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ION ACOUSTIC SOLITONS IN DENSE MAGNETIZED PLASMAS WITH NONRELATIVISTIC AND ULTRARELATIVISTIC DEGENERATE ELECTRONS AND POSITRONS

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ABSTRACT

The propagation of electrostatic waves in a dense magnetized electron–positron–ion (EPI) plasma with nonrelativistic and ultrarelativistic degenerate electrons and positrons is investigated. The linear dispersion relation is obtained for slow and fast electrostatic waves in the EPI plasma. The limiting cases for ion acoustic wave (slow) and ion cyclotron wave (fast) are also discussed. Using the reductive perturbation method, two-dimensional propagation of ion acoustic solitons is found for both the nonrelativistic and ultrarelativistic degenerate electrons and positrons. The effects of positron concentration, magnetic field, and mass of ions on ion acoustic solitons are shown in numerical plots. The proper form of Fermi temperature for nonrelativistic and ultrarelativistic degenerate electrons and positrons is employed, which has not been used in earlier published work. The present investigation is useful for the understanding of linear and nonlinear electrostatic wave propagation in the dense magnetized EPI plasma of compact stars. For illustration purposes, we have applied our results to a pulsar magnetosphere.

Key words: plasmas – pulsars: general – stars: atmospheres – waves

Online-only material: color figures

1. INTRODUCTION

The study of electron–positron (EP) plasmas has been the main focus of many plasma physics researchers for many decades (Mofiz 1989; Tajima & Taniuti 1990; Berezhiani et al. 1995; Polpel et al. 1995; Pokhetelov et al. 2001). Such EP plasmas may exist in active galactic nuclei (AGNs), pulsars, neutron stars, black holes, accretion disks, and the electrosphere of strange stars and are also believed to be in the early universe (Goldreich & Julian 1969; Sturrock 1971; Aron & Scharlemann 1979; Berezhiani et al. 1995; Wheeler et al. 2000; Hasegawa et al. 2002; Aksenov et al. 2004; Cassé et al. 2004; Daligault & Murillo 2005; Harko & Cheng 2006; Titarchuk & Chardonnet 2006; Thoma 2009; Kashiyama et al. 2011). In the early universe after the big bang, temperatures in the megaelectronvolt range prevailed for a time up to one second. During this time, the main constituent of the universe was EP plasmas filled with intense radiation, neutrinos, antineutrinos, and a small amount of ions (Misner 1973; Gibbons et. al 1983). Low-luminosity active galactic nuclei (LLAGN) are supposed to lose their energy coming from interiors in the form of teraelectronvolt radiation, resulting in the possible creation of EP pairs (Brodatzki et al. 2011). When a star loses its thermal energy, it tends to collapse due to gravitational pull until the electrons present inside the core become degenerate and stop the further collapse of the star due to degeneracy pressure. The density and temperature in the interior of these compact stars become very high, and the constituent particles may be accelerated to much higher energy either by gravitational collapse or by an electromagnetic field. These high-energy particles collide with each other to produce electrons and positrons like in white dwarfs; these pairs are produced during collapse to a neutron star (Misner 1973; Manchester & Taylor 1977; Koester & Chanmugam 1990; Kashiyama et al. 2011). In the core of many white dwarfs the number density may exceed $10^{29}\,\mathrm{cm}^{-3}$ (Shapiro & Tukolsky 1983; Koester & Chanmugam 1990; Daligault & Murillo 2005; Kashiyama et al. 2011; Lallement et al. 2011). Electron–positron pairs are also created in the polar gaps of the pulsars (Uson & Melrose 1996). Near the surface of a pulsar, EP pairs are produced through a cascade process (Weatherall 1997; Contopoulos et al. 1999). In the presence of a strong magnetic field $B \sim 10^{12}$ G near pulsars, electrons are accelerated to very high energies, and later they emit γ-rays due to the phenomenon known as curvature radiation. These γ-rays have energies more than twice the rest mass energy of electrons ($2m_e c^2$). These γ-rays are converted into electrons and positrons through pair production in the presence of an intense magnetic field. The pairs of electrons and positrons produced in this process are further accelerated and produce more γ-rays. However, the plasma is believed to contain ions besides electrons and positrons (Beloborodov & Thompson 2007a, 2007b). Ions can be created either inside the core of the compact stars or come from outside through an accretion process (Thompson & Beloborodov 2005; Beloborodov & Thompson 2007a, 2007b; Istomin & Sobyanin 2007).

The peculiarity of EP plasmas is that the two species have equal masses but opposite charges. This symmetry of charge and mass is broken in the presence of heavy ions, thus allowing the propagation of both fast (electron and positron dynamic scale) and slow (ion dynamic scale) timescale phenomena in plasmas. The dynamics of electron–positron–ion (EPI) plasmas are significantly different from EP plasmas. The electrostatic and electromagnetic modes of EP plasmas are modified due to the presence of heavy ions. For example, the Alfvén wave frequency, shear flow, instability conditions, and ion temperature gradient modes are studied in the literature for EPI plasmas (Mirza & Azeem 2001; Pokhetelov et al. 2001). The dynamics of drift Alfvén waves in relativistic nonuniform EP and multicomponent plasmas has been investigated (Onishchenko et al. 1999, 2000, 2001).

The study of quantum degeneracy in plasmas becomes important when the thermal de Broglie wavelengths of plasma species become equal or larger than the interparticle distance.

If the concentration of particles becomes too high, the interparticle distance becomes so small that quantum effects start to appear. The quantum effects in dense plasmas appear due to Heisenberg's uncertainty principle, i.e., localizing the particles in a small region, say ∇x, gives a momentum of $\nabla p \sim \hbar/\nabla x$ to the particle, and for high density ∇x becomes small. Therefore, electrons (positrons) have higher Fermi energy as compared to their thermal energy. Degenerate plasma pressure appears due to the combined effects of Pauli's exclusion principle (fermions) and Heisenberg's uncertainty principle, and it depends on the number densities of the constituent particles but is independent of temperature (Vernet 2007). The ions remain nondegenerate due to their heavier mass, and electrons (positrons) become degenerate due to less inertia in comparison with ions in dense EPI plasmas. During the last decade, there has been much interest in the plasma physics community in studying the wave dynamics of the relativistic degenerate plasmas (Mamun et al. 2010; Khan 2012; Zobair et al. 2012; Nahar et al. 2013; Rahman et al. 2013a, 2013b; Mahmood et al. 2014). At the high density of compact stars, the electrons (positrons) remain nonrelativistic if their Fermi energy remains less than their rest mass energy. For number densities in the range 10^{30}–10^{34} cm^{-3}, the electrons have a Fermi energy that is comparable to or greater than their rest mass energy, and the electron Fermi speed turns out to be comparable to the speed of light in a vacuum. The equation of state for the degenerate electrons having nonrelativistic and ultrarelativistic energies in the compact stars has already been derived by Chandrasekhar (Chandrasekhar 1935, 1939). In the case of nonrelativistic degenerate electrons (positrons), the equation of state is such that $P_{Fe} \propto n_e^{5/3}$ and for ultrarelativistic degenerate electrons (positrons) $P_{Fe} \propto n_e^{4/3}$, where P_{Fe} is the degenerate electron (positron) pressure and n_e is the electron (positron) number density. It is significant to note that the degenerate pressure depends only on the number density. Recently, Zeba et al. (2012) studied the nonlinear ion acoustic waves in the dense unmagnetized EPI plasmas with ultrarelativistic degenerate electrons and positrons. However, we are investigating the propagation of electrostatic waves in dense magnetized EPI plasmas with both nonrelativistic and ultrarelativistic degenerate electrons (positrons). The presence of a magnetic field plays an important role in the characteristics of waves in compact stars. The proper forms of Fermi temperatures for nonrelativistic and ultrarelativistic energies of degenerate electrons (positrons) are used in this manuscript (details in Appendix A), which have not been used in the earlier published work. Because we are considering the very dense nonrelativistic and ultrarelativistic EPI plasmas, it is necessary to check the validity of the model to ignore the annihilation process to study the collective behavior (oscillations) of plasmas. The details are discussed in Appendix B.

This article is organized in the following manner. In Section 2, the basic dynamic equations are described for studying the electrostatic waves in magnetized dense EPI plasmas. The dispersion relation of both slow and fast electrostatic waves propagating in dense magnetized EPI plasmas is obtained with nonrelativistic and ultrarelativistic degenerate electron (positron) energy limits in Sections 3 and 4, respectively. The ZK (Zakharov–Kuznetsov) equation for the propagation of two-dimensional ion acoustic solitons is obtained using the reductive perturbation method in Section 5. The numerical plots and results using the dense plasma parameters of a compact star are discussed in Section 6. Finally, the conclusion is presented in Section 7.

2. SET OF NONLINEAR DYNAMIC EQUATIONS

We are considering a dense EPI plasma embedded in a constant magnetic field $\mathbf{B}_0$, which is assumed along the $\hat{x}$-axis, i.e., $\mathbf{B}_0 = B_0\hat{x}$. The ions are taken to be inertial and nondegenerate due to their heavy mass in comparison with electrons and positrons. The electrons and positrons are assumed to be inertialess and fully degenerate to behave as a quantum fluid. The basic set of nonlinear dynamic equations for ions are written as follows:

$$\frac{\partial n_i}{\partial t} + \nabla.(n_i\mathbf{u}_i) = 0, \tag{1}$$

$$\frac{\partial \mathbf{u}_i}{\partial t} + (\mathbf{u}_i.\nabla)\,\mathbf{u}_i = -\frac{e}{m_i}\nabla\phi + \frac{e}{m_i c}(\mathbf{u}_i \times \mathbf{B}_0).$$

The momentum equations for inertialess degenerate electrons and positrons are given by

$$\nabla\phi = \frac{1}{en_e}\nabla p_{Fe,} \tag{2}$$

$$\nabla\phi = -\frac{1}{en_p}\nabla p_{Fp}, \tag{3}$$

where p_{Fe} and p_{Fp} are Fermi pressures of degenerate electrons and positrons, respectively. The electrons and positrons also gyrate around the magnetic field, but due to their lesser mass, small gyro-radius, and very high gyro-frequency as compared with the massive ions, they are more tightly bound to magnetic field lines. Therefore, the one-dimensional motion of degenerate electron and positron fluids is a valid assumption.

