1, because $\lambda_{s+1} = 0$ and λ_s do not have the same parity. If it is even, following the same idea, s has multiplicity 2. According to this result, we remove the first one or the first two columns of λ's Young Diagram, and also what is left from its s^{th} row. Save this parts we removed. We repeat the previous step for every remaining part in order to discover all dark gray parts of the resulting partition.

Consider the removed columns and rows from each of the s steps above. Organize the columns by joining them side by side, and the rows by joining them one above the other, in the same order they were removed in the previous steps. These rows are of even size, so split them into two equal parts. After all this adjustments, conjugate the remaining partitions and add the partition $(1, 1, ..., 1)$ with n parts to the light gray one. Finally, join both partitions together.

In order to illustrate the inverse map, we describe it step-by-step in the following example.

Example 4.4. *Again with* $n = 5$, *consider* $(6, 4, 1)$ *which lies in* $P_d(11)$. *According to the inverse map, its image is* $(\boxed{5}, 3, 2, \boxed{1}, \boxed{1}, \boxed{1}, \boxed{1}, 1, 1)$, *as we can see next.*



So, the map we described is a bijection and the equality in item (i) holds. □

4.2 Mock Theta Function $\sigma(q)$

In this section we consider the Mock Theta function of order 6

$$\sigma(q) = \sum_{n=0}^{\infty} \frac{(-q;q)_n q^{\binom{n+2}{2}}}{(q;q^2)_{n+1}}. \tag{4.6}$$

Its general term

$$\frac{(1+q)(1+q^2)\cdots(1+q^s)q^{1+2+3+\cdots+(s+1)}}{(1-q)(1-q^3)(1-q^5)\cdots(1-q^{2s+1})},$$

generates the partitions of n into parts of two different colors:

- dark gray parts, ranging from 1 to $s+1$, with no gaps, the largest one with multiplicity exactly one and the others with multiplicity 1 or 2;
- any number of light gray odd parts less than or equal to $2s+1$.

Some partition identities we find in this section are similar to those for partitions generated by $\rho(q)$. In addition, the columns of the table for $\sigma(q)$, constructed similarly to the previous one, reveals a relation between partitions generated by $\sigma(q)$ and $\rho(q)$.

$\sigma(q)$ may also be interpreted in a combinatorial way, as we find in [9].

Theorem 4.4. *The coefficient of q^n in the expansion of* (4.6) *is equal to the number of elements in the set of matrices of the form*

$$A = \begin{pmatrix} c_1 & c_2 & \cdots & c_s \\ d_1 & d_2 & \cdots & d_s \end{pmatrix}, \tag{4.7}$$

with non-negative integer entries satisfying

$$c_s = 1;\ d_t \geqslant 0; \tag{4.8}$$
$$c_t = i_t + c_{t+1} + 2d_{t+1},\ \textit{with}\, i_t \in \{1,2\},\ \forall t < s; \tag{4.9}$$
$$n = \sum c_t + \sum d_t. \tag{4.10}$$

Proof. According to the general term of (4.6), we can decompose n as

$$n = ((1+j_1)\cdot 1 + (1+j_2)\cdot 2 + \cdots + (1+j_s)\cdot s + (s+1)) + (d_1 + 3d_3 + \cdots + (2s+1)\cdot d_{s+1})$$

with $j_t \in \{0,1\}$ and $d_t \geqslant 0$, or, equivalently, as the sum of the entries of the matrix

$$A = \begin{pmatrix} (s+1) + j_1 + \cdots + j_s + 2d_2 + \cdots + 2d_{s+1} & \cdots & 2 + j_s + 2d_{s+1} & 1 \\ d_1 & \cdots & d_s & d_{s+1} \end{pmatrix},$$

Noting that the entries satisfy conditions (4.8) to (4.10), the theorem is proved. □

Example 4.5. *Looking at the first few terms of the expansion*

$$\sigma(q) = q + q^2 + 2q^3 + 3q^4 + 3q^5 + 5q^6 + 7q^7 + 8q^8 + 11q^9 + \cdots ,$$

one can see that there are 11 *partitions of* 9 *into parts we described above. Consequently, there are* 11 *matrices of type (*4.7*) whose sum of their entries is equal to* 9*. They are shown below.*

Partitions from $\sigma(q)$	*Matrices of type* (4.7)
$(3,2,2,1,1)$	$\begin{pmatrix} 5 & 3 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
$(3,2,2,\boxed{1},1)$	$\begin{pmatrix} 4 & 3 & 1 \\ 1 & 0 & 0 \end{pmatrix}$
$(3,2,\boxed{1},\boxed{1},1,1)$	$\begin{pmatrix} 4 & 2 & 1 \\ 2 & 0 & 0 \end{pmatrix}$
$(3,2,\boxed{1},\boxed{1},\boxed{1},1)$	$\begin{pmatrix} 3 & 2 & 1 \\ 3 & 0 & 0 \end{pmatrix}$
$(\boxed{3},3,2,1)$	$\begin{pmatrix} 5 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix}$
$(\boxed{3},\boxed{3},2,1)$	$\begin{pmatrix} 7 & 1 \\ 0 & 2 \end{pmatrix}$
$(\boxed{3},2,\boxed{1},\boxed{1},1,1)$	$\begin{pmatrix} 5 & 1 \\ 2 & 1 \end{pmatrix}$
$(2,\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},1,1)$	$\begin{pmatrix} 3 & 1 \\ 5 & 0 \end{pmatrix}$
$(2,\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},1)$	$\begin{pmatrix} 2 & 1 \\ 6 & 0 \end{pmatrix}$
$(\boxed{3},2,\boxed{1},\boxed{1},\boxed{1},1)$	$\begin{pmatrix} 4 & 1 \\ 3 & 1 \end{pmatrix}$
$(\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},1)$	$\begin{pmatrix} 1 \\ 8 \end{pmatrix}$

The entries in the second row of the matrices of type (4.7) describe the light gray odd parts of the partition associated to each matrix. To know how many of these parts the partition has, we have to sum the d_i, for $i = 1, 2, \ldots, s+1$.

