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Simplified self-consistent model for emittance growth in charged beams with mismatched envelopes

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This paper analyzes the envelope dynamics of magnetically focused, high-intensity charged particle beams. As known, mismatched envelopes decay into equilibrium with simultaneous emittance growth. To describe the emittance growth we develop a simplified self-consistent macroscopic model: emittance is evaluated in a partially analytical way which invokes the beam profile, with self-consistency resulting from the inclusion of the emittance growth into the envelope equation. The model is then compared with full N -particle beam simulations and the agreement is shown to be quite reasonable. The model helps to understand the physics of the problem and is computationally faster than full simulations. Other aspects are discussed in the paper. © 2007 American Institute of Physics. [DOI: 10.1063/1.2472294]

I. INTRODUCTION

Magnetically focused beams of charged particles can relax from nonstationary into stationary flows with concomitant emittance growth.¹ Such is the case of beams with an initially mismatched envelope, flowing along the magnetic symmetry axis of focusing systems. Here the initial oscillating envelope relaxes into the equilibrium state with the corresponding growth of emittance. Gluckstern² shows that mismatched beams induce formation of large scale resonant islands³ beyond the beam border. Beam particles could be captured by the resonant islands, departing from the beam vicinity. That would cause the noticeable emittance growth, along with the associated decay of the whole system into equilibrium. At the final relaxed state the entire system could be seen as formed by a dense core of cold particles plus a tenuous population of hotter particles called halo. The natural splitting of the initial cold beam into a cold dense and hot tenuous populations suggests describing the latter as a test particle population. Further steps can be attempted to include into the theory, under various degrees of approximation, the reaction of the hot population on the cold one.

An initial approach to study the self-consistent dynamics of the core plus halo is to make use of conservation laws for the entire system formed by the beam particles and fields, along with a few general assumptions on the beam aspect. Some estimates on upper limits for the final relaxed state can be thus obtained.¹ This is most useful since with a discrete amount of information based on initial conditions one can predict experimentally relevant features like beam size, beam emittance, and others. Such is the case of a recent investigation where mismatched space-charge dominated beams were seen to relax into matched hotter beams for which emittance, beam size, and the number of particles in the halo could be evaluated.⁴

One additional issue would be on the decay rate of the initial beam into the relaxed final state. We note here that in order to describe the final state solely, no information on beam evolution is actually needed. But if one wishes to study in more detail the decay from the initial state, knowledge of the mechanism controlling beam heating is needed.

The purpose of the present paper is to extend previous results including those of Ref. 4, improving the estimates on the characteristics of final relaxed beams and describing in an approximate way the decay dynamics towards this relaxed state. With regards to the latter issue, we intend to develop a semianalytical method which, besides being much faster than the direct fully numerical particle simulation, helps to understand halo formation. Our model is based on a feedback loop connecting the dynamics of a small group of test particles representing the halo with its associated emittance, and a single dynamical variable representing the average radius of the particle distribution. The small number of dynamical entities is critical in shortening the numerical runs.

The paper is organized in the following form: in Sec. II we introduce the basic model as well as the macroscopic model for beam dynamics. In Sec. III we study the microscopic particle dynamics comparing result of the purely test particle approach with full self-consistent simulations. In Sec. IV we look into the dynamics of emittance growth and let the test particles to react back on the core to allow for a simple model encompassing self-consistent aspects of the problem. In Sec. V we conclude the work. Full self-consistent simulations make use of Gauss's method with a varying number of particles N , up to $N=20000$;⁵ all full simulations start with round homogeneous beams (flat top beams) without emittance (cold beams) in order to represent space-charge dominated beams.

II. THE MODEL AND GENERAL EQUATIONS

Our system is formed by a round beam of charged particles moving along the inner channel of a circular conducting pipe; the beam is focused by a constant solenoidal mag-