The Poisson equation is written as

$$\nabla^2\phi = 4\pi e(n_e - n_p - n_i). \tag{4}$$

The electric field intensity is defined as $\mathbf{E} = -\nabla\phi$ (where ϕ is the electrostatic potential). The velocity of the ion fluid is $\mathbf{u}_i = (u_{ix}, u_{iy}, u_{iz})$; n_i, n_e, and n_p are the densities of ions, electrons, and positrons, respectively; m_i, e, and c are the ions mass, charge of an electron, and speed of light, respectively. The plasma rotational effects are ignored in the present model because the rotation of the pulsars is very low (on the order of millisecond^{-1}) in comparison with the ion gyro-frequency, which is on the order of 10^{13}–10^{16} s^{-1} (Gregory et al. 1994).

Consider the obliquely propagating wave in the X-Y plane, i.e., $\nabla = (\partial/\partial x, \partial/\partial y, 0)$ in magnetized dense EPI plasmas. The above set of dynamic equations can be written in the component form as follows:

$$\frac{\partial n_i}{\partial t} + \frac{\partial(n_i u_{ix})}{\partial x} + \frac{\partial(n_i u_{iy})}{\partial y} = 0, \tag{5}$$

$$\frac{\partial u_{ix}}{\partial t} + \left(u_{ix}\frac{\partial}{\partial x} + u_{iy}\frac{\partial}{\partial y}\right)u_{ix} = -\frac{e}{m_i}\frac{\partial\phi}{\partial x}, \tag{6}$$

$$\frac{\partial u_{iy}}{\partial t} + \left(u_{ix}\frac{\partial}{\partial x} + u_{iy}\frac{\partial}{\partial y}\right)u_{iy} = -\frac{e}{m_i}\frac{\partial\phi}{\partial y} + \Omega_i u_{iz}, \tag{7}$$

$$\frac{\partial u_{iz}}{\partial t} + \left(u_{ix}\frac{\partial}{\partial x} + u_{iy}\frac{\partial}{\partial y}\right)u_{iz} = -\Omega_i u_{iy}, \tag{8}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi = 4\pi e\,(n_e - n_p - n_i). \tag{9}$$

On the ion dynamics scale the electrons and positrons are assumed to be inertialess and follow the magnetic field lines. Therefore, the $\hat{x}$ component of the normalized momentum equations for electrons and positrons is given by

$$\frac{\partial}{\partial x}\phi = \frac{1}{n_e e}\frac{\partial}{\partial x}P_{Fe} \tag{10}$$

$$\frac{\partial}{\partial x}\phi = -\frac{1}{n_p e}\frac{\partial}{\partial x}P_{Fp}, \tag{11}$$

where $\Omega_i = eB_0/m_i c$ is the ion gyro-frequency.

The expression for the Fermi pressure P_{Fe} of electrons in dense plasmas is derived by Chandrasekhar (1939), which is given as $P_{Fe} = (\pi m_e^4 c^5/3h^3) \times [(2\alpha_F^3 - 3\alpha_F)(\alpha_F^2 + 1)^{1/2} + 3\sinh^{-1}\alpha_F]$, where $\alpha_F = p_F/m_e c$ is the normalized relativistic factor and is related to plasma number density by $\alpha_F = (n_e/n_c)^{1/3}$ with $n_c \simeq 5.9 \times 10^{29}\ \mathrm{cm}^3$ being a normalizing density defined with quantities such as c (speed of light), m_e (mass of electron), and h (Planck's constant). The two limiting cases of nonrelativistic and ultrarelativistic degenerate electron pressures can be retrieved from the above expression by imposing the conditions $\alpha_F \to 0$ and $\alpha_F \to \infty$, respectively, which gives $P_{Fe} = (3/\pi)^{2/3}(h^2/20\,m_e)\,n_e^{5/3}$ (nonrelativistic degenerate case) and $P_{Fe} = (3/\pi)^{1/3}(hc/8)n_e^{4/3}$ (ultrarelativistic degenerate case). Because a positron has the same mass as an electron but only has the opposite charge, the same expressions of Fermi pressures for degenerate positrons in the nonrelativistic and ultrarelativistic limits remain valid, and only electron density n_e changes to positron density n_p.

3. DISPERSION RELATION FOR NONRELATIVISTIC DEGENERATE EPI PLASMA CASE

In order to find the dispersion relation for the nonrelativistic degenerate case of electron and positron quantum fluids, we assume that the perturbations are of the form $e^{i(\mathbf{k}.\mathbf{x}-\omega t)}$ (where $k^2 = k_x^2 + k_y^2$). The x-components of the momentum equation for an inertialess degenerate electron and positron quantum fluid in the nonrelativistic limit are given as

$$\frac{\partial}{\partial x}\phi = \frac{h^2}{12\,em_e}\left(\frac{3}{\pi}\right)^{2/3} n_e^{-1/3}\frac{\partial}{\partial x}n_e, \tag{12}$$

$$\frac{\partial}{\partial x}\phi = -\frac{h^2}{12\,em_p}\left(\frac{3}{\pi}\right)^{2/3} n_p^{-1/3}\frac{\partial}{\partial x}n_p. \tag{13}$$

Here $m_e = m_p$, and the Fermi pressure for a nonrelativistic degenerate fluid of the j^{th} species is $P_{Fj} = (3/\pi)^{2/3}(h^2/20\,m_j)\,n_j^{5/3}$ ($j = e, p$).

Using the set of dynamic Equations (5)–(9) and (12) and (13) in the linearized form, we find the dispersion relation as follows:

$$\omega^4 - \omega^2\left(\Omega_i^2 + \frac{C_s^2 k^2}{\lambda_F^2 k^2 + C_1}\right) + \frac{C_s^2 k_x^2 \Omega_i^2}{\lambda_F^2 k^2 + C_1} = 0. \tag{14}$$

The four roots of the quartic equation are given as

$$\omega_\pm^2 = \frac{1}{2}\left(\Omega_i^2 + \frac{C_s^2 k^2}{\lambda_F^2 k^2 + C_1}\right) \pm \frac{1}{2}\left[\left(\Omega_i^2 + \frac{C_s^2 k^2}{\lambda_F^2 k^2 + C_1}\right)^2 - \frac{4C_s^2 k_x^2}{\lambda_F^2 k^2 + C_1}\right]^{1/2}, \tag{15}$$

where $C_s = \sqrt{2k_B T_{Fe}/m_i}$ is the ion acoustic speed, $\lambda_F = \sqrt{2k_B T_{Fe}/4\pi e^2 n_{i0}}$ is the Fermi length of the system, $p = n_{p0}/n_{e0}$ is the ratio of positron to electron equilibrium density, and $C_1 = 3(1 + p^{1/3})/2(1 - p)$. The Fermi temperature T_{Fe} is defined for a nonrelativistic degenerate electron gas by the following relation: $T_{Fe} = E_{Fe}/k_B = p_F^2/2m_e k_B = (\hbar^2/2m_e k_B)(3\pi^2 n_{e0})^{2/3}$ (Chandrasekhar 1939). The effect of positron density appears in the constant C_1. The positive sign of the frequency (ω_+) corresponds to the fast electrostatic wave, whereas the negative sign (ω_-) corresponds to the slow electrostatic wave in dense magnetized EPI plasmas. The limiting cases for obliquely propagating waves in the parallel and perpendicular directions to the magnetic field $\mathbf{B}_0$ are described below.

3.1. *Wave Propagation Parallel to the Magnetic Field*

In order to study wave propagation in a direction parallel to the magnetic field $\mathbf{B}_0$, we put $k_y = 0$ in Equation (15) to obtain the following two roots:

$$\omega_+^2 = \Omega_i^2, \quad \omega_-^2 = \frac{C_s^2 k_x^2}{\lambda_F^2 k_x^2 + C_1}. \tag{16}$$

The plus root is just a frequency, so it can be neglected; the second root is the dispersion relation for an ion acoustic wave propagating along the magnetic field. The effect of positron density appears in the constant C_1. Applying the condition $\lambda_F^2 k_x^2 \ll C_1$ (long wavelength limit), we get for the ion acoustic wave $\omega_-/k_x = C_s/\sqrt{C_1}$, which is the phase speed of the IAW in a dense EPI plasma. The phase speed of the IAW decreases with the increase in positron density in a dense EPI plasma because the value of C_1 increases. Similarly for the short wavelength case, we have $\lambda_F^2 k_x^2 \gg C_1$ and $\omega_-/k_x = \omega_{pi}$, which is the ion plasma frequency.

3.2. *Wave Propagation Perpendicular to the Magnetic Field*

In order to find the electrostatic wave propagating in a direction perpendicular to the magnetic field $\mathbf{B}_0$, we put $k_x = 0$ in Equation (15) and get

$$\omega_+^2 = \Omega_i^2 + \frac{C_s^2 k_y^2}{\lambda_F^2 k_y^2 + C_1} \quad \omega_-^2 = 0. \tag{17}$$

The plus root gives the dispersion relation for an ion cyclotron wave (ICW), whereas the slow wave vanishes. The effect of positron density appears in the constant C_1. Again applying the condition $\lambda_F^2 k_x^2 \ll C_1$ of a long wavelength limit, we obtain $\omega_+/k_y = \sqrt{\Omega_i^2/k_y^2 + C_s^2/C_1}$, which is the phase speed of an ICW in a dense EPI dense plasma. Similarly for a short wavelength limit, we have $\lambda_F^2 k_x^2 \gg C_1$ and $\omega_+ = \sqrt{\Omega_i^2 + \omega_{pi}^2}$, which is the upper hybrid ion frequency in a dense magnetized EPI plasma.

4. DISPERSION RELATION FOR ULTRARELATIVISTIC DEGENERATE PLASMA

In order to find the dispersion relation for ultrarelativistic degenerate electrons and positrons, we can write the momentum equations for electron and positron quantum fluids, respectively:

$$\frac{\partial}{\partial x}\phi = \frac{hc}{6e}\left(\frac{3}{\pi}\right)^{1/3} n_e^{-2/3}\frac{\partial}{\partial x}n_e \tag{18}$$

$$\frac{\partial}{\partial x}\phi = -\frac{hc}{6e}\left(\frac{3}{\pi}\right)^{1/3} n_p^{-2/3}\frac{\partial}{\partial x}n_p, \tag{19}$$

where Fermi pressure for an ultrarelativistic degenerate fluid of the *j*th species is $P_{Fj} = (3/\pi)^{1/3}(hc/8)n_j^{4/3}$ $(j = e, p)$.

