UNIVERSIDADE FEDERAL DO RIO GRANDE DO SUL

INSTITUTO DE INFORMÁTICA

CURSO DE PÓS-GRADUAÇÃO EM CIÊNCIA DA COMPUTAÇÃO

# Interval Approximation Theory

por

Dalcidio Moraes Claudio

RP- 226 ABRIL/1994

Relatório de Pesquisa



UFRGS-II-CPGCC
Caixa Postal 15064 - CEP 91501-970
Porto Alegre - RS - BRASIL
Telefone: (051) 336-8399 e 339-1355
Fax: (051) 336-5576
E-mail: PGCC@INF.UFRGS.BR

# Contents

*RESUMO*

Este trabalho tem como objetivo a resolução de equações à coeficientes intervalares utilizando a Teoria de Domínios. É definida uma nova igualdade intervalar e definido um corpo dinâmico onde as equações são resolvidas. É também obtida uma condição para não existencia de soluções.

Palavras-chave:

Equações intervalares, domínios, raizes, ordens intervalares.

*ABSTRACT*

This article deals with the solution of a few types of interval coefficient equations, where we make use of the concepts of approximation and information as seen from Domain Theory. To this end, we introduce a new definition of interval equality, which leads to the construction of a dynamic field, in which the equations are solved. The resulting structure allows a new treatment of both linear and nonlinear interval coefficient systems of equations. As a particular application of the theory developed, we derive a condition for the non-existence of solutions.

Key Words:

Domain theory, interval equations, intervals orders.

# 1 INTRODUCTION

The interval spaces ( $\mathbf{IR}, +, *, \subseteq$ ) , ( $V_n\mathbf{IR}, +, \subseteq$ ), etc. are not algebraically complete, in that algebraic equations cannot in general be solved. Thus, for example, in ( $\mathbf{IR}, +, =$ ) there does not exist in general an inverse element, so that an equation as simple as $A + X = B$ can only be solved in special cases [ Rat 70 ]. In interval theory, in view of the way the relation of equality is defined, we cannot find directly the solution of the set of equations of the form $a + x = b$, when $a \in A$ , $b \in B$, $A$ and $B$ being intervals. This equation also breaks down the monotonicity property, as we can see in the following example:

Let $X$ , $Y$ be intervals such that $[2, 5] + X = [1, 10]$ and $[3, 4] + Y = [1, 10]$. We have

$$[2, 5] \supseteq [3, 4]$$

but

$$X = [-1, 5] \subset [-2, 6] = Y$$

Although there is no question about the contribution of interval mathematics to the solution of mathematical problems, particularly through the use of the so-called self-validating methods ( [ KUL 88 ], [ RUM 91 ] ), intervals have been used mostly to construct algorithms for the solution of mathematical problems and not to the foundation of scientific computing [ SMA 90 ]. This attitude is based upon the arguments mentioned above and also on the incompatibility between the Hausdorff topology and the order used in Moore spaces ([ MOO ]). We have observed that the only order compatible with this topology is the trivial order, that is, the equality. In this context, we face a big dilemma:

If continuity according to the topology will not imply monotonicity according to the order, either the function is not computable [ SMY 90 ] or the notion of function computability (linked to the topologic notion of continuity) cannot be defined within this structure.

Since the order used in interval spaces is given by the set inclusion relation, which allows the concept of inclusion-monotonicity, we can make use each time of only one of either concept for the development of the theory.

Bearing in mind that we are interested in the foundations of scientific computing [ SMA 90 ], the option we suggest is to take the monotonic-inclusion order and to consider the Scott topology, which is compatible with that order. In Section 2, we describe the basic concept of intervals and their characterization as an order-theoretic structure. Next in Section 3 we introduce a dynamic relation to replace the usual notion of interval equality so as to make it possible to solve general equations and systems of equations, and not only particular cases. The characterization of the dynamic field itself is described in Section 4, where we study the main properties. We then solve a number of equations in this space, to give an important illustration of the theory.