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netic field and is aligned with the symmetry axis of the pipe. The beam is initially cold, which means that its initial emittance can be neglected. Since in this case we have a space-charge dominated beam, and since space-charge beams are fairly homogeneous,¹ we suppose that the density distribution over the beam cross section initially obeys a flat top profile

$$n(r) = \begin{cases} \text{Const.} & \text{if } r \leq r_0, \\ 0 & \text{if } r > r_0. \end{cases} \quad (1)$$

r is the radial variable measured from the symmetry center and r_0 denotes the initial value for the beam radius. As the beam envelope evolves, particles are expelled from the core and start to populate an extended hot halo surrounding the cold core. At this point expression (1) is no longer valid because not only particles are not restricted to live within r_0 , but also because the density becomes inhomogeneous. Notwithstanding the complexity of the dynamics, an exact governing equation can still be obtained, now for the evolving beam envelope which we represent by r_b . r_b is related to the rms radius $\langle x^2 + y^2 \rangle^{1/2}$ through

$$r_b^2 \equiv 2\langle x^2 + y^2 \rangle, \quad (2)$$

where the brackets denote particle average or, equivalently, phase-space average, the definition of which will be made more precise and operational later. The envelope equation itself reads^{6,7}

$$\frac{d^2 r_b(s)}{ds^2} = -\kappa r_b(s) + \frac{K}{r_b(s)} + \frac{\varepsilon^2(s)}{r_b^3(s)}, \quad (3)$$

where $\kappa \equiv (qB/2\gamma m\beta c^2)^2$ is the focusing factor, B denotes the axial and constant focusing magnetic field. $K = 2Nq^2/\gamma^3 m\beta^2 c^2$ is the constant beam perveance, and $\varepsilon(s)$ is the beam emittance, which can depend on the axial distance s . N is the number of beam particles per unit axial length, q denotes the beam particles charge, m is the corresponding particle mass, $\gamma = (1 - \beta^2)^{-1/2}$ is the relativistic factor, $\beta = v_z/c$, where v_z is the constant axial beam velocity, and c denotes the speed of light. Beam emittance is defined in the form

$$\varepsilon^2 = 4 \left[\langle x^2 + y^2 \rangle \langle x'^2 + y'^2 \rangle - \frac{\langle x^2 + y^2 \rangle'^2}{4} \right] \quad (4)$$

where the primes indicate derivatives with respect to the axial distance s . Equation (4) is only defining, but it can be associated with the energy conserving relation⁶

$$\frac{1}{2} \langle x'^2 + y'^2 \rangle + \frac{\kappa}{2} \langle x^2 + y^2 \rangle + \mathcal{E}(s) = \text{Const.} \quad (5)$$

to provide information on emittance growth. In Eq. (5), $\mathcal{E}(s)$ is the average self-field energy per particle, $\mathcal{E}(s) = (1/4\pi K) \int |\nabla\psi|^2 d^2r$, where ψ is the dimensionless scalar electromagnetic potential governed by the Poisson equation

$$\nabla_{\perp}^2 \psi = -\frac{2\pi K}{N} n(\mathbf{r}, s), \quad (6)$$

and measured in units of $\gamma^3 m\beta^2 c^2 / q$.

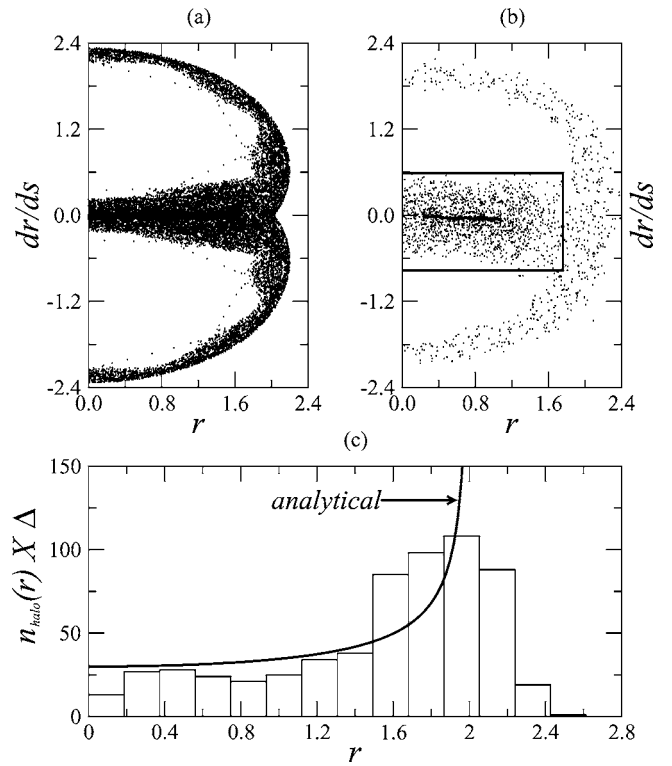


FIG. 1. Phase-spaces corresponding to a test particle population in (a), and to the results of a full simulation in (b). In panel (c) we display the histogram for halo particles computed from panel (b); $\Delta = 0.18$ is the bin size of the histogram, used to scale the density n_{halo} to the number of particles per bin.