Using the set of dynamic Equations (5)–(9), (18), and (19) in the linear limit, we find the dispersion relation for the sinusoidal electrostatic perturbations in a dense magnetized EPI plasma for the ultrarelativistic case as follows:

$$\omega^4 - \omega^2\left(\Omega_i^2 + \frac{C_s^2k^2}{\lambda_F^2k^2 + C_2}\right) + \frac{C_s^2k_x^2\Omega_i^2}{\lambda_F^2k^2 + C_2} = 0. \tag{20}$$

The four roots of the above quartic equation are

$$\omega_\pm^2 = \frac{1}{2}\left(\Omega_i^2 + \frac{C_s^2k^2}{\lambda_F^2k^2 + C_2}\right) \pm \frac{1}{2}\left[\left(\Omega_i^2 + \frac{C_s^2k^2}{\lambda_F^2k^2 + C_2}\right)^2 - \frac{4C_s^2k_x^2}{\lambda_F^2k^2 + C_2}\right]^{1/2}, \tag{21}$$

where $C_2 = (1/hc)(\pi/3)^{1/3}(6C_s^2n_{e0}^{-1/3}m_i)(1 + p^{2/3}/1 - p)$. The Fermi temperature (T_{Fe}) for an ultrarelativistic degenerate electron gas is proportional to $n_{e0}^{1/3}$, and it is described by the following relation: $T_{Fe} = E_{Fe}/k_B = c/k_B(3h^3n_{e0}/8\pi)^{1/3}$ (Vernet 2007) (for its derivation please see Appendix A as well). The effect of positron density appears in the constant C_2. The positive sign of the frequency (ω_+) corresponds to the fast electrostatic wave, whereas the negative sign (ω_-) corresponds to the slow electrostatic wave in dense magnetized EPI plasmas for ultrarelativistic degenerate electrons and positrons. The limiting cases for obliquely propagating electrostatic waves in the parallel and perpendicular directions to the magnetic field $\mathbf{B}_0$ are obtained as follows.

4.1. *Wave Propagation Parallel to the Magnetic Field*

In order to find the electrostatic waves propagating in the parallel direction to B_0, we put $k_y = 0$ in Equation (21) to get

$$\omega_+^2 = \Omega_i^2 \quad \omega_-^2 = \frac{C_s^2k_x^2}{\lambda_F^2k_x^2 + C_2}. \tag{22}$$

The plus root of ω gives the ion gyro-frequency, which is nonpropagating, so it can be neglected. The minus root of ω gives the dispersion relation of an ion acoustic wave (IAW). The long wavelength condition $\lambda_F^2k_x^2 \ll C_2$ gives the phase speed of an IAW as $\omega_-/k_x = C_s/\sqrt{C_2}$. The effect of positron density on an IAW appears in the constant C_2. The phase speed of the IAW decreases with the increase of positron concentration in dense EPI plasmas. This happens because C_2 increases by increasing the value of p, and as a result the phase velocity of the IAW decreases. For the short wavelength condition $\lambda_F^2k_x^2 \gg C_2$, we have $\omega_- = \omega_{pi}$, which is of course the ion plasma frequency.

4.2. *Wave Propagation Perpendicular to the Magnetic Field*

To study the propagation of electrostatic waves in the perpendicular direction to the magnetic field in a dense EPI magnetized plasma, we put $k_x = 0$ in Equation (21), which gives

$$\omega_+^2 = \Omega_i^2 + \frac{C_s^2k_y^2}{\lambda_F^2k_y^2 + C_2} \quad \omega_-^2 = 0. \tag{23}$$

The positive root gives the dispersion relation for an ion cyclotron wave (ICW), whereas the negative root vanishes. Now applying the long wavelength condition $\lambda_F^2k_x^2 \ll C_2$, we get $\omega_+/k_y = \sqrt{\Omega_i^2/k_y^2 + C_s^2/C_2}$, which is the phase speed of an ICW in ultrarelativistic EPI dense magnetized plasmas. The positron density effect appears in C_2, which increases by increasing the positron density p in EPI plasmas, and as a result the phase speed of an ICW decreases. In the limit of short wavelength, we have $\lambda_F^2k_x^2 \gg C_2$ and $\omega_+ = \sqrt{\Omega_i^2 + \omega_{pi}^2}$, which is the upper ion hybrid frequency in a dense magnetized EPI plasma.

5. DERIVATION OF ZAKHAROV–KUZNESTOV (ZK) EQUATION

In this section we derive the Zakharov–Kuznetsov (ZK) equation using a reductive perturbation method to study the two-dimensional propagation of a nonlinear electrostatic wave in the *X*–*Y*-plane. The ZK equation will be derived in a dense magnetized EPI plasma with two extreme conditions of nonrelativistic and ultrarelativistic degenerate electrons and positrons. The set of Equations (5)–(9) is written in the dimensionless form as

$$\frac{\partial\tilde{n}_i}{\partial\tilde{t}} + \frac{\partial(\tilde{n}_i\tilde{u}_{ix})}{\partial\tilde{x}} + \frac{\partial(\tilde{n}_i\tilde{u}_{iy})}{\partial\tilde{y}} = 0, \tag{24}$$

$$\frac{\partial\tilde{u}_{ix}}{\partial\tilde{t}} + \left(\tilde{u}_{ix}\frac{\partial}{\partial\tilde{x}} + \tilde{u}_{iy}\frac{\partial}{\partial\tilde{y}}\right)\tilde{u}_{ix} = -\frac{\partial\tilde{\phi}}{\partial x}, \tag{25}$$

$$\frac{\partial\tilde{u}_{iy}}{\partial\tilde{t}} + \left(\tilde{u}_{ix}\frac{\partial}{\partial\tilde{x}} + \tilde{u}_{iy}\frac{\partial}{\partial\tilde{y}}\right)\tilde{u}_{iy} = -\frac{\partial\tilde{\phi}}{\partial y} + \Omega\tilde{u}_{iz}, \tag{26}$$

$$\frac{\partial\tilde{u}_{iz}}{\partial\tilde{t}} + \left(\tilde{u}_{ix}\frac{\partial}{\partial\tilde{x}} + \tilde{u}_{iy}\frac{\partial}{\partial\tilde{y}}\right)\tilde{u}_{iz} = -\Omega\tilde{u}_{iy}, \tag{27}$$

$$\left(\frac{\partial^2}{\partial\tilde{x}^2} + \frac{\partial^2}{\partial\tilde{y}^2}\right)\tilde{\phi} = \left(\frac{\tilde{n}_e}{1 - p} - \frac{p\tilde{n}_p}{1 - p} - \tilde{n}_i\right). \tag{28}$$

The variables appearing in the above equations having a tilde are all dimensionless, where time and space variables are normalized by inverse ion plasma frequency ω_{pi}^{-1} and system Fermi length λ_F, respectively (defined in the above section). The normalization of ion, electron, and positron density (n_i, n_e, n_p) is done by their equilibrium densities (n_{i0}, n_{e0}, n_{p0}), respectively. The perturbed velocity of the ion fluid and electrostatic potential are normalized by ion acoustic speed C_s and k_BT_{Fe}/e, respectively. The dimensionless parameter Ω is defined as Ω_i/ω_{pi}.

5.1. *Nonrelativistic Degenerate Electrons and Positrons Case*

In order to derive the ZK equation for a nonrelativistic fully degenerate EPI plasma, we use the following dimensionless *x*-components of momentum equations of an inertialess degenerate electron and positron quantum fluid:

$$\frac{\partial}{\partial\tilde{x}}\tilde{\phi} = \beta_{eNR}\tilde{n}_e^{-1/3}\frac{\partial}{\partial\tilde{x}}\tilde{n}_e \tag{29}$$

$$\frac{\partial}{\partial\tilde{x}}\tilde{\phi} = -p^{2/3}\beta_{eNR}\tilde{n}_p^{-1/3}\frac{\partial}{\partial\tilde{x}}\tilde{n}_p \tag{30}$$

where $\beta_{eNR} = (h^2/12k_Bm_eT_{Fe})(3/\pi)^{2/3}n_{e0}^{2/3}$.

We omit the tilde from all of the dimensionless variables in the above set of equations for simplicity in the calculations.

In order to find the nonlinear perturbation solution of the above Equations (24)–(30) and to obtain the ZK equation for a two-dimensional electrostatic soliton in dense EPI plasmas, we use the following stretching for independent variables (space and time) (Kourakis et al. 2009):

$$X = \varepsilon^{1/2}(x - \lambda t), \quad Y = \varepsilon^{1/2} y, \quad \tau = \varepsilon^{3/2} t, \tag{31}$$

where all of the perturbed quantities are expanded in a power series of ε given below

$$\begin{aligned}
u_{ix} &= \varepsilon u_{ix}^{(1)} + \varepsilon^2 u_{ix}^{(2)} + \varepsilon^3 u_{ix}^{(3)} + \cdots,\\
u_{iy} &= \varepsilon^{3/2} u_{iy}^{(1)} + \varepsilon^2 u_{iy}^{(2)} + \varepsilon^{5/2} u_{iy}^{(3)} + \cdots,\\
u_{iz} &= \varepsilon^{3/2} u_{iz}^{(1)} + \varepsilon^2 u_{iz}^{(2)} + \varepsilon^{5/2} u_{iz}^{(3)} + \cdots,\\
n_i &= 1 + \varepsilon n_i^{(1)} + \varepsilon^2 n_i^{(2)} + \varepsilon^3 n_i^{(3)} + \cdots,\\
n_e &= 1 + \varepsilon n_e^{(1)} + \varepsilon^2 n_e^{(2)} + \varepsilon^3 n_e^{(3)} + \cdots,\\
n_p &= 1 + \varepsilon n_p^{(1)} + \varepsilon^2 n_p^{(2)} + \varepsilon^3 n_p^{(3)} + \cdots
\end{aligned} \tag{32}$$