Definition 4.3. *Let* $p_\sigma(n,k)$ *be the number of partitions of* n *into parts of two different colors, counted by the general term of (4.6), having* k *light gray odd parts. We denote by* $P_\sigma(n,k)$ *the set of partitions counted by* $p_\sigma(n,k)$*.*

For a fixed n, we classify its partitions of type described in definition 4.3 according to the sum on the second row of the matrix associated to each partition. Similar to the one in the previous section, we construct a table whose entry in line n and column $n-j$ is the number of times j appears as sum of the entries of the second row in type (4.7) matrices.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
1	1																																							
2	1	0																																						
3	1	0	1																																					
4	1	0	1	1																																				
5	1	0	1	1	0																																			
6	1	0	1	1	1	1																																		
7	1	0	1	1	1	2	1																																	
8	1	0	1	1	1	2	1	1																																
9	1	0	1	1	1	2	2	2	1																															
10	1	0	1	1	1	2	2	3	2	1																														
11	1	0	1	1	1	2	2	3	2	3	1																													
12	1	0	1	1	1	2	2	3	3	4	3	1																												
13	1	0	1	1	1	2	2	3	3	5	4	3	2																											
14	1	0	1	1	1	2	2	3	3	5	4	5	4	1																										
15	1	0	1	1	1	2	2	3	3	5	5	6	6	3	2																									
16	1	0	1	1	1	2	2	3	3	5	5	7	7	6	5	2																								
17	1	0	1	1	1	2	2	3	3	5	5	7	7	8	8	5	1																							
18	1	0	1	1	1	2	2	3	3	5	5	7	8	9	10	8	6	2																						
19	1	0	1	1	1	2	2	3	3	5	5	7	8	10	11	11	10	6	2																					
20	1	0	1	1	1	2	2	3	3	5	5	7	8	10	11	13	13	10	7	2																				
21	1	0	1	1	1	2	2	3	3	5	5	7	8	10	12	14	15	14	13	7	3																			
22	1	0	1	1	1	2	2	3	3	5	5	7	8	10	12	15	16	17	18	13	8	3																		
23	1	0	1	1	1	2	2	3	3	5	5	7	8	10	12	15	16	19	21	18	16	9	2																	
24	1	0	1	1	1	2	2	3	3	5	5	7	8	10	12	15	17	20	23	22	23	17	9	3																
25	1	0	1	1	1	2	2	3	3	5	5	7	8	10	12	15	17	21	24	25	28	24	19	11	3															
26	1	0	1	1	1	2	2	3	3	5	5	7	8	10	12	15	17	21	24	27	31	30	28	21	11	3														
27	1	0	1	1	1	2	2	3	3	5	5	7	8	10	12	15	17	21	25	28	33	34	36	31	23	13	3													
28	1	0	1	1	1	2	2	3	3	5	5	7	8	10	12	15	17	21	25	29	34	37	41	39	36	26	13	4												
29	1	0	1	1	1	2	2	3	3	5	5	7	8	10	12	15	17	21	25	29	34	39	44	45	46	39	27	15	4											
30	1	0	1	1	1	2	2	3	3	5	5	7	8	10	12	15	17	21	25	29	35	40	46	49	54	50	44	32	15	4										
31	1	0	1	1	1	2	2	3	3	5	5	7	8	10	12	15	17	21	25	29	35	41	47	52	59	59	58	50	33	17	5									
32	1	0	1	1	1	2	2	3	3	5	5	7	8	10	12	15	17	21	25	29	35	41	47	54	62	65	69	65	53	38	18	4								
33	1	0	1	1	1	2	2	3	3	5	5	7	8	10	12	15	17	21	25	29	35	41	48	55	64	69	77	77	73	61	40	19	5							
34	1	0	1	1	1	2	2	3	3	5	5	7	8	10	12	15	17	21	25	29	35	41	48	56	65	72	82	86	88	82	66	45	21	5						
35	1	0	1	1	1	2	2	3	3	5	5	7	8	10	12	15	17	21	25	29	35	41	48	56	65	74	85	92	99	98	91	74	48	22	5					
36	1	0	1	1	1	2	2	3	3	5	5	7	8	10	12	15	17	21	25	29	35	41	48	56	66	75	87	96	107	111	112	102	80	53	24	6				
37	1	0	1	1	1	2	2	3	3	5	5	7	8	10	12	15	17	21	25	29	35	41	48	56	66	76	88	99	112	120	128	125	112	91	56	26	6			
38	1	0	1	1	1	2	2	3	3	5	5	7	8	10	12	15	17	21	25	29	35	41	48	56	66	76	88	101	115	126	139	142	140	127	96	62	28	6		
39	1	0	1	1	1	2	2	3	3	5	5	7	8	10	12	15	17	21	25	29	35	41	48	56	66	76	89	102	117	130	147	155	162	158	137	109	67	29	7	
40	1	0	1	1	1	2	2	3	3	5	5	7	8	10	12	15	17	21	25	29	35	41	48	56	66	76	89	103	118	133	152	164	178	182	174	156	117	72	32	7

Table 9 – Table from the characterization given by Theorem (4.4)

Again, looking at the 5^{th} line, from the right to the left, there are:

- no partitions with no light gray odd parts.
- 1 partitions with 1 light gray odd part: $(2, \boxed{1}, 1, 1)$.
- 1 partitions with 2 light gray odd parts: $(2, \boxed{1}, \boxed{1}, 1)$.
- no partitions with 3 light gray odd parts.
- 1 partition with 4 light gray odd parts: $(\boxed{1}, \boxed{1}, \boxed{1}, \boxed{1}, 1)$.

Remark 4.1. *From the definition of the Mock Theta function of order* 5

$$1 + 2\Psi_0(q) = \sum_{n=0}^{\infty} (-1;q)_n q^{\binom{n+1}{2}},$$

which represents partitions whose parts appear at most twice, the largest part s appears once and every integer smaller than s is also a part, it is easy to see that $p_\sigma(n, 0)$ represents the coefficients of the Mock Theta function Ψ_0 above.

As it happens in the table of Mock Theta function $\rho(q)$, the columns in the table of Mock Theta function σ also become constant from certain values of n on, which leads us to the following theorem. Its proof is analogous to the one in Theorem 4.2.

Theorem 4.5. *For $n, i \geqslant 0$, we have*

(i) $p_\sigma(3n + 3 + i, n + 1 + i) = p_\sigma(3n + 3, n + 1)$;

(ii) $p_\sigma(3n + 1 + i, n + i) = p_\sigma(3n + 4, n + 1)$.

The fixed values in the columns in the table built for $\sigma(q)$ are also related to partitions of certain integers into distinct parts, but all of them must be greater than or equal to 2. This result is described next.