If one now supplements Eqs. (3)–(6) with the initial condition at beam entrance $r_b(0) = r_0$ augmented by a condition of straight injection $r_b'(0) = 0$, estimates on the final relaxed state of an initially mismatched beam become possible. With further information on the way particles are ejected from the beam core, helpful information on the dynamics of the relaxing beam can also be obtained.

III. ESTIMATES AND SIMULATIONS FOR THE RELAXED BEAM STATE

In this section we make use of the conserved quantities discussed in the previous section to make predictions on the final, or relaxed beam states, where variations of average quantities vanish. Our basic assumption on the final state will be of a matched cold core surrounded by a hot, but otherwise stationary halo. In general the halo is taken to be produced as the mismatched beam decays into its final matched state, but halo structure must be better specified so we can obtain refined information on this final state. A better specification of the halo is one of the purposes of the present section.

As the beam oscillates a small population of particles at the beam borders sees the beam from the viewpoint of test particles and basically are ejected via Gluckstern's resonance mechanism.² For a perfectly round beam the dynamics is purely radial and Fig. 1(a) reveals the phase-space appearance for those test particles after a sufficiently long time. In the figure we display a Poincaré plot of the dynamical variables r , r' after a large number of oscillations of a mis-

matched core. Test particles are initially distributed within a small region \mathcal{I} at beam border, $\mathcal{I}=[r_0, r_0(1+\delta)]$, where $\delta \ll 1$. As we mentioned, particles at the beam border are the ones forming the halo. Cold particles initially well inside the beam are actually part of the beam and tend to stay there, and at large distances away from the beam densities are so small that real particles cannot be found. This allows us to conclude that particles preferentially carried away by the resonances are precisely those located close to the beam borders.⁸ The dynamics of each test particle is then computed from

$$\mathbf{r}'' = -\kappa\mathbf{r} - \nabla_{\perp}\psi(\mathbf{r}, s), \quad (7)$$

with ψ as the potential solving a “test particle version” of the Poisson equation (6),

$$\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial \psi}{\partial r} = \begin{cases} -\frac{2\pi K}{N} \frac{N}{\pi r_c^2}, & r < r_c(s) \\ 0, & r > r_c(s) \end{cases}; \quad (8)$$

test particles are thus subjected only to the action of a bulk flat top core of radius $r_c(s)$. For the moment we suppose the flat top core to be negligibly affected by the small ejected population, so $r_c(s)$ representing the beam’s cold core envelope is governed by an equation similar to Eq. (3) with ε set to zero. Since emittance then vanishes, the equilibrium beam radius r_{eq} is computed from Eq. (3) with $\varepsilon=0$, as $r_{\text{eq}}^2 = K/\kappa$.

Now a relevant scaling issue. In the remaining, and in particular to obtain the dynamics subjacent to the phase-space pictured in Fig. 1(a), we first rescale all transverse coordinates so they are measured in units of r_{eq} ; in addition, $\kappa^{1/2}s \rightarrow s$. This is equivalent to set $K \rightarrow 1$, $\kappa \rightarrow 1$, which explicitly leaves us with the only free parameter of the theory, the initial mismatch r_0/r_{eq} , now represented as the initial radius r_0 . We note that under this scaling condition, the emittance is measured in units of $K/\kappa^{1/2}$. We also take $\delta=0.04$ in the present case and record particle variables each time the envelope attains a maximum. Under straight beam injection $r'_b(0)=0$ this maximum naturally coincides with the initial size r_0 provided $r_0 > r_{\text{eq}}$. We take here $r_{\text{max}}=r_0=1.6$ without any loss of generality and observe that these relatively large mismatches (=60% in this instance) are quite appropriate to describe the beam dynamics in new generation high-power microwave sources.⁹ From the figure one readily obtains information on the halo dimensions. One sees that the halo thus formed basically lies on the separatrix of the corresponding dynamical system, occupying a narrow band determined by the resonance width.¹⁰ The separatrix has one semicircular branch of average radius $r_{\text{separatrix}} \approx 2.0$ in this particular case, and a central horizontal segment extending up to $r_{\text{separatrix}}$. For further investigation, we shall take the resonance width and the particle density within the resonance region as approximately constants, which seems to be reasonably compatible with the aspect of Fig. 1(a). One is then referred to panel (b) where the result of a full self-consistent simulation is displayed. Simulations are made with help of Gauss’s law⁵ for a number up to $N=20000$ of macroparticles; convergence is attained at somewhat smaller values of N . This kind of simulation is very convenient for