Using Equations (31) and (32) in the set of dynamic Equations (24)–(30) and collecting the coefficients of different powers of ε, we get the following equations:

$$-\lambda \partial_X u_{ix}^{(1)} + \partial_X \phi^{(1)} = 0, \tag{33}$$

$$-\lambda \partial_X n_i^{(1)} + \partial_X u_{ix}^{(1)} = 0, \tag{34}$$

$$u_{iz}^{(1)} = \frac{1}{\Omega} \partial_Y \phi^{(1)}. \tag{35}$$

After integrating the above equations and applying the boundary conditions, i.e., $X \to \infty$, $u_{ix}^{(1)}$, and $n_i^{(1)} \to 0$, we have

$$u_{ix}^{(1)} = \frac{\phi^{(1)}}{\lambda} \tag{36}$$

$$n_i^{(1)} = \frac{\phi^{(1)}}{\lambda^2}. \tag{37}$$

The lowest order terms of the electron and positron momentum Equations (29) and (30) give

$$n_e^{(1)} = \frac{\phi^{(1)}}{\beta_{eNR}} \tag{38}$$

$$n_p^{(1)} = -\frac{\phi^{(1)}}{p^{2/3}\beta_{eNR}}. \tag{39}$$

The lowest order term of the Poisson equation is given as follows:

$$\frac{n_e^{(1)}}{1-p} - \frac{p n_p^{(1)}}{1-p} = n_i^{(1)}. \tag{40}$$

Using Equations (37), (38), and (39) in the above equation, we get an expression of phase speed of the wave λ as follows:

$$\lambda = \pm \left(\frac{\beta_{eNR}(1-p)}{1+p^{1/3}} \right)^{1/2}. \tag{41}$$

Here it is important to note that λ depends on positron density through $p = n_{p0}/n_{e0}$. The phase velocity of the wave decreases by increasing the positron density in dense EPI plasmas. The dimensional form for phase speed of the wave is given by

$$\lambda_{NR} = \pm \left[\frac{(3/\pi)^{2/3} h^2 n_{e0}^{2/3}(1-p)}{6 m_i m_e (1+p^{1/3})} \right]^{1/2}, \tag{42}$$

which is exactly equal to the phase speed of an IAW in dense magnetized EPI plasmas as derived in Section 3.1, i.e., $\omega_-/k_x = C_s/\sqrt{C_1}$ is obtained in the long wavelength limit.

Now the collection of next higher order terms of ε from the ion continuity and momentum equations relates the first-order and second-order perturbed quantities as follows:

$$\lambda \partial_X n_i^{(2)} - \partial_X u_{ix}^{(2)} - \partial_Y u_{iy}^{(2)} = \partial_\tau n_i^{(1)} + \partial_X \left(u_{ix}^{(1)} n_i^{(1)} \right) = f_1, \tag{43}$$

$$\lambda \partial_X u_{ix}^{(2)} - \partial_Y \phi^{(2)} = \partial_\tau u_{ix}^{(1)} + u_{ix}^{(1)} \partial_X u_{ix}^{(1)} = f_2, \tag{44}$$

$$u_{iy}^{(2)} = \lambda \partial_X u_{iz}^{(1)} = f_3, \tag{45}$$

$$u_{iz}^{(2)} = -\lambda \partial_X u_{ix}^{(1)} = f_4, \tag{46}$$

$$\partial_X \phi^{(2)} - \beta_{eNR} \partial_X n_e^{(2)} = -\frac{1}{3} \beta_{eNR} n_e^{(1)} \partial_X n_e^{(1)} = f_5, \tag{47}$$

$$\partial_X \phi^{(2)} + p^{2/3} \beta_{eNR} \partial_X n_p^{(2)} = \frac{1}{3} p^{2/3} \beta_{eNR} n_p^{(1)} \partial_X n_p^{(1)} = f_6, \tag{48}$$

$$\left(\frac{n_e^{(2)}}{1-p} - \frac{p n_p^{(2)}}{1-p} - n_i^{(2)} \right) = \partial_X^2 \phi^{(1)} + \partial_Y^2 \phi^{(1)} = f_7. \tag{49}$$

Using Equations (43)–(49) and (41), we obtain the following relation:

$$\lambda f_1 + f_2 + \lambda \partial_Y f_3 + \frac{f_5}{1+p^{1/3}} + \frac{p^{1/3} f_6}{1+p^{1/3}} + \lambda^2 \partial_X f_7 = 0. \tag{50}$$

Using the relations for the first-order quantities given in Equations (35)–(39) in terms of one variable $\phi^{(1)}$, we get the following ZK equation in dimensional form:

$$\partial_\tau \phi^{(1)} + A_{NR} \phi^{(1)} \partial_X \phi^{(1)} + C_{NR} \partial_X \partial_Y^2 \phi^{(1)} + B_{NR} \partial_X^3 \phi^{(1)} = 0, \tag{51}$$

where the nonlinear coefficient A_{NR}, the dispersive coefficients B_{NR} (along the magnetic field) and C_{NR} (perpendicular to the magnetic field) in the dimensional form are given by

$$A_{NR} = \frac{e}{\lambda_{NR} m_i} \left[\frac{3}{2} + \frac{(1-p)(p^{-1/3}-1)}{6(1+p^{1/3})^2} \right], \tag{52}$$

$$B_{NR} = \frac{\lambda_{NR}^3}{2} \frac{\lambda_F^2}{C_s^2}, \tag{53}$$

$$C_{NR} = \frac{\lambda_{NR}^3}{2\omega_{pi}^2} \left(1 + \frac{\omega_{pi}^2}{\Omega_i^2} \right). \tag{54}$$

The independent variables of space (X, Y) and time (τ) have the dimensions in ZK Equation (51). The dependent variable $\phi^{(1)}$ has the dimension of the electrostatic potential. The anisotropy of the problem in the presence of magnetic field under consideration is manifested by the fact that only the coefficient C_{NR} is dependent on the magnetic field through ion gyro-frequency Ω_i, whereas the coefficients A_{NR} and B_{NR} in the ZK equation are independent of an external magnetic field.

The travelling wave solution of Equation (51) in the form of obliquely propagating ion acoustic solitons in a dense magnetized EPI plasma is obtained as follows:

$$\phi^{(1)} = \frac{3U}{A_{NR} l_x} \sec h^2 \left[\frac{l_x X + l_y Y - UT}{W} \right], \tag{55}$$

where the transformed coordinate ξ in the comoving frame is defined as $\xi = l_x X + l_y Y - UT$, where U is the speed of the nonlinear structure and l_x and l_y are the direction cosines that satisfy the condition $l_x^2 + l_y^2 = 1$. The following boundary conditions are used for the localized structure: $\phi^{(1)} \to 0$, $\nabla\phi^{(1)} \to 0$, and $\nabla^2\phi^{(1)} \to 0$ as $\xi \to \infty$. The width of the soliton structure is $W = \sqrt{4l_x(B_{NR}l_x^2 + C_{NR}l_y^2)/U}$, and the amplitude is given by $(3U/A_{NR}l_x)$. Therefore, the presence of positrons in a dense magnetized EI plasma modifies the IA soliton's amplitude as well as its width.

5.2. Ultrarelativistic Degenerate Electrons and Positrons Case

In order to derive the ZK equation for ultrarelativistic fully degenerate EPI plasmas, we use the following equations for the x components of momentum equations for degenerate electrons and positrons:

$$\frac{\partial}{\partial \tilde{x}}\tilde{\phi} = \beta_{eUR}\tilde{n}_e^{-2/3}\frac{\partial}{\partial \tilde{x}}\tilde{n}_e \tag{56}$$

$$\frac{\partial}{\partial \tilde{x}}\tilde{\phi} = -p^{2/3}\beta_{eUR}\tilde{n}_p^{-1/3}\frac{\partial}{\partial \tilde{x}}\tilde{n}_p, \tag{57}$$

where $\beta_{eUR} = hc\,(3/\pi)^{1/3}\, n_{e0}^{1/3}/6k_B T_{Fe}$. We will omit the tilde from all of the dimensionless quantities for simplicity in the following calculations.

The various perturbed quantities of Equations (24)–(28) and (56) and (57) are expanded in different orders of ε with the use of the reductive perturbation method. After collecting the first-order terms of ε of Equations (56) and (57), we find

$$n_e^{(1)} = \frac{\phi^{(1)}}{\beta_{eUR}}, \tag{58}$$

$$n_p^{(1)} = -\frac{\phi^{(1)}}{p^{1/3}\beta_{eUR}}. \tag{59}$$

Similarly, the next higher order terms of ε give

$$\partial_X\phi^{(2)} - \beta_{eUR}\partial_X n_e^{(2)} = -\frac{2}{3}\beta_{eUR}n_e^{(1)}\partial_X n_e^{(1)}, \tag{60}$$

$$\partial_X\phi^{(2)} + p^{1/3}\beta_{eUR}\partial_X n_p^{(2)} = \frac{2}{3}p^{1/3}\beta_{eUR}n_p^{(1)}\partial_X n_p^{(1)}. \tag{61}$$

The ion continuity and ion momentum equations along with the Poisson equation are the same as we previously used in the case of the nonrelativistic degenerate EPI plasma in the previous section. Here, we use the expressions for ion continuity, momentum, and the Poisson Equations (24)–(28) along with Equations (58)–(59) to get the phase speed of the wave as follows:

$$\lambda = \pm\left(\frac{\beta_{eUR}\,(1-p)}{1+p^{2/3}}\right)^{1/2}.$$

The dimensional form of the wave phase speed is written as

$$\lambda_{UR} = \pm\left[\frac{(3/\pi)^{1/3}}{6m_i}\frac{(1-p)}{(1+p^{2/3})}n_{e0}^{1/3}hc\right]^{1/2},$$

which is same phase speed as for an IAW for a dense magnetized plasma with ultrarelativistic degenerate electrons and positrons as obtained in the limiting case in Section 4.1, i.e., $\omega_-/k_x = C_s/\sqrt{C_2}$.

Using the higher order set of dynamic Equations (43)–(46), (49), (60), and (61), we obtain the ZK equation for a dense magnetized EPI plasma for the case of ultrarelativistic degenerate electrons and positrons in terms of $\phi^{(1)}$ as follows:

$$\partial_\tau\phi^{(1)} + A_{UR}\phi^{(1)}\partial_X\phi^{(1)} + C_{UR}\partial_X\partial_Y^2\phi^{(1)} + B_{UR}\partial_X^3\phi^{(1)} = 0, \tag{62}$$

where the nonlinear coefficient A_{UR} and the dispersive coefficients in the parallel and perpendicular directions to the magnetic field, i.e., B_{UR} and C_{UR}, respectively, in the dimensional form are given by

$$A_{UR} = \frac{e}{\lambda_{UR}m_i}\left[\frac{3}{2} + \frac{(1-p)\left(p^{1/3}-1\right)}{3(1+p^{2/3})^2}\right] \tag{63}$$

$$B_{UR} = \frac{\lambda_{UR}^3}{2}\frac{\lambda_F^2}{C_s^2} \tag{64}$$

$$C_{UR} = \frac{\lambda_{UR}^3}{2\omega_{pi}^2}\left(1 + \frac{\omega_{pi}^2}{\Omega_i^2}\right). \tag{65}$$

The independent variables of space (X, Y) and time (τ) have dimensions in ZK Equation (51), and the dependent variable $\phi^{(1)}$ has the dimension of the electrostatic potential. The anisotropy of the problem under consideration is manifested by the fact that only the coefficient C_{UR} is dependent on the magnetic field through ion gyro-frequency Ω_i, whereas coefficients A_{UR} and B_{UR} in the ZK equation are independent of the external magnetic field.