Theorem 4.6. *Denoting by $p_d(n,\ parts\ \geqslant k)$ the number of partitions into distinct parts larger than or equal to k, for $n \geqslant 1$ we have*

(i) $p_\sigma(3n, n - 1) = p_d(2n,\ parts\ \geqslant 2)$;

(ii) $p_\sigma(3n + 1, n - 1) = p_d(2n + 1,\ parts\ \geqslant 2)$.

Proof. Again, this theorem is proved by setting a bijection between two different sets of partitions. In order to avoid extensive details, we only highlight some adjustments we make in the proof of Theorem 4.3, so that we have an analogous proof for Theorem 4.6.

When in Theorem 4.3 we decreased by one each part from the light gray partition, now we must also decrease the largest dark part by one. It is possible since we always have one dark part. The remaining steps are equal to the original map.

In the inverse map, we set how many dark gray parts we would have according to the parity of the parts. Unlike there, now if λ_i is odd, there will be two parts of size i in the resulting partition. Otherwise there will be just one part of size i. This occurs because now we have the dark part $s+1$, which does not appear in ρ. □

By observing the constant values in the columns of Tables 8 and 9, we note that the sum of two certain numbers in table for $\sigma(q)$ appears in the table for $\rho(q)$. This property is proved next.

Corollary 4.1. *For all $n \geqslant 1$, we have:*

(i) $p_\sigma(3n, n-1) + p_\sigma(3n+1, n-1) = p_\rho(3n+1, n)$;

(ii) $p_\sigma(3n+1, n-1) + p_\sigma(3n+3, n) = p_\rho(3n+2, n)$.

Proof. Both items have analogous proofs, so we present only one of item (i).

By Theorems 4.3 and 4.6, statement (i) is equivalent to

$$p_d(2n+1, \text{ parts } \geqslant 2) + p_d(2n, \text{ parts } \geqslant 2) = p_d(2n+1).$$

This identity can be easily proved by splitting the set whose cardinality is counted by $p_d(2n+1)$ into two new ones: one of partitions of $2n+1$ into distinct parts with no part of size 1, and the other of partitions of $2n+1$ into distinct parts having 1 as a part. Noting that the last set has cardinality equal to $p_d(2n, \text{ parts } \geqslant 2)$, the corollary follows. □

Combining Corollary 4.1 and Theorems 4.2 and 4.5, we get another relation between Mock Theta functions ρ and σ.

Corollary 4.2. *For all $n \geqslant 1$, we have*

$$p_\sigma(2n-1, n-1) + p_\sigma(2n+1, n) = p_\rho(2n-1, n-1).$$

Although Corollary 4.2 can be obtained from previous results, alternatively we can demonstrate it by a bijective proof, which is described next.

Bijective proof for Corollary 4.2. Let $P_\rho(2n-1, n-1)$ be the set of all partitions counted by $p_\rho(2n-1, n-1)$, and define its two disjoint subsets as follows.

- $P_\rho(2n-1, n-1)^*$: the set of partitions such that, if s is the largest dark part, it appears twice or the largest light part is $2s+1$.
- $P_\rho(2n-1, n-1)^\#$: the set of partitions such that the largest dark part s appears once and the largest light part is smaller than or equal to $2s-1$.

  So we have

$$P_\rho(2n-1, n-1) = P_\rho(2n-1, n-1)^\# \cup P_\rho(2n-1, n-1)^*$$

Now, separate all the partitions according to the set they belong. As an example, take $n = 9$:

$P_\rho(17,8)$
$(\boxed{3},\boxed{3},\boxed{3},\boxed{3},\boxed{1},\boxed{1},\boxed{1},\boxed{1},1)$
$(\boxed{3},\boxed{3},\boxed{3},2,\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},1)$
$(\boxed{5},\boxed{3},2,\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},1)$
$(\boxed{3},\boxed{3},2,2,\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},1)$
$(\boxed{5},2,2,\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},1)$
$(\boxed{3},3,2,\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},1,1)$
$(3,2,2,\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},1,1)$
$(3,3,2,\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},1)$

$P_\rho(17,8)^{\#}$	$P_\rho(17,8)^{*}$
$(\boxed{3},\boxed{3},\boxed{3},2,\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},1)$	$(\boxed{3},\boxed{3},\boxed{3},\boxed{3},\boxed{1},\boxed{1},\boxed{1},\boxed{1},1)$
$(\boxed{3},3,2,\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},1,1)$	$(\boxed{5},\boxed{3},2,\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},1)$
$(3,2,2,\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},1,1)$	$(\boxed{3},\boxed{3},2,2,\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},1)$
	$(\boxed{5},2,2,\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},1)$
	$(3,3,2,\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},1)$

It is easy to see that $P_\rho(2n-1,n-1)^{\#} = P_\sigma(2n-1,n-1)$. So, it remains to be proved that

$$|P_\rho(2n-1,n-1)^{*}| = |P_\sigma(2n+1,n)|,$$

which is done by exhibiting a bijection between both sets. Let λ be a partition lying in $P_\rho(2n-1,n-1)^{*}$. We define the map according to the following steps:

- If the largest dark part s of λ appears twice, increase one of them by 1 and add $\boxed{1}$ as a light gray part. Thus we have a partition of $2n+1$ into n light gray parts and the dark ones ranging from 1 to $s+1$, with possible repetition of parts between 1 and $s-1$. Hence, a partition lying in $P_\sigma(2n+1,n)$.

- In case s appears once and the largest light gray part of λ is $\boxed{2s+1}$, split it into two new dark parts s and $s+1$. Then add two light parts $\boxed{1}$. Once again, we have a partition lying in $P_\sigma(2n+1,n)$ with dark parts $1,...,s,s,s+1$.

For example, for $n = 9$ it works as follows.

$$(\boxed{3},\boxed{3},\boxed{3},\boxed{3},\boxed{1},\boxed{1},\boxed{1},\boxed{1},1) \longrightarrow (\boxed{3},\boxed{3},\boxed{3},2,\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},1,1)$$

$$(\boxed{5},\boxed{3},2,\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},1) \longrightarrow (\boxed{3},3,2,2,\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},1)$$

$$(\boxed{3},\boxed{3},2,2,\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},1) \longrightarrow (\boxed{3},\boxed{3},3,2,\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},1)$$

$$(\boxed{5},2,2,\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},1) \longrightarrow (\boxed{5},3,2,\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},1)$$

$$(3,3,2,\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},1) \longrightarrow (4,3,2,\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},\boxed{1},1).$$

The inverse map depends on how many dark parts of size s the partition lying in $P_\sigma(2n+1,n)$ has. In case it has two, merge one part of size s and one of size $s+1$, getting a light gray part $\boxed{2s+1}$, and remove two parts $\boxed{1}$. Otherwise, decrease the part $s+1$ by 1 and remove one part $\boxed{1}$.