round beams because it is based on collective effects; the field at a certain radial coordinate r depends on the total number of particles with coordinates smaller than r , which precludes the effects of collisions between individual particles. Instabilities and profile distortions around the round shape are small here,^{11,12} so the Gauss’s approach is indicated. The beam is quite stable for small deviations around the symmetry axis and centroid dynamics can be safely ignored as well.^{13,14} Note that the seed for any linear or dynamical instabilities is provided by random positional beam loading.

Panel (b), constructed at time $s \sim 1000$ approximately corresponding to 150 envelope cycles, reveals similarity with the test particle computations. The remarkable difference lies on the aspect of the horizontal branch where the ultradense thin line representing the cold core in the full simulations is absent in the test particle approach (when comparing both plots, we recall that Poincaré plots are cumulative while a single snapshot represents results of the full simulation). The density of the tenuous population around the dense core is small, so if one evaluates average quantities depending on coordinates or velocities, these quantities tend to be dominantly determined by the core. With this remark in mind, and to simplify calculations, we will take the tenuous population along the horizontal branch as only adding density to the core but not affecting dynamical quantities corresponding to the core, like its contribution to the total emittance for instance. In other words we stick to the view that the core is a flat top entity of radius r_c . The halo itself shall be taken as formed by the ejected population lying on the semicircular branch of the separatrix, off the box indicated in the figure.

If we therefore take the halo as formed by this outermost semicircular branch and assume phase-space density and resonance width as approximately constants, as discussed above, the radial density of the halo population can be cast into the form

$$n_{\text{halo}}(r) \equiv 2 \int_{r'_1}^{r'_2} \sigma(r, r') dr' \approx \frac{2\sigma r_h w}{\sqrt{r_h^2 - r^2}}, \quad (9)$$

where w is the resonance width assumed to be small, $r'_2 - r'_1$ measures the vertical extension of the resonance strip at coordinate r , $\sigma(r, r') = \sigma$ is the phase-space density within the resonance width associated with the halo population, and the factor of 2 stems from the mirror symmetry across the horizontal axis. Coefficient $2\sigma r_h w$ can be expressed in terms of the halo fraction $f \equiv N_{\text{halo}}/N$ as we write $N_{\text{halo}} = \int_{r=0}^{r_h} n_{\text{halo}}(r) dr$; the final expression reads

$$n_{\text{halo}}(r) = \frac{2fN}{\pi\sqrt{r_h^2 - r^2}}. \quad (10)$$

As a quantitative test we compare Eq. (10) with results of a full simulation. We display a histogram of the final population discarding the central region which, as remarked earlier, is dominated by the core. The comparison is found in Fig. 1(c) where one sees that the agreement is relatively good for $r_h = r_{\text{separatrix}} = 2.0$. In particular, radial densities really tend to peak at $r=r_h$ owing to the geometrical aspects of the particles distribution along the resonances.

Given the nice agreement, which in turn suggests that the halo is well defined this way, we now turn to obtaining characteristics of the final relaxed states both from estimates and from full simulations. In order to do so we will need to break up all the statistic summations into two pieces: one coming from the cold core and the other coming from the halo particles. To be specific, we shall represent an average quantity $\langle G \rangle$ of a one-particle quantity $G_n = G_n(r_n, r'_n)$ in the form

$$\langle G \rangle \equiv \sum_n \frac{G_n(r_n, r'_n)}{N}, \quad (11)$$

where the sum extends over all the N particles present in the system. Expression (11) can be cast into the form