The traveling wave solution $\phi^{(1)}$ of Equation (62) for an obliquely propagating ion acoustic soliton is obtained as follows (Kourakis et al. 2009):

$$\phi^{(1)}(X, Y, \tau) = \frac{3U}{A_{UR}l_x}\sec h^2\left[\frac{l_x X + l_y Y - UT}{W}\right], \tag{66}$$

where the transformed coordinate ξ in the comoving frame is defined as $\xi = l_x X + l_y Y - UT$, where U is the speed of the nonlinear structure and l_x and l_y are the direction cosines that satisfy the condition $l_x^2 + l_y^2 = 1$. The following boundary conditions are used for the localized structure: $\phi^{(1)} \to 0$, $\nabla\phi^{(1)} \to 0$, and $\nabla^2\phi^{(1)} \to 0$ as $\xi \to \infty$. The width of the soliton structure is $W = \sqrt{4l_x(B_{UR}l_x^2 + C_{UR}l_y^2)/U}$, and the amplitude is given by $(3U/A_{NR}l_x)$. Therefore, the presence of positrons in a dense EI magnetized plasma modifies the IAW soliton's amplitude as well as its width.

6. NUMERICAL PLOTS AND RESULTS

In this section, we obtain the numerical plots for limiting cases of nonrelativistic and ultrarelativistic degenerate dense EPI plasmas that can exist in a pulsar magnetosphere. The density regimes for the plasma to be nonrelativistic and degenerate (electrons and positrons) is roughly $10^{26}\,\mathrm{cm}^{-3} < n_{e0} < 10^{29}\,\mathrm{cm}^{-3}$, whereas at the densities well above $10^{30}\,\mathrm{cm}^{-3}$ the electrons as well as positrons become ultrarelativistic (Rasheed et al. 2001; Sabry et al. 2012). We now consider the application of the present study to the pulsar atmosphere, and it is necessary to mention here that the density parameters of the pulsar atmosphere used for the plots are not certain (Aron & Scharlemann 1979; Shapiro & Tukolsky 1983; Tajima & Taniuti 1990; Gurevich et al. 1993; Thoma 2009; Laing & Diver 2013). However, it is likely to appear in these regions. The characteristics of the linear and nonlinear propagation of electrostatic waves are of importance for developing a deeper insight into the underlying pulsar radiation physics.

6.1. Results for Nonrelativistic Degenerate Electrons and Positrons

In order to study the nonrelativistic degenerate plasma case, we have considered the magnetic field 10^8 G, density of electrons $n_{e0} = 10^{28}\,\mathrm{cm}^{-3}$, and thermal temperature $T = 2\times 10^7$ K of a dense star. In order to check the degeneracy condition for electrons, the value of T_{Fe} should be greater than the system thermal temperature T. At the given value of electron density $n_{e0} = 10^{28}\,\mathrm{cm}^{-3}$ and a typical value of positron concentration $p = 0.6$, the Fermi temperatures of the electrons, positrons, and ions turn out to be $T_{Fe} = 1.95\times 10^8$ K, $T_{Fp} = 1.38\times 10^8$ K, and $T_{Fi} = 1.109\times 10^5$ K, respectively. The Fermi temperature of ions remains less than the system thermal temperature; therefore, the ion fluid remains nondegenerate, whereas electron–positron fluids become degenerate. The parameter corresponding to degeneracy $\chi_j = T_{Fj}/T$ (where $j = e, p, i$) comes out to be $\chi_e \gg 1$ and $\chi_p \gg 1$, whereas for ions $\chi_i \ll 1$. The Coulomb coupling parameter for ions is defined as $\Gamma_i = e^2/k_B T d_i$ where $d_i = (3/4\pi n_{i0})^{1/3}$ is the mean interionic distance. In the case of hydrogen ion H^+ fluid, using the same dense plasma parameters, we have the numerical values of the Coulomb coupling parameter for ions, electrons, and positrons, $\Gamma_i = 0.290$, $\Gamma_e = 0.029$, and $\Gamma_p = 0.029$, respectively. Therefore, $\Gamma_j \ll 1$ $(j = e, p, i)$ holds, so we can assume that the correlations among ions (i.e., ion crystallization effect), electrons, and positrons in dense plasmas are ignored, and we can apply a fluid model for these three different species.

The nonrelativistic condition for degenerate electrons and positrons holds if their Fermi energy (which is related to the densities of the species) is much less than the rest mass energy of the species, i.e., $k_B T_{F\alpha} \ll m_\alpha c^2$ $(\alpha = e, p)$. The Fermi energies for degenerate electrons and positrons for the above dense EPI plasma turn out to be $k_B T_{Fe} = 2.7\times 10^{-8}$ erg and $k_B T_{Fp} = 1.91\times 10^{-8}$ erg, respectively, which are much less than their rest mass energies, i.e., $m_e c^2 = 8.18\times 10^{-7}$ erg (where $m_e = m_p$). In the presence of a very strong magnetic field, the energy of degenerate electrons and positrons in the direction perpendicular to the magnetic field is not continuous. The energy of the degenerate electrons and positrons in the perpendicular direction is quantized into Landau levels. The critical value of magnetic field for which the energy of degenerate electrons begins to split into Landau levels is $B_c = m_e^2 c^2/e\hbar = 4.414\times 10^{13}$ G. Istomin & Sobyanin (2007). For the nonrelativistic degenerate electrons case, the energy of these Landau levels is given by $(2l+1)\hbar\Omega_{ce}$ Chabrier et al. (2006), where l is the number of the Landau level, $l = 0, 1, 2, 3, \ldots$. The lowest Landau level $l = 0$ energy for degenerate electrons is $\hbar\Omega_{ce}$, which turns out to be 1.84×10^{-11} erg. Therefore, $\hbar\Omega_{ce} \ll (k_B T_{Fe}, k_B T_{Fp})$ (where $m_e = m_p$) holds, and the quantization into Landau energy level for electrons and positrons can safely be ignored in the present model.

The values of the remaining parameters such as plasma frequency $\omega_{pj} = (4\pi e^2 n_{j0}/m_j)^{1/2}$ $(j = e, p, i)$ for ions, electrons, and positrons with $n_{e0} = 10^{28}\,\mathrm{cm}^{-3}$, $p = 0.6$, and $B_0 = 10^9$ Gauss are $\omega_{pi} = 8.5\times 10^{16}\ \mathrm{s}^{-1}$, $\omega_{pe} = 5.6406\times 10^{18}\ \mathrm{s}^{-1}$, and $\omega_{pp} = 4.36919\times 10^{18}\ \mathrm{s}^{-1}$, respectively. The ion and electron (positron) cyclotron frequencies come out to be $\Omega_i = eB_0/m_i c = 1.0\times 10^{13}\,\mathrm{s}^{-1}$ and $\Omega_{e,p} = eB_0/m_{e,p}c = 1.75883\times 10^{16}\ \mathrm{s}^{-1}$, respectively. The Fermi lengths for electrons and positrons are $\lambda_{Fe} = 2.15\times 10^{-9}$ cm and $\lambda_{Fp} = 1.76\times 10^{-9}$ cm, respectively, and the ion gyro-radius at electron Fermi temperature is $\rho_s = C_s/\Omega_i = 1.83\times 10^{-5}$ cm. These values show that $\rho_s > \lambda_{Fe,p} > d_i$; therefore our assumption of the fluid model holds for a magnetized dense EPI plasma.

In order to check whether the pair annihilation in EPI plasma can be ignored in this model, we calculate the annihilation time $T_{\mathrm{ann}} = 5.0515\times 10^{-15}$ s using Equation (B1) of Appendix B (because this formula is valid for all energies) and the electron plasma frequency $\omega_{pe} = 5.6406\times 10^{18}\ \mathrm{s}^{-1}$ with the plasma parameters of nonrelativistic degenerate electrons. Therefore, $T_{\mathrm{ann}} \gg \omega_{pe}^{-1}$ as we have discussed in Appendix B, and we can ignore the pair annihilation process.

The dispersion relation (15) for a nonrelativistic degenerate magnetized EPI is plotted in Figures 1(a) and (b) by varying positron concentration and magnetic field intensity. In the figures, the slow phase velocity curve corresponds to the IAW (ion acoustic wave), while the fast phase velocity curve corresponds to the ICW (ion cyclotron wave). The variation of positron concentration on phase velocities of IAW and ICW is shown in Figure 1(a). It is obvious from the plot that the phase velocity of both waves IAW and ICW decreases by increasing the positron density in dense EPI plasmas. Also it is clear from Figure 1(a) that the dispersion effects of the wave are shifted to large values of k (or small wavelengths of the wave) by increasing the value of parameter p, i.e., enhancing the concentration of positrons. The effect of magnetic field intensity on the phase velocity of the ion acoustic and ion cyclotron waves is shown in Figure 1(b). The phase velocities of both IAW and ICW increase by enhancing the magnetic field intensity in dense EPI plasmas. The wave dispersion effects also shift to large value of k (or small wavelengths) with the increase in the value of magnetic field intensity.