# 2 BASIC CONCEPTS

**DEFINITION 2.1** **[Interval]** Let **R** be the ordered field of real numbers. Given real numbers $a$ , $b$ with $a \leq b$, the set

$$\{ x \in \mathbf{R} : a \leq x \leq b \}$$

will be called **interval** and will be denoted by

$$X := [ a , b ]$$

In this work, the closed set of all real numbers will be considered an interval and will be denoted it by

$$\perp := [-\infty , +\infty]$$

**DEFINITION 2.2** [ **IR** ] The set of all intervals of real numbers will be denoted by **IR** . The capital letters $X$, $Y$, $Z$,... will be used to denote interval variables in **IR** . Real numbers are identified with single-point intervals, so that

$$\mathrm{R} \subset \mathrm{IR}$$

where we identify

$$x = [ x , x ]$$

**DEFINITION 2.3** Given $A$ , $B$ $\in$ **IR**, the operations $+$, $-$, $\cdot$, $/$ are defined by

$$A * B = \{ a * b : a \in A, b \in B \},$$

where $*$ denote any operator in $\{+, -, \cdot, /\}$ ; for the division ( / ) operation, one must assume $0 \notin B$.

The above definition is motivated by the fact that, in general, the intervals $A$ and $B$ will contain the exact numerical values $\alpha \in A$, $\beta \in B$ , which are not really known. The only thing we do know are the including intervals $A$ and $B$.

From definition 2.3 we derive the so-called **inclusion principle of interval arithmetics**:

$$\alpha * \beta \in A * B \text{ whenever } \alpha \in A, \beta \in B.$$

Also, one can easily check the following properties [ MOO ], assuming $\perp = [-\infty, +\infty] \notin$ **IR** :

$$[ a, b ] + [ c, d ] = [ a + c, b + d ]$$

$$[\,a,\,b\,] - [\,c,\,d\,] = [\,a - d,\,b - c\,]$$

$$[\,a,\,b\,] \cdot [\,c,\,d\,] = [\,\min(ac,\,ad,\,bc,\,bd), \max(ac,\,ad,\,bc,\,bd)\,]$$

$$[\,a,\,b\,] / [\,c,\,d\,] = [\,a,\,b\,] \cdot [\,1/d,\,1/d\,]\,,\ \text{provided} 0 \notin [\,c,\,d\,],$$

which show in particular that the interval subtraction and division operations do not play the role of inverse to the addition and multiplication. Another difference between **R** and **IR** is that the distributive law which holds in **R** is not valid either; for the space **IR** , we can only show the so-called **subdistributive law**

$$A \cdot (B + C) \subseteq A \cdot B + A \cdot C$$

when $A, B, C \in$ **IR** .

We will now characterize **IR** as an **order-theoretic structure**, which will allow us to make the concepts of information and approximation precise in our theory.

**DEFINITION 2.4 [partial order]** Let D be a set and $\sqsubseteq$ a relation on D. Then $(D, \sqsubseteq)$ is said to be a **partial order** iff $\sqsubseteq$ is reflexive, transitive, and anti-symetric.

**DEFINITION 2.5 [complete partial order]** Let $(D, \sqsubseteq)$ be a partial order. A non-empty set $X \subseteq D$ is said to be **directed** iff for all $x, y \in X$ , we have $x \sqsubseteq z$ and $y \sqsubseteq z$ for some $z \in X$. $(D, \sqsubseteq)$ is said to be a **complete partial order** iff D has a least element $\perp$, and for all $X \subseteq D$ directed, X has a least upper bound, denoted by $\bigsqcup X$.

**DEFINITION 2.6 [$\omega$-continuous complete partial order]** Let $(D, \sqsubseteq)$ be a complete partial order. The way-below order $<<$ on $D$ is defined by $d << e$ iff for all $X \subseteq D$ directed, $e \sqsubseteq \bigsqcup X$ implies $d \sqsubseteq x$ for some $x \in X$. $A$ set $B \subseteq D$ is said to be a **basis** for $D$ iff for all $x \in D$, the set $\{\,b \in B \mid b << x\,\}$ is directed and its least upper bound is $x$. $(D, \sqsubseteq)$ is said to be an **$\omega$-continuous complete partial order** iff $D$ has a denumerable basis.

**DEFINITION 2.7 [Domain]** Let $(D, \sqsubseteq)$ be an $\omega$-continuous complete partial order. A set $X \subseteq D$ is said to be **bounded** iff $X$ has an upper bound in $D$. $D$ is said to be **bounded-complete** iff every bounded subset of $D$ has a least upper bound. A **domain** is a bounded-complete, $\omega$-continuous complete partial order.