$$\langle G \rangle \equiv \frac{N_{\text{core}}}{N} \sum_{\text{core}} \frac{G_n(r_n, r'_n)}{N_{\text{core}}} + \frac{N_{\text{halo}}}{N} \sum_{\text{halo}} \frac{G_n(r_n, r'_n)}{N_{\text{halo}}}, \quad (12)$$

where we explicitly divide the right-hand side into contributions from the core and halo particles, respectively, $N = N_{\text{core}} + N_{\text{halo}}$. The final form for the average of G is thus

$$\langle G \rangle = (1 - f)\langle G \rangle_{\text{core}} + f\langle G \rangle_{\text{halo}}, \quad (13)$$

where the fraction $f = N_{\text{halo}}/N$ was introduced earlier. At this point it is interesting to evaluate the squared beam envelope r_b^2 , which was defined in Eq. (2) as twice the rms beam radius,

$$r_b^2 \equiv 2\langle r^2 \rangle = (1 - f)2\langle r^2 \rangle_{\text{core}} + f2\langle r^2 \rangle_{\text{halo}}. \quad (14)$$

We specialize the calculation to the case of cores with uniform density. Then, as we evaluate the averages equivalently from integral over particle densities, we obtain

$$2\langle r^2 \rangle_{\text{core}}(s) = \frac{2 \int_0^{r_c} \sigma_{\text{core}}(s) 2\pi r r^2 dr}{\int_0^{r_c} \sigma_{\text{core}}(s) 2\pi r dr} = r_c^2(s). \quad (15)$$

For the halo, looking at the asymptotic relaxed state and thus making use of expression (10) for the radial halo density one obtains

$$2\langle r^2 \rangle_{\text{halo}}(s \rightarrow \infty) \equiv \frac{2 \int_0^{r_h} n_{\text{halo}}(r) r^2 dr}{\int_0^{r_h} n_{\text{halo}}(r) dr} = r_h^2. \quad (16)$$

The final result for the relaxed state can be simply written in the form

$$r_b^2 = (1 - f)r_c^2 + fr_h^2. \quad (17)$$

This form is identical to the one used in a previous paper⁴ where, however, uniform density for the halo component was assumed. We will see shortly that splitting of the averaged radial coordinate is critical to evaluate fraction f , and the similar aspects of the previous and present approaches in regard to the halo density suggests that both results fortunately should agree numerically. Slight discrepancies arise owing to the halo inhomogeneity and also owing to the

fact that, for sake of completeness, the self-field energy of the halo is included in the present investigation but left out in the previous. However, halo density is so small that the discrepancies are almost negligible. We also observe that throughout the paper we use the fraction f as its asymptotic value; $f = f(s \rightarrow \infty)$. This is equivalent to saying that halo particles are such from the start, but the assumption may not be completely accurate. The fact is that although most of the halo particles are indeed ejected from beam borders at early stages of the dynamics, some of them come from inner regions more gradually.⁸ Even so, taking f constant and equal to the asymptotic value provides an easy way to reach fairly precise values as shall be shown here. We elaborate more on this point in our final remarks.

Let us now apply the conservation laws linking initial and final states to obtain information on the state of the relaxed beam. As mentioned earlier, we suppose that an initially mismatched, space-charge dominated beam, decays into a matched beam with simultaneous emittance growth. Also as commented before, to simplify matters, and given the space-charge dominance, we neglect the initial emittance as a vanishing small quantity. In addition, the space-charge character of the initial distribution allows us to accurately represent the initial beam as a flat top charge distribution with uniform density, Eq. (1). Then, as the beam evolves the flat top quality is lost, emittance grows and the beam attains a matched state where the rms radius evolves to $r_b(s \rightarrow \infty) < r_b(s=0) \equiv r_0$.

In analytical terms the self-field energy is simply evaluated via Gauss' law. For the initial state the beam is a flat cold distribution of initial radius r_0 , and at final state the beam is to be taken as the superposition of a cold flat component of radius r_{eq} and perveance $K(1 - f)$ plus the hot halo. Halo density given by Eq. (10) is nonuniform but still preserves azimuthal symmetry, which allows application of Gauss' law. We can input all the variables into the energy conserving expression Eq. (5) from which, in association with the asymptotic stationary forms of Eqs. (3) and (4) plus expression (14) or (17), a final numerical result can be obtained for the fraction f . Further details of the steps needed in the analytical approach to evaluate f can be found in the Appendix.