The soliton structures of an IAW are plotted from Equation (55) with dense astrophysical parameters for nonrelativistic degenerate EPI plasmas as shown in Figures 1(c), (d), and (e), respectively, by varying positron concentration, magnetic field intensity, and mass of the ions. The decrease in the amplitude and the width of the soliton by increasing the concentration of positrons in dense magnetized EPI plasmas is shown in Figure 1(c). In Figure 1(d), the IAW solitons are plotted by varying the magnetic field intensity. The width of the soliton decreases with the increase in the magnetic field strength. However, the amplitude of the soliton remains the same when varying the magnetic field. The IAW solitons are plotted for hydrogen (H) and helium (He) ions in Figure 1(e). The width of

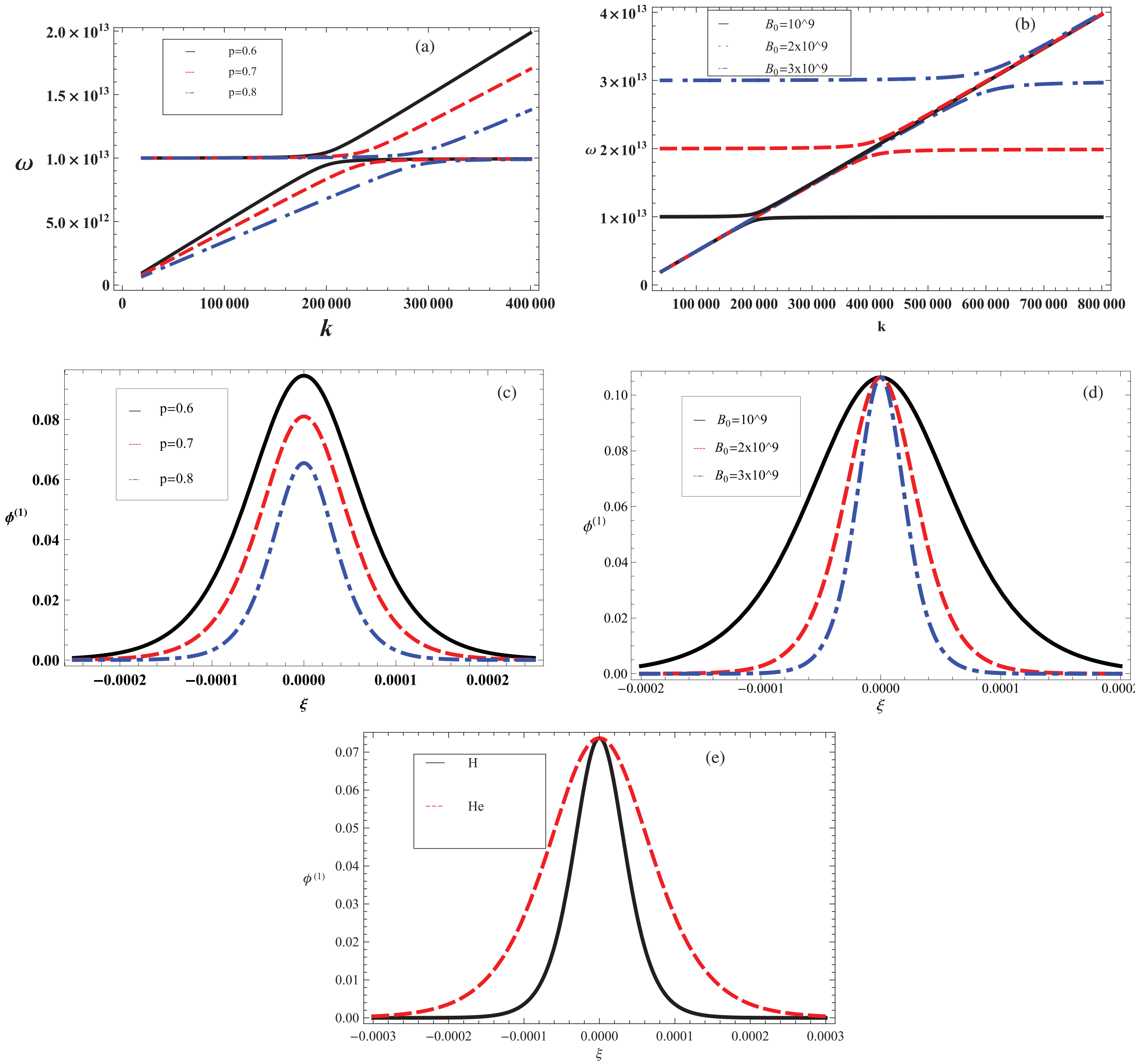

Figure 1. (a) Dispersion relation (15) for a dense magnetized EPI plasma is plotted by varying positron concentration $p = 0.6$ (solid curve), $p = 0.7$ (dashed curve), and $p = 0.8$ (dotted-dashed curve) at $n_{e0} = 10^{28}\ \mathrm{cm}^{-3}$, $B_0 = 10^9$ G, for nonrelativistic degenerate electrons and positrons. (b) The dispersion relation (15) for a dense magnetized plasma is plotted by varying magnetic field intensity, $B_0 = 10^9$ G (solid curve), $B_0 = 2 \times 10^9$ G (dashed curve), and $B_0 = 3 \times 10^9$ G (dotted-dashed curve) at $n_{e0} = 10^{28}\ \mathrm{cm}^{-3}$, $p = 0.6$, for nonrelativistic degenerate electrons and positrons. (c) The IAW soliton pulses from Equation (55) are plotted by varying positron concentration, i.e., $p = 0.6$ (solid curve), $p = 0.7$ (dashed curve), and $p = 0.8$ (dotted-dashed) at $n_{e0} = 10^{28}\ \mathrm{cm}^{-3}$, $B_0 = 10^9$ G, in a nonrelativistic dense magnetized EPI plasma. (d) The IAW solitons from Equation (55) are plotted by varying the magnetic field intensity, i.e., $B_0 = 10^9$ G (solid curve), $B_0 = 2 \times 10^9$ G (dashed curve), and $B_0 = 3 \times 10^9$ G (dotted-dashed curve) with $n_{e0} = 10^{28}\ \mathrm{cm}^{-3}$, $p = 0.6$, in a nonrelativistic dense magnetized EPI plasma. (e) The IAW soliton pulses from Equation (55) are plotted for hydrogen ions (solid curve) and helium ions (dashed curve) in a nonrelativistic dense magnetized EPI plasma.

(A color version of this figure is available in the online journal.)

the soliton increases with the increase in the mass of the ions in dense magnetized nonrelativistic EPI plasmas.

6.2. Results of Ultrarelativistic Degenerate Electrons and Positrons

We study the ultrarelativistic degenerate plasma case with the expected astrophysical parameters of the dense EPI plasma to be produced near the surface of pulsars (Shapiro & Tukolsky 1983; Tajima & Taniuti 1990; Gurevich et al. 1993; Beloborodov & Thompson 2007b; Thoma 2009), i.e., $n_{0e} = 10^{32}\ \mathrm{cm}^{-3}$, $B_0 = 10^{12}$ G, and the thermal temperature of the system is $T = 9 \times 10^9$ K (Rasheed et al. 2001). The positron concentrationis varied as $p = 0.6$, 0.7, and 0.8. To make the study of ultrarelativistic degenerate electrons and positrons appropriate we take the value as roughly $n_{e0} = 10^{32}\ \mathrm{cm}^{-3}$. The positron concentration is varied from $p = 0.6$, 0.7, and 0.8. At the given value of electron, positron, and ion densities, i.e., $n_{e0} = 10^{32}\ \mathrm{cm}^{-3}$, $n_{p0} = 0.6 n_{e0}$, and $n_{i0} = 0.4 n_{e0}$, the corresponding Fermi temperatures of electrons, positrons, and ions come out to be $T_{Fe} = 3.2766 \times 10^{10}$ K, $T_{Fp} = 2.7636 \times 10^{10}$ K,