**THEOREM 2.8 [order-theoretic structure of IR ]** For $X,\ Y \in$ **IR** define $X \sqsubseteq Y$ iff $X \subseteq Y$. Then (**IR**, $\sqsubseteq$) is a domain. Moreover, the least element of **IR** is $\perp := [-\infty, +\infty]$ ; the way-below order for **IR** satisfies $X << Y \text{iff} X = \perp$

or$x_1 < y_1 \leq y_2 < x_2$, where$X = [x_1;\ x_2]$and$y = [y_1;\ y_2]$ ; and a basis for **IR** is $\{[p, q] / p, q \in Q\, p < q\}$.

An interval $[a;\ b]$ can be seen as a **totally defined** real number if $a = b$, and as a **partially defined** real number if $a < b$ ; the interval $\perp$ can be seen as a **completely undefined** real number. Totality and partiality correspond to respectively maximality and non-maximality for $\sqsubseteq$ . The fact that $\perp$ is a completely undefined real number corresponds to the fact that it is the least element for $\sqsubseteq$ .

# 3 INTERVAL APPROXIMATIONS

Following Scott [ SCO 72 ], when one works in scientific computing it is important that one is able to consider the relation of qualitative approximation. It would be important if one could discuss degrees of approximation when the objects under study were numerical methods. It would also be important to have a definition for convergence based on the behavior of the objects in the sense of qualitative approximation. The relationship between the elements which make up the structure is interpreted as follows. The relation $x \sqsubseteq y$ may be intuitively read as $y$ giving more information (or at least as much as) than $x$ in respect to a real $z$ self-contained in both $x$ and $y$. We understand that $x$ gives less information than $y$, but we cannot state how much less in quantitative terms. We can think of these objects as something "containing" information, not complete information but only partial information.

We will now define a family of relations for the elements of **IR** . Because of its dynamical behavior and its special features it will be called afterwards a **dynamic equivalence relation**. The structure obtained from **IR** using this dynamic equivalence relation will be called from now on a **dynamic field**. The basic idea is to take "equivalence classe" for equality.

**DEFINITION 3.1** Consider the complete partial order (R, $\sqsubseteq$ , $\perp$). Given $A, B \in$ **IR**, $x \in$ **R** , we say that $A$ and $B$ are $x$-related, denoted by $A \equiv_x B$, iff $A \sqsubseteq x$ and $B \sqsubseteq x$ , i.e., $A$ and $B$ are approximations to $x$.

This relation, which will be called **dynamic equivalence relation** (DER), has the following properties.

**THEOREM 0.3.2** Consider (**R**, $\sqsubseteq$ , $\equiv_x$ $\perp$) introduced above. Then the following holds :

a) $\equiv_x$ is $x$-reflexive, i.e.,

$\forall\, A \in$ **IR**, $\forall\, x \in A$, we have $A \equiv_x A$.

b)$\equiv_x$ is symmetric, i.e.,

$\forall\, A, B \in$ **IR** we have $A \equiv_x B \;\rightarrow\; B \equiv_x A$.

c)$\equiv_x$ is $x$-transitive, i.e.,

$\forall\, A,\, B,\, C\, \in \mathbf{IR}$, we have $A \equiv_x B\ \wedge\ B \equiv_x C\ \rightarrow\ A \equiv_x C$.

The indexing element (i.e., $x$) may be seen as a commom property that the elements (objects) which are related (in respect to the indexing element) share. These objects come, in general, from physical measurements, observations, etc., that is, they are inexact quantities which are then represented by intervals.

We can establish the following relationship between $\equiv_x$ and $=$ in $\mathbf{IR}$ :

**THEOREM 3.3** Let $A, B \in \mathbf{IR}$. Then

$\forall\, \lambda\, \in\, A\, \delta\, \in\, B\ A \equiv_{\lambda,\delta} B$ iff $A = B$.

$\square$ Proof:

$\forall\, \delta\, \in\, B, \lambda\, \in\, A\ \ A \equiv_x B \Longleftrightarrow\ \{ \forall\, \delta\, \in\, B\ A \sqsubseteq \delta \forall\, \lambda\, \in\, A\ A \sqsubseteq\, \lambda$

$A \sqsubseteq B\ \wedge\ B \sqsubseteq A\ \Longleftrightarrow\ A = B$ $\square$

The monotonicity law holds, as we see from the following result:

**THEOREM 3.4** Let $A, B, C, D\ \in \mathbf{IR}$, $x,\ y\ \in \mathbf{R}$ , and let $*$ denote an arithmetic operator in $\{ +,\ -,\ \cdot,\ / \}$. Then

$A \equiv_x B\ \wedge\ C \equiv_y D\ \Longrightarrow\ A * C \equiv_{x*y} B * D$

where, for the case of $/$, we assume $0\ \notin\ C$. $\square$ Proof: The result is a straightforward consequence of the definition for $\sqsubseteq$ , since $A \sqsubseteq x$ , $B \sqsubseteq x$ , $C \sqsubseteq y$ and $D \sqsubseteq y$ imply $A * C \sqsubseteq x * y$ and $B * D \sqsubseteq x * y$.