On the simulation side, we let the self-consistent full code run for a long enough time to at least reach approximate equilibrium. From that state, we numerically compute the fraction f according to the scheme advanced earlier: we discard particles populating the central region of the respective phase-space, those inside the box of Fig. 1(b), and count the remaining particles as the halo. The definition of the box is a bit arbitrary, but once one defines it such as to remove the central population leaving the semicircular branch fairly untouched, results are not much affected by the arbitrariness. Our results are condensed in Table I, from which is seen that the analytical estimates match nicely the results arising from full simulations.

Up to this point all the analysis has been focused on the connection involving initial and final states of the dynamical system. With further information on halo dynamics we may create a simple model describing the decay dynamics to-

TABLE I. Comparison of full simulations, and analytical estimates for asymptotic states based on conserved quantities.

r_0	$\varepsilon_{\text{analytical}}$	$\varepsilon_{\text{simulation}}$	$f_{\text{analytical}}$	$f_{\text{simulation}}$
2.0	1.443	1.408	0.124	0.120
1.6	0.768	0.715	0.102	0.091
1.2	0.208	0.187	0.026	0.025

wards the relaxed state. This is done in the next section.

IV. DECAY DYNAMICS TOWARD THE FINAL RELAXED STATE

Once again let us observe that what has been done so far was to predict the final state from initial conditions with the help of conserved quantities. If one wishes to go a step further and study the decay process itself, some dynamics must be incorporated into the theory. We will do this in two stages. As a first and simpler step we shall investigate emittance growth in the field of a given oscillating cold core. Then we shall attempt to construct a self-consistent model which includes the feedback effects of emittance growth on the beam dynamics. In both cases, the halo particles shall be treated like a test particle population.

A. Emittance growth driven by a periodically oscillating cold core

As mentioned before, it is reasonable to suppose that as the beam core oscillates, particles in the vicinity of the beam border are ejected according to the Gluckstern's mechanism.² The overall emittance for the whole beam system is defined in Eq. (4). As we did earlier for the beam envelope, Eq. (14), we divide the summation of the velocity term in Eq. (4) into its core and halo components. The resulting expression takes the form

$$\varepsilon^2 = r_b^2 [(1-f)r_c'^2 + 2f\langle r'^2 \rangle_{\text{halo}} - r_b'^2], \quad (18)$$

where the connection between r_b and r_c is defined by Eq. (14) and where the averages subscripted with *halo* now denote averages only over the test particle population, which is the one modeling the halo,

$$\langle G \rangle_{\text{halo}} \rightarrow \frac{1}{N_{\text{test particles}}} \sum_{\text{test particles}} G_i. \quad (19)$$

The core is governed by Eq. (3) with ε set to zero, and the dynamics of each test particle is simply governed by Eqs. (7) and (8). We still have to decide what value should be assigned to the fraction δ associated with the radial interval containing the test particle population. Fortunately the answer is not involved. We simply observe that since we already know the fraction f of emitted particles and since we also know that the initial beam profile is a flat top one, it is easy to conclude that $\delta = f/2$.

Our results for emittance growth in this case where core oscillation does not decay are summarized by comparisons of Figs. 2(a) and 2(b). In panel (a) we plot the results of full

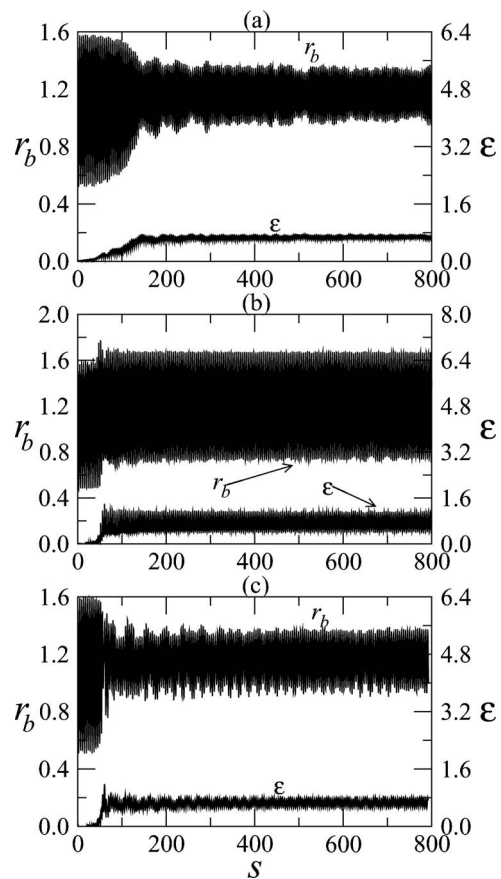


FIG. 2. Emittance growth and envelope (r_b) dynamics in: (a) full simulations; (b) the case where the core r_c oscillates with fixed amplitude [$r_b \rightarrow r_c$ and $\varepsilon \rightarrow 0$ in Eq. (3)]; (c) in the self-consistent simplified model.