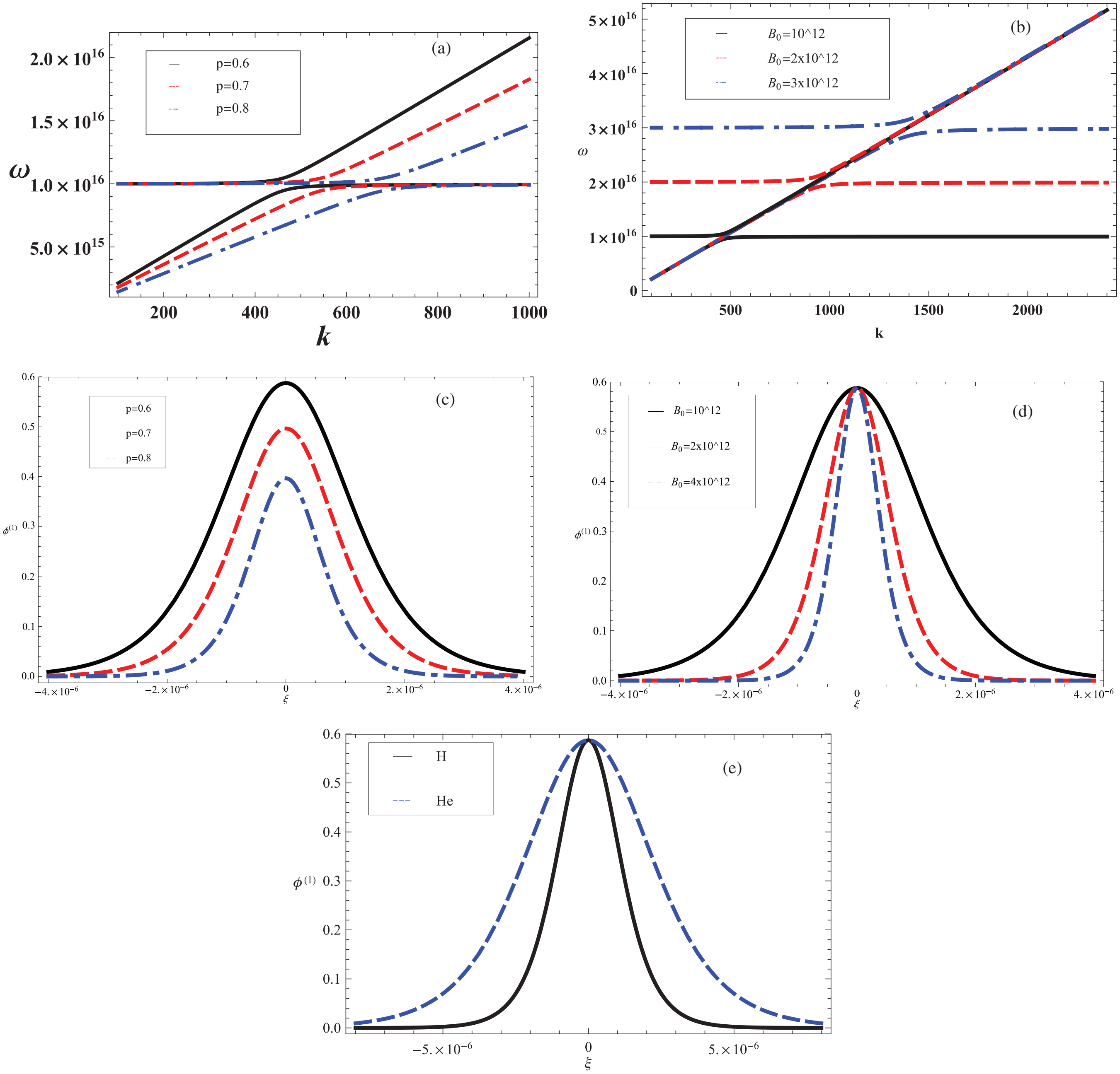

Figure 2. (a) Dispersion relation (21) for an ultradense magnetized EPI plasma is plotted by varying positron concentration $p = 0.6$(solid curve), $p = 0.7$ (dashed curve), and $p = 0.8$ (dotted-dashed curve) at $n_{e0} = 10^{32}\ \mathrm{cm}^{-3}$, $B_0 = 10^{12}$ G, for ultrarelativistic degenerate electrons and positrons. (b) The dispersion relation (21) for an ultradense magnetized plasma is plotted by varying magnetic field intensity, i.e., $B_0 = 10^{12}$ G (solid curve), $B_0 = 2 \times 10^{12}$ G (dashed curve), and $B_0 = 3 \times 10^{12}$ G (dotted-dashed curve) at $n_{e0} = 10^{32}\ \mathrm{cm}^{-3}$, $p = 0.6$, for ultrarelativistic degenerate electrons and positrons. (c) The IAW soliton pulses from Equation (66) are plotted by varying positron concentration, i.e., $p = 0.6$ (solid curve), $p = 0.7$ (dashed curve), and $p = 0.8$ (dotted-dashed) at $n_{e0} = 10^{32}\ \mathrm{cm}^{-3}$, $B_0 = 10^{12}$ G, for ultrarelativistic electrons and positrons in a dense magnetized EPI plasma. (d) The IAW solitons from Equation (66) are plotted by varying the magnetic field intensity, i.e., $B_0 = 10^{12}$ G (solid curve), $B_0 = 2 \times 10^{12}$ G (dashed curve), and $B_0 = 3 \times 10^{12}$ G (dotted-dashed curve) with $n_{e0} = 10^{32}\ \mathrm{cm}^{-3}$, $p = 0.6$, in an ultrarelativistic dense magnetized EPI plasma. (e) The IAW soliton pulses from Equation (66) are plotted for hydrogen ions (solid curve) and helium ions (dashed curve) in an ultrarelativistic dense magnetized EPI plasma.

(A color version of this figure is available in the online journal.)

and $T_{Fi} = 5.14 \times 10^7$ K, respectively. Because $T_{Fe} \gg T$, $T_{Fp} \gg T$, and $T_{Fi} \ll T$, the plasma parameters corresponding to degeneracy for the species, i.e., $\chi_j = T_{Fj}/T (j = e, p, i)$, come out to be $\chi_e \gg 1$, $\chi_p \gg 1$, and $\chi_i \ll 1$. Therefore the ions will remain nondegenerate while electrons and positrons become degenerate.

In order to calculate the ultrarelativistic electron and positron energy limits, we need to compare the Fermi energies of the electrons and positrons, $k_B T_{F\alpha}$ ($\alpha = e, p$), with their rest mass energies $m_\alpha c^2$ (where $m_e = m_p$). For the ultrarelativistic energy of degenerate electrons and positrons, we have the following expression for the Fermi temperature $T_{F\alpha} = 1/k_B\,(3c^3h^3n_{\alpha 0}/8\pi)^{1/3}$ (Vernet 2007) of the species. The Fermi energies of electrons and positrons turn out to be $k_B T_{Fe} = 4.5217 \times 10^{-6}$ erg and $k_B T_{Fp} = 3.8137 \times 10^{-6}$ erg, respectively, whereas their rest mass energies are

$m_{e,p}c^2 = 8.18 \times 10^{-7}$ erg, which shows that $k_B T_{Fe}, (k_B T_{Fp}) \gg m_{e,p}c^2$ (or $v_{F\alpha} \simeq c$ where $v_{F\alpha}$ is the Fermi velocity of degenerate plasma species). Therefore, the degenerate electrons and positrons have ultrarelativistic energies in dense magnetized EPI plasmas. In this case the Coulomb coupling parameter for H^+ ions is $\Gamma_i = (Z_i e)^2 / k_B T d_i = 0.0138$, where $d_i = (3/4\pi n_{i0})^{1/3} = 1.813 \times 10^{-11}$ cm, and for electrons and positrons we have $\Gamma_e = 0.00138$, $\Gamma_p = 0.00138$. Therefore, ion crystallization effects can be ignored in the present model in such a dense magnetized plasma.

As far as the Landau quantization of degenerate electron and positron energies in the presence of strong magnetic field is concerned, we use the following formula to calculate the quantization energy of Landau levels for ultrarelativistic electrons and positrons: $m_{e,p}c^2((1 + (2l + 1)\hbar\omega_{ce,p}/m_{e,p}c^2)^{1/2} - 1)$ (Chabrier et al. 2006),where l is the number of the Landau level $l = 0, 1, 2, 3, \ldots$ in dense magnetized EPI plasmas. The value of quantized energy at the lowest Landau level ($l = 0$) comes out to be 9.1×10^{-9} erg, which is much less than the Fermi energies of electrons and positrons, i.e., $k_B T_{Fe}$ and $k_B T_{Fp}$, and the effect of quantization of Landau energy levels for degenerate electrons and positrons in the ultrarelativistic limits can also be ignored in the model.

The numerical values of the plasma parameters at $n_{e0} = 10^{32}\,\mathrm{cm}^{-3}$ and magnetic field $B_0 = 10^{12}$ G are given as follows: $\omega_{pi} = (4\pi e^2 n_{i0}/m_i)^{1/2} = 8.50 \times 10^{18}\ \mathrm{s}^{-1}$ (ion plasma frequency), $\omega_{pe} = (4\pi e^2 n_{e0}/m_e)^{1/2} = 5.6406 \times 10^{20}\ \mathrm{s}^{-1}$ (electron plasma frequency), and $\omega_{pp} = (4\pi e^2 n_{p0}/m_e)^{1/2} = 4.36919 \times 10^{20}\ \mathrm{s}^{-1}$ (positron plasma frequency). The ion, electron, and positron cyclotron frequencies at such a high magnetic field turn out to be $\Omega_i = eB_0/m_i c = 1.0 \times 10^{16}\ \mathrm{s}^{-1}$ and $\Omega_{e,p} = eB_0/m_{e,p}c = 1.75883 \times 10^{19}\ \mathrm{s}^{-1}$, respectively. The numerical values of Fermi lengths for electrons and positrons are $\lambda_{Fe} = 1.97 \times 10^{-10}$ cm and $\lambda_{Fp} = 1.61 \times 10^{-10}$cm, respectively. The ion gyro-radius at the electron Fermi temperature is $\rho_s = C_s/\Omega_i = 1.68 \times 10^{-7}$ cm. The present values imply that $\rho_s > \lambda_{Fe} > d_i$; therefore we can use the fluid model for an ultradense magnetized EPI plasma at very short scale length.

In order to check whether pair annihilation in an EPI plasma can be ignored in this model or not, we calculate the annihilation time T_{ann} using Equation (B1) of Appendix B and the parameters of ultrarelativistic degenerate electrons to get a numerical value of $T_{\mathrm{ann}} = 9.2964 \times 10^{-18}$ s at density $n_{e0} = 3 \times 10^{32}\ \mathrm{cm}^{-3}$, the value of electron plasma frequency $\omega_{pe} = 9.7698 \times 10^{20}\ \mathrm{s}^{-1}$. Therefore, $T_{\mathrm{ann}} \gg \omega_{pe}^{-1}$; as we have discussed in Appendix B we can ignore the pair annihilation process even at such a high density.

The numerical plots of the dispersion relation (21) are plotted for electrostatic waves in a magnetized dense EPI plasma with ultrarelativistic energies of electrons and positrons in Figures 2(a) and (b) for positron concentration and magnetic field strength, respectively. In these figures, the upper curves correspond to the phase velocities of the ICW (ion cyclotron wave), while the lower curves correspond to the phase velocities of the IAW in magnetized dense EPI plasmas. In Figure 2(a), the phase velocity of both the ICW and IAW decreases by increasing the positron concentration in an ultradense magnetized EPI plasma. The dispersion effects for both of the waves shift to higher values of k (i.e., at small wavelengths) by increasing the positron concentration in such an ultradense plasma. The increase in the phase velocity of both of the electrostatic waves, ICW and IAW, by increasing the strength of the intense magnetic field is shown in Figure 2(b). The wave dispersion effect also shifts to the large values of k (or at small wavelengths) by increasing the strength of the magnetic field in an ultradense EPI plasma.

The soliton structures for an IAW (slow wave in a magnetized plasma) from Equation (66) are plotted in Figures 2(c), (d), and (e) for ultrarelativistic degenerate EPI magnetized plasmas by varying positron concentration, magnetic field, and ion mass, respectively. The decrease in the amplitude as well as the width of the IAW soliton by increasing the positron concentration in an ultradense magnetized EPI plasma is shown in Figure 2(c). In Figure 2(d), the amplitude of the IAW soliton remains the same; however, the width of the soliton decreases with the increase in the magnetic field strength. The increase in the width of the IAW soliton by increasing the mass of the ions (i.e., from hydrogen to helium ions) in an ultradense EPI plasma is shown in Figure 2(e). The amplitude of the IAW soliton remains the same but the width increases with the increase in mass of the ions in an ultradense EPI plasma.

7. CONCLUSION

We have studied the propagation of linear electrostatic waves and IAW solitons in a dense magnetized EPI plasma with nonrelativistic and ultrarelativistic degenerate electrons and positrons. The linear dispersion relations of electrostatic waves (slow and fast waves) in dense magnetized EPI plasmas are obtained for both nonrelativistic and ultrarelativistic degenerate electron and positron energy limits. The limiting cases of both the fast and slow waves in the perpendicular and parallel directions to the magnetic field are also discussed. Using a reductive perturbation method, the ZK equation for an IAW soliton for a dense magnetized EPI plasma is obtained in dimensional form instead of dimensionless form (as done in most of the existing literature) in order to maintain the uniform behavior of plots in nonrelativistic and ultrarelativistic energy limits. The appropriate form of Fermi temperatures of degenerate electrons and positrons and their relation with densities of the plasma species for nonrelativistic and ultrarelativistic pressures are used in the numerical analysis. It is noticed that the effects of positron concentration, magnetic field strength, and different masses of ions on IAW solitons in a dense magnetized EPI plasma are significant.