Before we list the main algebraic properties of $(\mathbf{IR}\,, \equiv_x\,)$, it is important to establish the relationship between different interval evaluations of an expression. Including sets for the range of a function $f$ plays a very important role in Interval Analysis, since exact values of a function are not known in general. We should notice that the use of intervals make it possible in a simple way to obtain such including sets, i.e. sets which contain the range, from which we can derive results about the non-existence of solutions by showing that these larger sets do not contain the element zero.

**DEFINITION 3.5** Let $f : D \subseteq \mathbf{R} \longrightarrow \mathbf{R}$ be continuous and let $X$ be a subset of $D$. The **interval image** of $f$ in $X$ is the set

$I_f(X)\ =\ \{\, f(x)\ \mid\ x\ \in\ X\, \}\ =\ [\min_{x\in X} f(x);\ \max_{x\in X} f(x)]$.

In view of the difficulty to compute exactly this interval, one might instead compute upper bounds to this set by replacing the real constants and variables by their interval counterparts, giving an interval evaluation which will be denoted by $f(X)$ .

**THEOREM 3.6** [ ALE 80 ] Let $I_f(X)$and$f(X)$ be as defined above. Then there holds

$$f(x) \in I_f(X) \subseteq f(X) \quad \forall\, x \in X.$$

example:
Let $f : \mathbf{R} \longrightarrow \mathbf{R}$
$x \longmapsto x - x$
For $X = [-2, 2]$, we get

$$I_f([-2,2]) = [0,0] \subseteq [-4,4] = [-2,2] - [-2,2] = f([-2,2])$$

We will not be concerned here with the techniques to improve the interval evaluation (for a discussion of this point, see e.g. [ KUL 88 ]). The distinct forms of interval evaluations and the relation $\equiv$ bear the following interconnection :

**THEOREM 3.7** Let $f : D \subseteq \mathbf{R} \longrightarrow \mathbf{R}$ be continuous and let $X \subseteq D$. Let $f_1(X)$ and $f_2(X)$ be two evaluations of $f$ in $X$. Then, for each $i \in I_f(x)$, there holds

$$f_1(X) \equiv_i f_2(X).$$

□ Proof: Straight from the previous theorem. □

# 4 CHARACTERIZING THE DYNAMIC FIELD $(\mathbf{IR}, +, \cdot, \equiv_x, \perp)$

According to the definition of $\equiv_x$, when we name an element $A$ we are indeed dealing with a class of intervals which have the property $x$. We will now proceed to verify that the algebraic properties are valid within this context.

**THEOREM 4.1** Consider the set **IR** of intervals endowed with the operations given by definition 2.3. Let $A, B, C, D \in \mathbf{IR}$. Then the following properties hold:

a)$A + B \equiv_\lambda B + A \quad \forall\, \lambda = a + b\,, a \in A,\ b \in B$ and $A \cdot B \equiv_\beta B \cdot A \quad \forall\, \beta = a \cdot b,\ a \in A,\ b \in B$.

b)$A + (B + C) \equiv_\lambda (A + B) + C \qquad \forall\, \lambda = a + b + c,\ a \in A,\ b \in B\,, c \in C$ and $A \cdot (B \cdot C) \equiv_\beta (A \cdot B) \cdot C \qquad \forall \beta = a \cdot b \cdot c,\ a \in A,\ b \in B,\ c \in C$.

c) there exists an interval $\mathcal{O}$ (namely: $\mathcal{O} = [0\,, 0]$) such that

$$A + \mathcal{O} \equiv_a A \qquad \forall\, A \in \mathbf{IR}, \forall\, a \in A.$$

and there exists an interval **1** (namely: **1**=[1,1]) such that

$$A \cdot \mathbf{1} \equiv_a A \quad \forall A \in \mathbf{IR}, \forall a \in A.$$

d)$\forall A \in \mathbf{IR}$, $\exists -A \in \mathbf{IR}$ such that $A - A \equiv_0 \mathcal{O}$ and $\forall A \in \mathbf{IR}$, with $0 \notin A$, $\exists \frac{1}{A}$ such that $A \cdot \frac{1}{A} \equiv_1 \mathbf{1}$.

e) $A \cdot (B + C) \equiv_\lambda A \cdot B + A \cdot C$, $\lambda = a \cdot (b + c)$, $a \in A$, $b \in B$, $c \in C$.