simulations and in panel (b) we plot results for the rms envelope radius of our simplified model, all for the initial mismatch $r_0=1.6$ -rms envelope, r_b , on the left vertical axis and emittance, ε , on the right vertical axis. We promptly notice from panel (b) that as soon as phase-mixing takes place due to the chaotic dynamics of the ejected test particles, emittance ceases to grow even though the core continues its oscillatory dynamics. Emittance approximately tends to the value predicted by full simulations, albeit exhibiting much larger oscillations around the average asymptotic value, the latter being in fact slightly larger than in the full case. This is to be expected because the oscillatory amplitude and energy of the driving core do not decay in this particular instance. The rms radius obtained from Eq. (14) tends to grow instead of recede because the core oscillates with fixed amplitude and the halo grows. The observed rms growth is clearly a wrong result which must be fixed. Fortunately this can be done when the feedback cycle is closed like in panel (c) with the backwards effect of a growing emittance on the rms envelope. This is the subject of the next subsection.

B. Closing the feedback loop

As emittance becomes noticeable, some effect on the cold core should be noticed. One expects that the extra kinetic energy of halo particles must affect the initial energy source which is the oscillating beam core. The inclusion of

coupling involving the core and the halo is done in a simple way, building on the approach of the previous subsection. In other words, what we do to close the feedback loop is simply to take into proper account the emittance term in the exact envelope equation for $r_b(s)$, Eq. (3); the emittance itself is again evaluated by means of Eq. (18) where now the cold core oscillatory behavior slowly decays as its energy is drained by the test particles representing the halo population. In more precise terms, the core r_c is no longer governed by Eq. (5) with $\varepsilon \rightarrow 0$. Rather, it can be obtained at any particular instant from Eq. (14) as

$$r_c(s) = \sqrt{\frac{r_b^2(s) - f\langle r^2 \rangle_{\text{halo}}(s)}{1 - f}}, \quad (20)$$

the dynamics of each test particle being again governed by Eqs. (7) and (8). Emittance evaluated from Eq. (18) is carried into the envelope equation Eq. (3) for $r_b(s)$, from which one is enabled to solve the system in a self-consistent fashion. The point here is that once the emittance term is turned on in the envelope equation, one converts the system from one where the envelopes oscillates freely into one where the rms beam radius $r_b(s)$ couples with the kinetic energy associated with the emittance term. The coupling is exactly represented by Eq. (3). Emittance growth is associated with halo production and in our simplified approach the dynamics of the test particles representing halo is evaluated on basis of the interaction of these particles and the dense core with radius r_c as commented earlier—the weak interaction of halo particles among themselves is neglected. We thus have to write r_c in terms of r_b which is now the variable naturally coming from the differential equation (3), and $\langle r^2 \rangle_{\text{halo}}$ coming from the statistics of test particles. The connection involving these quantities is provided by the above expression (20) which comes from Eq. (14) $r_b^2 = (1-f)r_c^2 + f\langle r^2 \rangle_{\text{halo}}$.

Our simplified procedure has advantages over a full simulation: the core component is represented by one single variable r_c and the halo can be represented by a relatively small number of test particles—typically, 1000 test particles. In addition, owing to the small number of variables, numerical runs are much faster than in the case of full codes.