□

4.3 Mock Theta Function $\nu(q)$

Overlooking signs in coefficients of Mock Theta function of order 5

$$\nu(q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(-q;q^2)_{n+1}},$$

we see that they represent some kind of partitions, as before, related to certain partitions into distinct parts. In this section, we present this relations and also an identity that easily follows from a formula found in [2].

Consider the unsigned version of $\nu(q)$:

$$\nu(-q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q^2)_{n+1}} = \sum_{s=0}^{\infty} \frac{q^{2+4+\cdots+2s}}{(1-q)(1-q^3)\cdots(1-q^{2s+1})}. \tag{4.11}$$

Its general term generates the partitions of n containing exactly one copy of each one of the even parts $2, 4, \cdots, 2s$ and any number of odd parts less than or equal to $2s+1$. For example, the partitions of 10 that satisfy the conditions are

$$(4, \boxed{3}, 2, \boxed{1});$$

$$(4, 2, \boxed{1}, \boxed{1}, \boxed{1}, \boxed{1});$$

$$(\boxed{3}, \boxed{3}, 2, \boxed{1}, \boxed{1});$$

$$(\boxed{3}, 2, \boxed{1}, \boxed{1}, \boxed{1}, \boxed{1}, \boxed{1});$$

$$(2, \boxed{1}, \boxed{1}, \boxed{1}, \boxed{1}, \boxed{1}, \boxed{1}, \boxed{1}, \boxed{1});$$

$$(\boxed{1}, \boxed{1}, \boxed{1}, \boxed{1}, \boxed{1}, \boxed{1}, \boxed{1}, \boxed{1}, \boxed{1}, \boxed{1}).$$

We can also find in [9] an interpretation in terms of two-line matrices for this partitions.

Theorem 4.7. *The coefficient of q^n in the expansion of* (4.11) *is equal to the number of elements in the set of matrices of the form*

$$A = \begin{pmatrix} c_1 & c_2 & \cdots & c_{s+1} \\ d_1 & d_2 & \cdots & d_{s+1} \end{pmatrix}, \tag{4.12}$$

with non-negative integer entries satisfying

$$c_{s+1} = 0;\ d_t \geqslant 0 \tag{4.13}$$

$$c_t = 2 + c_{t+1} + 2d_{t+1},\ \forall t \leqslant s; \tag{4.14}$$

$$n = \sum c_t + \sum d_t. \tag{4.15}$$

Proof. According to the general term of (4.11), we can decompose n as

$$n = (2 + 4 + \cdots + 2s) + (d_1 + 3d_2 + \cdots + (2s+1) \cdot d_{s+1}).$$

Organizing it in a matrix, we get

$$A = \begin{pmatrix} 2s + 2d_2 + \cdots + 2d_{s+1} & \cdots & 4 + 2d_s + 2d_{s+1} & 2 + 2d_{s+1} & 0 \\ d_1 & \cdots & d_{s-1} & d_s & d_{s+1} \end{pmatrix},$$

which clearly satisfies conditions (4.13) to (4.15). □

Example 4.6. *Looking at the first few terms of the expansion*

$$\nu(-q) = 1 + q + 2q^2 + 2q^3 + 2q^4 + 3q^5 + 4q^6 + 4q^7 + 5q^8 + 6q^{10} + 8q^{11} + \cdots$$

one can see that there are 6 *partitions of* 10 *into parts we described above. Consequently, there are* 6 *matrices of type* (4.12) *whose sum of their entries is equal to* 10*. They are shown below.*

Partitions from $\nu(-q)$	*Matrices of type* (4.12)
$(4, \boxed{3}, 2, \boxed{1})$	$\begin{pmatrix} 6 & 2 & 0 \\ 1 & 1 & 0 \end{pmatrix}$
$(4, 2, \boxed{1}, \boxed{1}, \boxed{1}, \boxed{1})$	$\begin{pmatrix} 4 & 2 & 0 \\ 4 & 0 & 0 \end{pmatrix}$
$(\boxed{3}, \boxed{3}, 2, \boxed{1}, \boxed{1})$	$\begin{pmatrix} 6 & 0 \\ 2 & 2 \end{pmatrix}$
$(\boxed{3}, 2, \boxed{1}, \boxed{1}, \boxed{1}, \boxed{1}, \boxed{1})$	$\begin{pmatrix} 4 & 0 \\ 5 & 1 \end{pmatrix}$
$(2, \boxed{1}, \boxed{1}, \boxed{1}, \boxed{1}, \boxed{1}, \boxed{1}, \boxed{1}, \boxed{1})$	$\begin{pmatrix} 2 & 0 \\ 8 & 0 \end{pmatrix}$
$(\boxed{1}, \boxed{1}, \boxed{1}, \boxed{1}, \boxed{1}, \boxed{1}, \boxed{1}, \boxed{1}, \boxed{1}, \boxed{1})$	$\begin{pmatrix} 0 \\ 10 \end{pmatrix}$

The entries in the second row of matrices like (4.12) describe the odd parts from 1 to $2s+1$ of the partition associated to each matrix. It means that entry d_t expresses how many light gray parts $2t-1$ the correspondent partition has. To know how many of these parts the partition has, we have to sum the d_i, for $i = 1, 2, \ldots, s$.

Definition 4.4. *Let $p_\nu(n,k)$ be the number of partitions of n counted by the general term of* (4.11)*, having k odd parts between* 1 *to* $2s+1$*. Write* $p_\nu(n) = \sum_k p_\nu(n,k)$.