□ Proof: (a), (b), and (c) come straight from Theorem 3.3 .

(d) We have just to note that $A - A \sqsubseteq 0$ and

$$A \cdot \frac{1}{a} \sqsubseteq \mathbf{1}.$$

(e) Taking in Theorem 3.7 $A \cdot (B + C) = f_1$and $A \cdot B + A \cdot C = f_2$, we get the claimed result if we observe that $\lambda \in I_f$ and make use of Theorem 3.7 .
□

We should make a few remarks about the theorem above. Although the neutral and inverse elements are not unique, they are equivalent with respect to the relation $\equiv$. The properties of the operations + end · are similar to the properties of a field; for this reason, this structure will be called dynamic field. We can think of **IR** as the "error domain" of a field. Since $\equiv$ is not an equivalence relation, we cannot speak of equivalence classes or quotient sets. From an algebraic viewpoint, the theorems refer more to classes of objects than to closed intervals. Its is also possible to see relation $\equiv$ as a weak equality relation: if $A$ and $B$ are error domains of $a$ and $b$, then when $A \equiv B$ we might have $a = b$ , and when $A \not\equiv B$ then we have $a \neq b$. A more realistic approach , according to the nature reality and the numerical means we use to measure it, would be to consider the intervals themselves as the basic elements. From this, the "real world" (i.e., **R** ) would be a particular case of the "interval world" (i.e., **IR** ). As our first goal now is the solution of nonlinear equations, we will gather a few properties of the dynamic field.

THEOREM 4.2 Consider (**IR**, +, ·, $\sqsubseteq$ , $\equiv$, $\perp$ ) as in the Theorem 4.1. Then the following properties hold: (a)

$$A \equiv B \wedge A' \sqsubseteq A \implies A' \equiv B$$

(b)

$$A \sqsubseteq B \implies A \equiv_b B \; \forall b \in B$$

(c)

$$(A + B) \cdot (A - B) \equiv A^2 - B^2$$

(d)

$$A \cdot B \equiv_0 \mathcal{O} \iff A \equiv_0 0 \vee B \equiv_0 \mathcal{O},$$

that is, there does not exist zero divisors (e)

$$A \equiv_c B\ ,\ A \geq 0,\ B \geq 0\ \rightarrow\ \pm\sqrt{A} \equiv_{\pm\sqrt{c}} \pm\sqrt{B}$$

□ Proof:

(c) comes straight from the fact that

$$(\,A\,+\,B\,)\cdot(\,A\,-\,B\,) \sqsubseteq A^2\,-\,B^2 \qquad (\text{ Berti })$$

or from Theorem ..... (d)

$$A\cdot B \equiv_0 \mathcal{O} \implies A\cdot B \sqsubseteq 0\ \rightarrow\ 0\in A \vee 0\in B.$$

The other items are trivial. □

We will consider now the solution of a few equations in this space . To reach this goal, we have to describe first what we mean by solution.

**DEFINITION 4.3** Given an equation

$$F\,(X) \equiv_y Y \qquad\qquad (\mathrm{I})$$

we will define its **optimal solution** $X_o$ (or simply **solution**) to be the set of all $x\in X$ which satisfy the equation $f(x) = y$ for some $y \in Y$ . We will call an **external solution** any $X_e \in \mathbf{IR}$ such that $X_e \in \mathbf{IR}$ such that $X_e \sqsubset X_o$ , and similarly an **internal solution** will be any $X_i \in \mathbf{IR}$such that$X_o \sqsubset X_i$. We form in this way an inclusion chain given by

$$X_e \sqsubset X_o \sqsubset X_i$$

THEOREM 4.4 Given $A, B \in \mathbf{IR}$, then $X = B - A$ is the optimal solution of

$$A\,+\,X \equiv_b B \qquad\qquad (I).$$

□ Proof:

$$A\,+\,X \equiv_b B \iff A\,+\,X\,-\,A \equiv_{b-a} B\,-\,A.$$

Since

$$X \equiv_x A\,+\,X\,-\,A,$$

we have $X \equiv_x B\,-\,A$ for all $x = b - a, b \in B,\ a \in A$. By theorem 3.3 , it follows that $X = B - A$, so that $X$ is the solution set of all equations of the form $a + x = b$, $b \in B$, $a \in A$, and so it is the optimal solution of (I). □