As mentioned above, the results of the model are displayed in panel (c) of Fig. 2. The overall aspect of the results is very similar to the full simulations seen in panel (a), so we conclude that the main aspects of these full simulations can be reproduced with the aid of our simplified self-consistent model. We point out that inclusion of emittance in the envelope equation is a nontrivial matter. Halo particles must accurately have the right relative phase with respect to the core so the core can really decay, delivering energy to the halo; our model reproduces well the effect. We note however that matching between model and full simulations is not perfect. The most prominent discrepancy is perhaps the faster decay towards the final relaxed state seen in the model, albeit the final states of model and full simulation are virtually the same. Apropos of this issue we recall that in our model we take all the halo particles sitting slightly away from the outermost beam border. This is not true in the real system, as it has been observed that a fraction of the halo comes from

inner regions of the beam due to inhomogeneous instabilities.⁸ The dominant population forming the halo still is that population at the borders, but the small fraction coming from the inner regions delays the ejection process and may smooth out the abrupt growth of emittance. Aside from this feature, agreement is nice.

V. CONCLUSIONS

The present investigation develops a simplified, but self-consistent model to describe emittance growth in the dynamics of space-charge dominated beams. Initial emittance is negligible and the beam starts from a flat top homogeneous radial profile. As emittance starts to grow due to nonlinear resonances whose presence results from beam mismatch, energy is extracted from the cold core which thus relax its vibrational state into a stationary one. Taking into account the typical profile of the final states, we first make use of the macroscopic constraints of the system to derive useful information on these final states. We then observe that emittance can be divided into a contribution from the core and another from the hot halo particles. With that on view we set up a simple technique to evaluate emittance growth. As a first step, given an oscillating core acting like a constant amplitude periodic driver, we investigate emittance behavior to conclude that after a certain amount of time needed to generate phase mixing, emittance stops to grow. The final emittance levels are comparable to those predicted by full simulations, but the everlasting core dynamics is incorrect and must be refined. To improve the description, we simply close a feedback loop by considering the envelope equation Eq. (3) with the emittance term previously calculated turned on. What we obtain is a very simple semianalytical model that captures the basic aspects of the relaxation process. Agreement between simulations and the model is accurate, a feature that not only enables to make predictions on asymptotic states, but also allows us to predict the time scales for the decay process. This is of practical interest because the model runs orders of magnitude faster than the full code, so one can obtain useful information within shorter periods of real time. We recall that the practical aspects result from the fact that the model operates with a small number of test particles that do not interact among themselves. Certain details can of course be improved. We notice, for instance, that in the model the time stretch during which emittance grows to its final state is a bit shorter than in the full simulations. This appears to be related to the way particles are extracted from the beam core; in the model the halo population is already placed on beam borders while in the real system some of the particles come from inner regions. While we can improve the model by taking this feature into account, the overall agreement between the present version of the model and full simulations is already nice. We shall therefore stop here and defer further investigation on this theme for future papers.

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APPENDIX: OBTAINING THE FRACTION f WITH THE ANALYTICAL APPROACH

- (1) Start with given values of r_0 , r_c , and r_h : in our case we took $r_0=1.6$, $r_c=1.0$, and $r_h=2.0$. We can right away construct the expression for the relaxed beam radius as a function of the yet unknown fraction f , as in Eq. (17): $r_b^2(f) = (1-f)r_c^2 + fr_h^2$.
- (2) For the initial beam assume round flat top cold distribution of radius r_0 and perveance K . For the relaxed beam assume round flat top cold core of perveance $(1-f)K$ and radius r_c plus halo of radius r_h with radial density given by expression (10).
- (3) Evaluate the potential $\psi(r)$ for initial and final relaxed states with help of: Poisson equation, Eq. (6); assumption of azimuthal symmetry; the densities obtained from the preceding item (2).
- (4) Evaluate the conserved energy given by expression (5) both for the initial and relaxed state. In doing so, note that $2\langle x'^2 + y'^2 \rangle \equiv r_b^2(f)[r_0^2]$ for the relaxed (initial) state.
- (5) Use conserved energy of the preceding expression to obtain $\langle x'^2 + y'^2 \rangle$ for the relaxed state in terms of the given initial conditions and the yet undetermined fraction f ; for the initial state, $\langle x'^2 + y'^2 \rangle_{\text{initial state}} = 0$.
- (6) Use expression (4) to write ε in terms of f through the preceding expression for $\langle x'^2 + y'^2 \rangle$. Note that for the relaxed state, $\langle x'^2 + y'^2 \rangle' = 0$.
- (7) Substitute ε from item (6) into expression (3) (with $r_b'' \rightarrow 0$) to finally obtain a closed expression for f whose roots can be obtained numerically.

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