Example 4.7. *By considering $n = 20$, we have* 5 *partitions satisfying the conditions and having* 6 *light gray odd parts. So* $p_\nu(20,6) = 5$ *and the set* $P_\nu(20,6)$ *is made of by*

$$(6, 4, \boxed{3}, 2, \boxed{1}, \boxed{1}, \boxed{1}, \boxed{1}, \boxed{1});$$

$$(\boxed{5}, \boxed{5}, 4, 2, \boxed{1}, \boxed{1}, \boxed{1}, \boxed{1});$$

$$(\boxed{5}, 4, \boxed{3}, \boxed{3}, 2, \boxed{1}, \boxed{1}, \boxed{1});$$

$$(4, \boxed{3}, \boxed{3}, \boxed{3}, 2, \boxed{1}, \boxed{1}, \boxed{1});$$

$$(\boxed{3}, \boxed{3}, \boxed{3}, \boxed{3}, \boxed{3}, \boxed{3}, 2).$$

For a fixed n, we classify its partitions of type described in Definition 4.4 according to the sum on the second row of the matrix associated to each partition. We construct a table in the same way we did for $\rho(q)$ and $\sigma(q)$ functions, now for Mock Theta function $\nu(q)$.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
1	1	0																																							
2	1	0	1																																						
3	1	0	1	0																																					
4	1	0	1	0	0																																				
5	1	0	1	0	1	0																																			
6	1	0	1	0	1	0	1																																		
7	1	0	1	0	1	0	1	0																																	
8	1	0	1	0	1	0	2	0	0																																
9	1	0	1	0	1	0	2	0	1	0																															
10	1	0	1	0	1	0	2	0	1	0	0																														
11	1	0	1	0	1	0	2	0	2	0	1	0																													
12	1	0	1	0	1	0	2	0	2	0	2	0	1																												
13	1	0	1	0	1	0	2	0	2	0	2	0	1	0																											
14	1	0	1	0	1	0	2	0	2	0	3	0	2	0	0																										
15	1	0	1	0	1	0	2	0	2	0	3	0	3	0	1	0																									
16	1	0	1	0	1	0	2	0	2	0	3	0	3	0	2	0	0																								
17	1	0	1	0	1	0	2	0	2	0	3	0	4	0	3	0	1	0																							
18	1	0	1	0	1	0	2	0	2	0	3	0	4	0	4	0	2	0	0																						
19	1	0	1	0	1	0	2	0	2	0	3	0	4	0	4	0	3	0	1	0																					
20	1	0	1	0	1	0	2	0	2	0	3	0	4	0	5	0	4	0	2	0	1																				
21	1	0	1	0	1	0	2	0	2	0	3	0	4	0	5	0	5	0	4	0	1	0																			
22	1	0	1	0	1	0	2	0	2	0	3	0	4	0	5	0	5	0	5	0	3	0	0																		
23	1	0	1	0	1	0	2	0	2	0	3	0	4	0	5	0	6	0	6	0	4	0	1	0																	
24	1	0	1	0	1	0	2	0	2	0	3	0	4	0	5	0	6	0	7	0	6	0	2	0	0																
25	1	0	1	0	1	0	2	0	2	0	3	0	4	0	5	0	6	0	7	0	7	0	4	0	1	0															
26	1	0	1	0	1	0	2	0	2	0	3	0	4	0	5	0	6	0	8	0	8	0	6	0	3	0	0														
27	1	0	1	0	1	0	2	0	2	0	3	0	4	0	5	0	6	0	8	0	9	0	8	0	5	0	1	0													
28	1	0	1	0	1	0	2	0	2	0	3	0	4	0	5	0	6	0	8	0	9	0	9	0	7	0	2	0	0												
29	1	0	1	0	1	0	2	0	2	0	3	0	4	0	5	0	6	0	8	0	10	0	10	0	9	0	5	0	1	0											
30	1	0	1	0	1	0	2	0	2	0	3	0	4	0	5	0	6	0	8	0	10	0	11	0	11	0	7	0	3	0	1										
31	1	0	1	0	1	0	2	0	2	0	3	0	4	0	5	0	6	0	8	0	10	0	11	0	12	0	10	0	5	0	1	0									
32	1	0	1	0	1	0	2	0	2	0	3	0	4	0	5	0	6	0	8	0	10	0	12	0	13	0	12	0	9	0	3	0	0								
33	1	0	1	0	1	0	2	0	2	0	3	0	4	0	5	0	6	0	8	0	10	0	12	0	14	0	14	0	11	0	6	0	1	0							
34	1	0	1	0	1	0	2	0	2	0	3	0	4	0	5	0	6	0	8	0	10	0	12	0	14	0	15	0	14	0	9	0	3	0	0						
35	1	0	1	0	1	0	2	0	2	0	3	0	4	0	5	0	6	0	8	0	10	0	12	0	15	0	16	0	16	0	13	0	6	0	1	0					
36	1	0	1	0	1	0	2	0	2	0	3	0	4	0	5	0	6	0	8	0	10	0	12	0	15	0	17	0	18	0	16	0	10	0	3	0	0				
37	1	0	1	0	1	0	2	0	2	0	3	0	4	0	5	0	6	0	8	0	10	0	12	0	15	0	17	0	19	0	19	0	14	0	6	0	1	0			
38	1	0	1	0	1	0	2	0	2	0	3	0	4	0	5	0	6	0	8	0	10	0	12	0	15	0	18	0	20	0	21	0	18	0	10	0	3	0	0		
39	1	0	1	0	1	0	2	0	2	0	3	0	4	0	5	0	6	0	8	0	10	0	12	0	15	0	18	0	21	0	23	0	21	0	15	0	7	0	1	0	
40	1	0	1	0	1	0	2	0	2	0	3	0	4	0	5	0	6	0	8	0	10	0	12	0	15	0	18	0	21	0	24	0	24	0	19	0	12	0	3	0	0

Table 10 – Table from the characterization given by Theorem (4.7)

By observing the table above we get some interesting results.

Remark 4.2. *It is clear to see that, for all $n \geqslant 1$, we have*

$$p_\nu(n,0) = \begin{cases} 1, & \text{if } n = s^2 + s \\ 0, & \text{otherwise.} \end{cases}$$

and

$$p_\nu(n,1) = \begin{cases} 1, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even.} \end{cases}$$

Theorem 4.8. *For all $n \geqslant 1$ and $i = 0, 2, 4$, we have*

$$p_\nu(4n^2 - 2n + i, 2) = n.$$

Proof. If $i = 0$, the largest even part of any partition counted by $p_\nu(4n^2 - 2n, 2)$ has to be $4n - 4$. So, we have to write

$$4n^2 - 2n = 2 + 4 + \cdots + 4n - 4 + r + s,$$

with odd r and s and $1 \leqslant s \leqslant r \leqslant 4n - 3$, which implies

$$r + s = 4n - 2.$$

Equivalently, writing $r = 2p - 1$ and $s = 2q - 1$, with $1 \leqslant q \leqslant p \leqslant 2n - 1$, we have to determine the number of solutions of equation $p + q = 2n$, with $1 \leqslant q \leqslant p \leqslant 2n - 1$.