Stefan Berti (see [ BER 69 ]) solves the equation $AX + B = C$ for a few cases where the equality holds. This will not work for us, since our purpose is to solve the whole family of equations $ax + b = c$, when $b \in B$, $a \in A$, and $c \in C$. The solution given by Berti's approach for the equation $[-3, -2]X + [2, 5] = [1, 7]$ is $X = [-\frac{2}{3}, \frac{1}{3}]$ , which does not contain the solution of, say ,the equation $-2x + 5 = 1$. The method we are discussing in this work solves this problem and contains all the solutions given by Berti.

**THEOREM 4.5** Given $A, B, C \in \mathbf{IR}$, with $0 \notin A$, then

$$X = (B - C) \cdot \frac{1}{A}$$

is the optimal solution of

$$A \cdot X + B \equiv_c C.$$

□ Proof: We have

$$A \cdot X + B \equiv_c C \implies A \cdot X \equiv_{ax} A \cdot X + B - B \equiv_{c-b} C - B$$

so that, taking $ax = c - b$, we get

$$A \cdot X \equiv_{c-b} C - B$$

hence,

$$X \equiv_{\frac{c-b}{a}} \frac{C-B}{A}$$

for all $a \in A$, $b \in B$ and $c \in C$, which shows that

$$X = \frac{C-B}{A}$$

solves the equation as we claimed; moreover, it has to be optimal, since it solves the entire family of equations. □

In a similar manner, the solution to the equation

$$A \cdot X + B = C \cdot X + D$$

in **IR** discussed in [ BER 73 ] is not big enough to solve the whole set of equations $ax + b = cx + d$, when $a, b, c, d$ belong to intervals on the real line. The solution in this case is given by

**THEOREM 4.6** Given $A, B, C, D, \in \mathbf{IR}$ , with $0 \notin A - C$, then

$$X = \frac{(D - B)}{(A - C)}$$

is the optimal solution of

$$A \cdot X + B \equiv C \cdot X + D.$$

We will now proceed to discuss the solution to the quadratic equation

$$ax^2 + bx + c = 0$$

when $a, b, c$ belong to $A, B, C \in \mathbf{IR}$, respectively. We will consider in this work only the case of real roots. The introduction of a convenient complex arithmetics, as well as a matrix arithmetics, will be the subject of another paper.

**THEOREM 4.7** Given $A, B, C \in \mathbf{IR}$ such that

$$0 \notin A$$

and

$$B^2 - 4A \cdot C = [\, d_1, d_2 \,] \, \text{with} d_1 \geq 0,$$

then

$$X = \frac{-B \pm \sqrt{B^2 - 4AC}}{2 \cdot A}$$

is an external solution of

$$a \cdot X^2 + B \cdot X + C \equiv_o \mathcal{O}.$$

□ Proof: As an interval way to evaluate $f(x) = Ax^2 + bx + c$ and to determine values which make it zero when $a$, $b$ and $c$ are intervals, we can write

$$\mathcal{O} \equiv_o AX^2 + BX + C \equiv_{I_f} (2AX + B)^2 - B^2 + 4AC,$$

so that

$$(2AX + B)^2 \equiv_{b^2 - 4ac} B^2 - 4AC,$$

which leads to

$$2AX + B \equiv_{\sqrt{b^2 - 4ac}} \sqrt{b^2 - 4AC},$$

giving

$$X \equiv_{\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}} \frac{-B \pm \sqrt{b^2 - 4AC}}{2 \cdot A}$$

The fact that this value, written as it is, is not the optimal solution, is due to the fact that we use the subdistributivity property,

$$\forall\, A, B, C \in \mathbf{IR}\,,\; A \cdot (B + C) \subseteq A \cdot B + A \cdot C,$$

so that we are not working with the range but instead with a (larger) interval evaluation. The following example will illustrate this point further.

example: Determine the solution set of

$$ax^2 + bx + c = 0$$

for $a \in [1,2]$, $b \in [3,5]$, and $c \in [0,1]$. Solution: First, we have to compute $B^2$, which is given by

$$B^2 = \begin{cases} [b_1^2\,,\, b_2^2\,], & \text{when } b_1 \geqslant b_2 \\ [b_2^2\,,\, b_1^2\,], & \text{when } b_1 < b_2 \\ [0\,,\, \max\{b_1^2, b_2^2\}], & \text{when } 0 \in B \end{cases}$$

According to Theorem 4.7 , we have

$$X \equiv \frac{-[3,5] \pm \sqrt{[9,25] - [0,8]}}{[2,4]},$$

so that

$$X_1 \equiv [-2,1]$$

and

$$X_2 \equiv [-5,-1]$$

Now examining the sign variation of the coefficients of the interval polynomial, we get by Descartes' rule that none of the real polynomials has any positive zeros. One way we can try to correct this is to use the product rule for the roots.

$$X_1\,,\, X_2 \equiv \frac{[0,1]}{[1,2]} = [0,1]:$$

taking

$$X_2 \equiv [-5,-1],$$

we get

$$X_1 \equiv \frac{[0,1]}{[-5,-1]} = [-1,0].$$

Although we have not obtained the optimal solution, we have got a far better solution than the previous one. A discussion of techiques to obtain the optimal solution, as well as iterative interval methods, will be the subject of other papers. For the example at hand, we get the optimal solution if we rewrite

$$X_1 \equiv \frac{-2C}{B + \sqrt{B^2 - 4AC}},$$

which yields

$$X_1 = [-0.5\,,\, 0].$$

The optimal solution is

$$X_o = [-5, -1], [-0.5, 0].$$

**THEOREM 4.8 [non-existence of solutions]** Let $I_f(X)$ and $f(X)$ be the range and an interval evaluation of a function $f$. If for a given interval $Y \subseteq X$, one can show that $0 \notin f(Y)$, then there does not exist any root of the function $f$ in the interval $Y$.

# 5 FINAL REMARKS

When $0 \in I_f(X)$, there exists an $x \in X$ such that $f(x) = 0$. As the range set is difficult to compute, we could be tempted to replace it by one interval evaluation. The following example illustrates this impossibility.

Let $f(x) = x^2 - 5x + 6$, which has $2, 3$ as its roots. Let $X = [0;\ 1.5]$ and consider the interval evaluations $f_1(X) = X^2 - 5X + 6$ and $f_2(X) = X \cdot (X - 5) + 6$. We get

$$f_1([0;\ 1.5]) = [-1.5, 8.25]$$

$$f_2([0;\ 1.5]) = [-1.5, 6]$$

$$I_f([0, 1.5]) = [0.75, 6]$$

One has thus established the following rule:

*In the search for a solution $x^*$ of an equation $f(x^*) = 0$, it is not possible to obtain an interval $X$ containg the root by simply requiring that $f(X) \sqsubseteq 0$.*

One should point out, however, that the technique we are using gives an important gain, in context above. Theorems stating the non-existence of solutions are rare gems in the real of arithmetics, and they are hard to derive. This is not the case here. Since the range of a function is contained in any of its interval evaluations, the following theorem holds.

# References


[RAT 70] RATSCHEK, H. **Teilbarkeitkriterium der Intervallmathematik.** Journal fur Mathematik, **252**, 1970, 128-138.

[KUL 88] KULISCH, U; STETTER, H. J. (eds.) **Scientific Computation with Automatic Result Verification.** Computing Supplementum 6, Springer Verlag, Winlny, 1988.

[RUM 91] RUMP, S.M. **Estimation of the sensivity of linear and nonlinear Algebraic Problems.** Linear Algebra and its Applications, **153**, 1991, 1-34.

[SMA 90] SMALE, S. **Some Remarks on the foundations of numerical analysis.** SIAM Review, vol.32 no2, 211-2301.

[ALE 80] ALEFELD, G ; Herzberger, J. **An introduction to interval computation.** Academic Press, NY, 1983.

[BER 69] BERTI S. **The solution of an interval equation.** Mathematica, cluj 11, (34), 2, 189-194, 1969.

[BER 73] BERTI, S. **On the interval equation** $Ax + B = Cx + D$ . Revue d'analyse numerique et de la theorie de l'aproximation, tome 2, pp. 11-26, 1973.

[SCO 72] SCOTT, D.S. **Lattice Theory, date types and semantics.** In : Formal simantics of programming languages. Englewood Cliffs, Prentice-Hall, 1970, pp. 66-106