First of all, the number of positive solutions of equation $p + q = 2n$ with no limitation for p and $q \leqslant p$ is $\left\lfloor \frac{2n}{2} \right\rfloor = n$. The solutions we do not want are those where $p > 2n - 1$. Although, note that $p > 2n - 1$ implies $p = 2n$, and so $q = 0$, which never occurs. So, there is no solution to eliminate and the number we are looking for is just n.

If $i = 2$ or 4, there is only one partition counted by $p_\nu(4n^2 - 2n + i, 2)$ with largest even part $4n - 2$. Indeed, $2 + 4 + \cdots + 4n - 2 + r + s = 4n^2 - 2n + i$, with odd r and s and $1 \leqslant s \leqslant r \leqslant 4n - 1$, implies $r + s = i$. As in those conditions 2 and 4 can only be written as $1 + 1$ and $3 + 1$, respectively, there is only one partition counted by $p_\nu(4n^2 - 2n + i, 2)$ with largest part $4n - 2$. The other partitions have largest even part equal to $4n - 4$. So,

$$4n^2 - 2n + i = 2 + 4 + \cdots + 4n - 4 + r + s,$$

with odd r and s and $1 \leqslant s \leqslant r \leqslant 4n - 3$, which implies

$$r + s = 4n - 2 + i.$$

Again writing $r = 2p - 1$ and $s = 2q - 1$, with $1 \leqslant q \leqslant p \leqslant 2n - 1$, and $i = 2j$, it is equivalent to determine the number of solutions of equation $p+q = 2n+j$, with $1 \leqslant q \leqslant p \leqslant 2n-1$.

The number of positive solutions of equation $p + q = 2n + j$ with no limitation for p and $q \leqslant p$ is $\left\lfloor \frac{2n + j}{2} \right\rfloor$. The solutions we do not want are those where $p > 2n - 1$. As q has to be at least 1 and $p > 2n - 1$, we can write

$$p = 2n - 1 + k \quad \text{with } 1 \leqslant k \leqslant j,$$

and so, for each value of k we get one value of p.

Then, the number of solutions we want is

$$\left\lfloor \frac{2n+j}{2} \right\rfloor - j = n-1.$$

Adding these to the other solution we already have, we get $p_\nu(4n^2-2n+i,2)=n$, for $i=0,2,4$.

□

As it happens in Tables 8 and 9, the columns in Table 10 become constant from certain values of n on. This fact can be proved analogously to those in Theorems 4.2 and 4.5. Hence, we omit the proof of the next theorem.

Theorem 4.9. *For all $n \geqslant 0$ and $i \geqslant 0$, we have*

$$p_\nu(3n+2+i, n+i) = p_\nu(3n+2, n).$$

The constant values in the columns of the table are also related to certain partitions into distinct parts.

Theorem 4.10. *For all $n \geqslant 0$ and $i \geqslant 0$, we have*

$$p_\nu(3n+2, n) = p_d(n+1).$$

Proof. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \in P_d(n+1)$. We describe how to associate λ to a partition $\mu \in P_\nu(3n+2, n)$ step-by-step and illustrate it with an example.

- Step 1: As λ has k distinct parts, subtract and save $k, k-1, k-2, \ldots, 1$ from $\lambda_1, \lambda_2, \ldots, \lambda_k$, respectively.
- Step 2: Conjugate the remaining partition and call the new parts $r_1, r_2, \ldots, r_{\lambda_1-k}$.
- Step 3: Double the partitions $(k, k-1, k-2, \ldots, 1)$ and $(r_1, r_2, \ldots, r_{\lambda_1-k})$ and add 1 to each part r_i.
- Step 4: Join the partitions.
- Step 5: Add $n-(\lambda_1-k)$ parts of size 1.

Overlooking the order of the parts, the partition μ obtained with the steps above is the partition

$$\mu = (2k, 2(k-1), 2(k-2), \ldots, 2, (2r_1+1), \ldots, (2r_{\lambda_1-k}+1), \underbrace{1, 1, \ldots, 1}_{n-(\lambda_1-k)}).$$

μ really lies in $P_\nu(3n+2, n)$, because every even part from 2 to $2k$ appears exactly once, $2r_i+1 \leqslant 2k+1$ for all $r_i \leqslant k$, the odd parts are in number of n and

$$2k + 2(k-1) + \cdots + 2 + (2r_1+1) + \cdots + (2r_{\lambda_1-k}+1) + \underbrace{1+1+\cdots+1}_{n-(\lambda_1-k)}$$

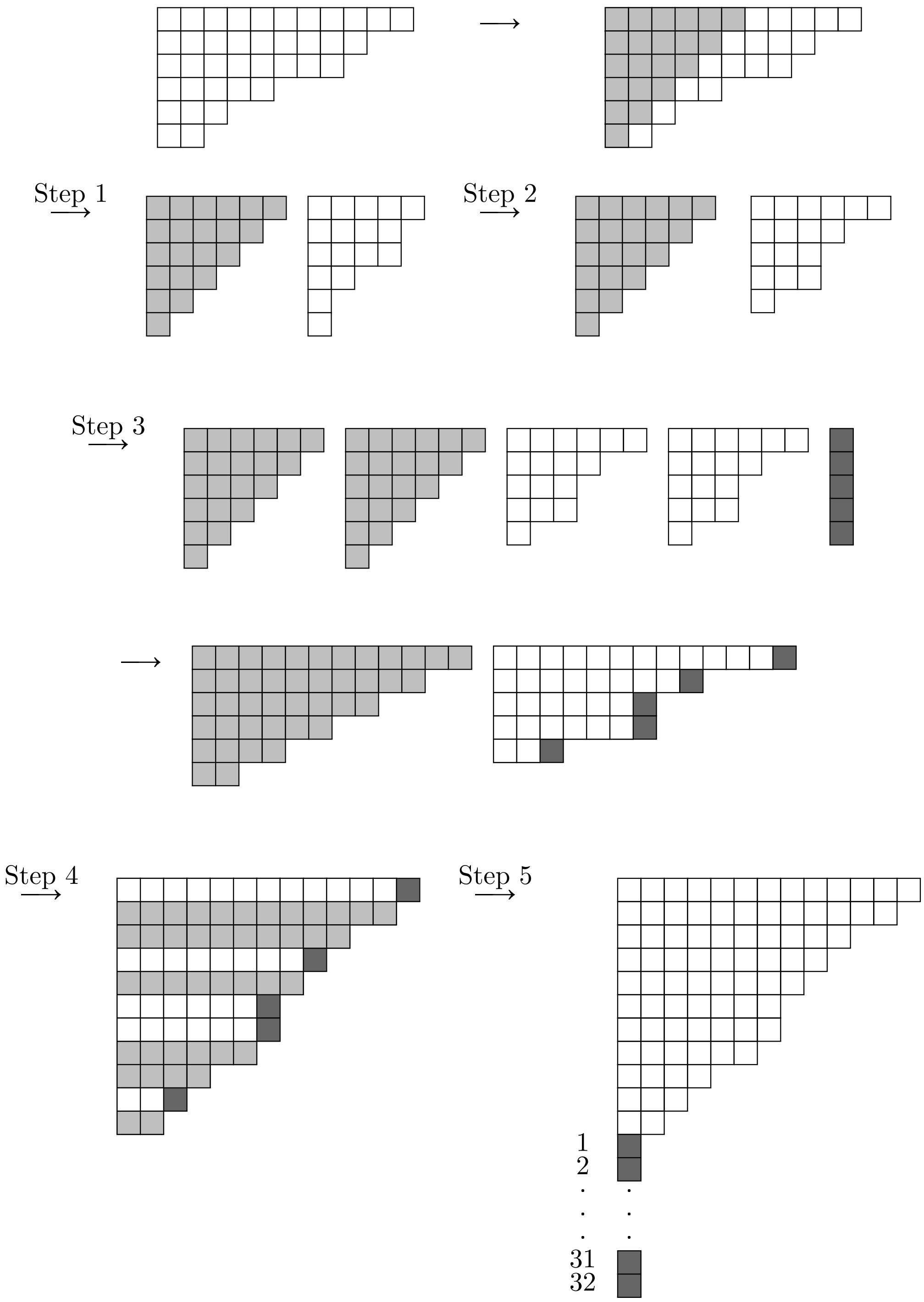
$$\begin{aligned}
&= k(k+1) + 2(r_1 + \cdots + r_{\lambda_1 - k}) + \lambda_1 - k + n - (\lambda_1 - k) \\
&= k(k+1) + 2(n + 1 - \frac{k(k+1)}{2}) + n \\
&= 3n + 2
\end{aligned}$$

The inverse map is easy to build.

□

Example 4.8. *For* $n = 37$*, consider* $\lambda = (11, 9, 8, 5, 3, 2)$ *and* $\mu = (13, 12, 10, 9, 8, 7, 7, 6, 4, 3, 2\underbrace{1, 1, \ldots, 1}_{32\ times})$.

The following diagram illustrates how to get from λ *to* μ.



By replacing x and y by q in Equation 6 at page 29 of [2], we get the next identity. Although it is already proved, we present a new combinatorial way to demonstrate it.

Theorem 4.11. *We have the identity*

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q^2)_{n+1}} = \sum_{n=0}^{\infty} (-q;q^2)_n q^n. \tag{4.16}$$

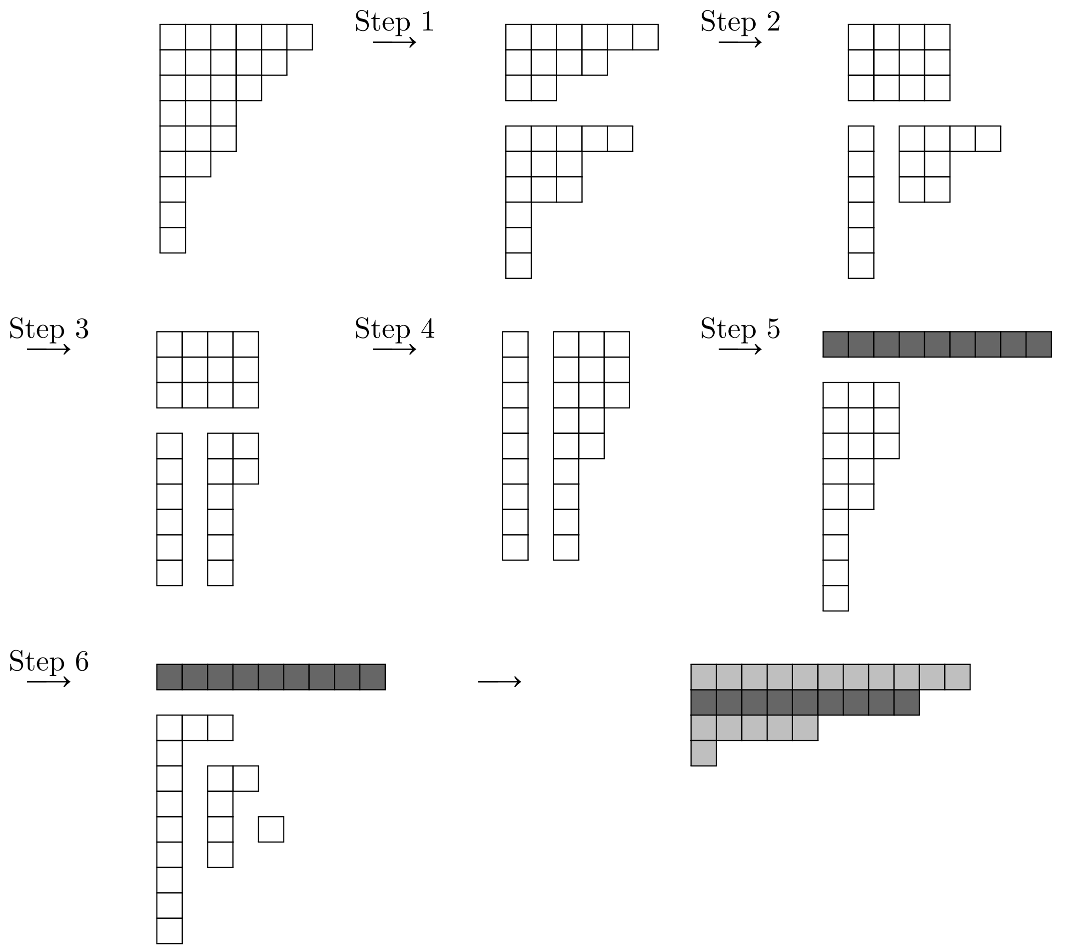
Proof. Let us consider the two following sets of partitions:

- $N(n)$: The set of partitions of n such that, if $2s$ is the largest even part, then all parts $2, 4, ..., 2s$ must appear exactly once, any odd part must be smaller than $2s+1$ and may appear in any number.
- $\Lambda(n)$: The set of partitions of n into dark gray and light gray parts such that there is only one dark part s and the light parts must be distinct odd parts, smaller than $2s-1$.

Note that the left and right hand sides of (4.16) are the generating functions for partitions lying in $N(n)$ and $\Lambda(n)$, respectively. If we prove that $|N(n)| = |\Lambda(n)|$, the theorem follows. So, we describe step-by-step a map from $N(n)$ to $\Lambda(n)$.

- Step 1: Given a partition counted by $N(n)$, consider its corresponding Young Diagram. Separate even and odd parts, getting two new diagrams.
- Step 2: As each even part appears exactly once and the greatest one is $2s$, change the s even parts for s parts of size $s+1$. Then, subtract and save one unit from each odd part.
- Step 3: As the partition into odd parts has turned into one with even parts, split each of these parts into two parts of equal size.
- Step 4: Join all the partitions together (for more details see Example 4.9) and then subtract and save one unit from each part of the new diagram.
- Step 5: Transform the removed units into a dark part.
- Step 6: The light gray parts are obtained by doing hooks with the remaining rows and columns: for the first part, take the first row together with the first column; for the second part, take what is left from the second row together with what is left from the second column. Keep doing this process until there are no more squares left in the old diagram.
- Step 6: Finally, get the parts together.

Example 4.9. *We illustrate how to get partition* $(\boxed{11}, 9, \boxed{5}, \boxed{1}) \in \Lambda(26)$ *from partition* $(6,5,4,3,3,2,1,1,1) \in N(26)$.



The resulting light gray parts are odd because the parts below the Durfee Square of the partition after step 5 appear in pairs. This also proves that these parts are distinct.

Note that the unique dark part of the resulting partition is equal to the number of parts of the original partition (s plus the number of odd parts). The largest light part is, at most, 2 times the number of odd parts plus $2s-1$. So, the resulting partition lies in $\Lambda(n)$.

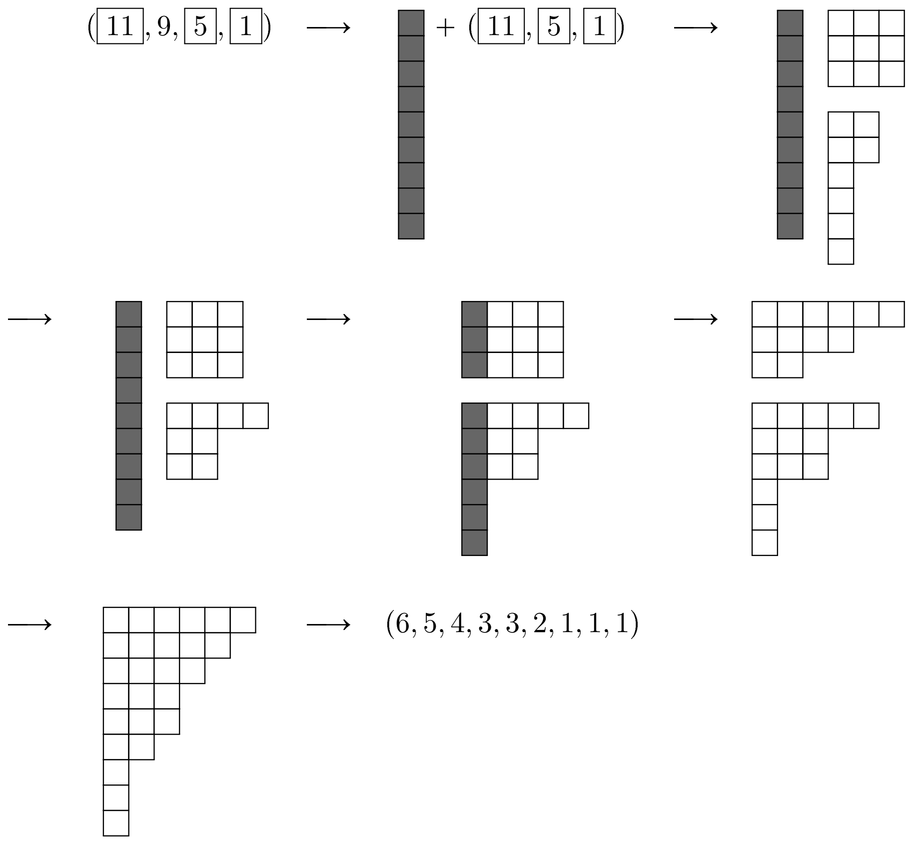
To describe the inverse map, take the dark part and draw it as a column. Beside it, build a square with side equal to the number of light gray parts of the partition. Over this square, display the remaining parts as hooks to complete the Young Diagram. The steps that complete the inverse map are easy to see and we illustrate them in the next example.

Example 4.10. *Considering the partition* $(\boxed{11}, 9, \boxed{5}, \boxed{1}) \in \Lambda(26)$*, we set the way back to* $(6,5,4,3,3,2,1,1,1) \in N(26)$*.*

Thus, with the maps we described above we have set a bijection between $N(n)$ and $\Lambda(n)$ and, therefore, statement (4.16) holds.

□

From Theorem 4.11, changing q by $-q$, we have the following



Corollary 4.3. *We have the identity*

$$\nu(q) = \sum_{n=0}^{\infty} (-1)^n (q;q^2)_n q^n.$$

Remark 4.3. *Knowing Mock Theta function*

$$\lambda(q) = \sum_{n=0}^{\infty} \frac{(-1)^n (q;q^2)_n q^n}{(-q;q)_n},$$

we set the two-variable generating function for $\lambda(q)$

$$\lambda(q,z) = \sum_{n=0}^{\infty} \frac{(-1)^n (q;q^2)_n q^n}{(-zq;q)_n}.$$

Considering the coefficient of z^0*, we get the generating function* $\sum_{n=0}^{\infty}(-1)^n(q;q^2)_n q^n$*, which is equal to* $\nu(q)$*.*

5 Concluding Remarks

Based on a matrix representation for partitions we could relate it to coefficients of some Mock Theta functions, by overlooking signs and interpreting the general terms as generating functions for some of those partitions. As it was possible to set this straight relation between partitions and two-line matrices, the second line provided us characteristics of the partitions. We classified the partitions in a table according to the sum of the elements of the second line of its matrix and we were able to see patterns suggested by it. Many identities were obtained by considering other known partitions and into distinct parts is the one appears more.

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