In the nonrelativistic dense EPI plasma having the astrophysical parameters described in previous sections, we find that the amplitude of $\phi_{NR}^{(1)}$ has a numerical value of 0.01 statvolts. If we calculate the energy density corresponding to this value of $\phi_{NR}^{(1)}$ to make a rough estimate, i.e., $\epsilon_{NR\phi} \sim e n_{e0}\phi_{NR}^{(1)} = 10^{17}\,\mathrm{erg\,cm}^{-3}$, it turns out to be a very huge value. Because a soliton is a shape-preserving structure, it may be one of the major sources of energy transport from the interior to the exterior of a star. When the nonlinearity term exceeds the dispersion term, as happens on the surface of the star, the soliton structure breaks and the energy is liberated into space. If we consider the plasma parameters of an ultrarelativistic dense EPI plasma, the approximate value of the energy density comes out to be $\epsilon_{UR\phi} \sim e n_{e0}\phi_{UR}^{(1)} = 5.87 \times 10^{22}\,\mathrm{erg\,cm}^{-3}$. So we see that the solitons in an ultrarelativistic regime have more energy density than in a nonrelativistic regime. Therefore, when these nonlinear electrostatic structures move from an ultrarelativistic (ultradense plasma) to a nonrelativistic (dense plasma) regime, from the interior to the surface of a compact star, they should liberate extra energy in the form of electromagnetic waves. Hence, energy can be transported from ultrarelativistic regimes to nonrelativistic regimes. Because the speed of ion acoustic solitons is very high, it is a very fast mode of energy transport from the interior to the exterior of dense stars (or from high-density regions to low-density regions) (Batani et al. 2001; Pakzad & Javidan 2011). Our present investigation is useful for understanding linear and nonlinear electrostatic waves in dense EPI plasmas in a pulsar atmosphere.

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APPENDIX A

According to the special theory of relativity, the general relation between energy E and momentum p of a free particle is given as

$$E = mc^2 \left\{ \left(1 + \frac{p^2}{m^2c^2}\right)^{1/2} - 1 \right\}, \tag{A1}$$

where m is the mass of the particle and c is the velocity of light in free space.

A.1. Fermi Energy and Temperature in Nonrelativistic Degenerate Plasma Case

If we assume that the particles are moving with nonrelativistic speed, $p \ll mc$, then Equation (A1) gives

$$E = mc^2 \left\{ \left(1 + \frac{p^2}{2m^2c^2}\right) - 1 \right\} = \frac{p^2}{2m}. \tag{A2}$$

For a degenerate plasma case, we use the expression of the Fermi momentum $p_F = \left(3h^3 n_0/8\pi\right)^{1/3}$ (Chandrasekhar 1935) in Equation (A2) (where n_0 is the number density of the particles) to get the expression of the Fermi energy E_F as

$$E_F = \frac{p_F^2}{2m} = \frac{1}{2m}(3h^3 n_0/8\pi)^{2/3}, \tag{A3}$$

and the Fermi temperature T_F as

$$T_F = \frac{E_F}{k_B} = \frac{1}{2mk_B}(3h^3 n_0/8\pi)^{2/3}. \tag{A4}$$

Thus the Fermi energy and Fermi temperature are proportional to $n_0^{2/3}$ in a nonrelativistic degenerate plasma.

A.2. Fermi Energy and Temperature in Ultrarelativistic Degenerate Plasma Case

If we assume that the particles are moving with ultrarelativistic speed, $p \gg mc$, and use this limit in Equation (A1), we have

$$E \simeq mc^2 \left\{ \sqrt{\frac{p^2}{m^2c^2}} \right\} = pc. \tag{A5}$$

Using the expression for the Fermi momentum $p_F = \left(3h^3 n_0/8\pi\right)^{1/3}$ as described in the above section for a degenerate plasma in Equation (A5), we get the following expression of Fermi energy E_F

$$E_F = p_F c = (3h^3 n_0/8\pi)^{1/3} c, \tag{A6}$$

and the Fermi temperature T_F is given as

$$T_F = \frac{E_F}{k_B} = \frac{c}{k_B}(3h^3 n_0/8\pi)^{1/3}. \tag{A7}$$

Thus we see that the Fermi energy and Fermi temperature are proportional to $n_0^{1/3}$ in an ultrarelativistic degenerate plasma, as is clear from the above equations.

APPENDIX B

The electron and positron, being antiparticles, have a strong tendency to annihilate one another. There are many processes under which these particles can annihilate. The most strong one is the pair annihilation process, in which both of the particles annihilate and produce two gammas as (Svensson 1982b)

$$e^+ + e^- \to \gamma + \gamma'.$$

In order to study the collective behavior of a plasma containing electrons and positrons, it is necessary to find the condition to neglect the annihilation process. Many authors have neglected the annihilation process in the ultrarelativistic dense plasmas (El-Taibany & Mamun 2012; Laing & Diver 2013). El-Taibany has discussed the condition for neglecting the annihilation El-Taibany & Mamun (2012). The general condition to ignore the annihilation processes is

$$\omega_{pj}^{-1} \ll T_{\rm ann}$$

where ω_{pj}^{-1} is the plasma frequency of the jth species (where $j = e, p, i$), and $T_{\rm ann}$ is the annihilation time. This condition shows that the time for plasma oscillations should be greater than the time for annihilation processes.

In the present case in order to neglect the annihilation process the condition $\omega_{pe}^{-1} \ll T_{\rm ann}$ must be satisfied. Thus the electrons/positrons must survive for enough time to complete one plasma period ω_{pe}^{-1} before annihilation takes place. This annihilation time can be above 1 s for a low-density laboratory plasma to study the collective effects in EPI plasmas (Surko & Murphy 1990). However, at much higher densities and temperatures one has to calculate the annihilation time.

The cross section for pair annihilation at nonrelativistic energies is given by Svensson (1982b) in Equation (55) as

$$\sigma(\gamma) = \frac{r_e^2 \pi}{\beta} \quad \alpha \ll \beta \ll 1,$$

where γ is the Lorentz factor, α is the fine structure parameter, and $\beta = \sqrt{\gamma^2 - 1}/\gamma$. For the ultrarelativistic case

$$\sigma(\gamma) = \frac{r_e^2 \pi}{\gamma}(\ln(2\gamma) - 1) \quad \gamma \gg 1.$$

It is clear from this expression that the cross section for annihilation decreases with the increase in γ (at higher energies).

The expression for the pair annihilation rate $\dot{n}_p$ (cm^{-3} s^{-1}) has been derived by Svensson (1982b) in Equation (57) as

$$\dot{n}_p = c n_{e0} n_{p0} \pi r_e^2 A(\theta),$$

where $\eta = 0.5616$, $\theta = k_B T_{Fe}/m_e c^2$ (for nonrelativistic case $\theta \ll 1$, and for the ultrarelativistic case $\theta \gg 1$) and the asymptotic forms of $A(\theta)$ are given by (Svensson 1982a) as $A(\theta) = \pi$ for $\theta \ll 1$ and $A(\theta) = (\pi/2)\theta^{-2}\ln(2\eta\theta)$ for

$\theta \gg 1$ whereas combing into a single expression, as in Equation (59) of the reference (Svensson 1982b) gives the value of $A(\theta)$ as

$$A(\theta) = \pi/[1 + 2\theta^2/\ln(2\eta\theta + 1.3)].$$

Basically this is for relativistic energies, but it is clearly mentioned on p. 342 of said reference that this formula is true for all temperatures. The expression deviated no more than 2% at all energies.

The annihilation time $T_{\rm ann}$ is defined as

$$T_{\rm ann} = n_{p0}(\dot{n}_p)^{-1}.$$

Thus we have,

$$T_{\rm ann} = 1/(cn_{e0}\sigma_T)[1 + 2\theta^2/\ln(1.12\theta + 1.3)], \qquad (B1)$$

where $\sigma_T = \pi r_e^2 = 6.65 \times 10^{-25}\,{\rm cm}^2$ is the electron cross section.

For the sake of application of the above formula (B1) we discuss the following two cases.

B.1. Nonrelativistic Case

For illustration purposes, we use the following numerical values to calculate the annihilation time for nonrelativistic degenerate electrons and positrons:

$n_{e0} = 10^{28}\,{\rm cm}^{-3}$, $k_B = 1.38 \times 10^{-16}\,{\rm erg/deg\,(K)}$, $c = 2.997 \times 10^{10}\,{\rm cm\,s}^{-1}$, $m_e = 9.109 \times 10^{-28} g$, $h = 6.6261 \times 10^{-27}$ erg−s, $T_{Fe} = (1/2m_e k_B)(3h^3 n_{e0}/8\pi)^{2/3} = 1.95 \times 10^8$ K, $\theta = k_B T_{Fe}/m_e c^2 = 0.0328$, $\omega_{pe} = (4\pi e^2 n_{e0}/m_e)^{1/2} = 5.6406 \times 10^{18}\ {\rm s}^{-1}$, $\sigma_T = \pi r_e^2 = 6.65 \times 10^{-25}\,{\rm cm}^2$.

Put the above numerical values in Equation (B1) to get

$$T_{\rm ann} = \frac{[1 + 2\,(0.0328)^2/\ln(1.12\,(0.0328) + 1.3)]}{(2.997 \times 10^{10})(10^{28})(6.65 \times 10^{-25})} = 5.0515 \times 10^{-15}\,{\rm s},$$

which shows that $T_{\rm ann} \gg \omega_{pe}^{-1}$.

B.2. Ultrarelativistic Case

For illustration purposes, we use the following numerical values to calculate the annihilation time for ultrarelativistic degenerate electrons and positrons:

$n_{e0} = 3 \times 10^{32}\,{\rm cm}^{-3}$, $k_B = 1.38 \times 10^{-16}\,{\rm erg/deg\,(K)}$, $c = 2.997 \times 10^{10}\,{\rm cm\,s}^{-1}$, $m_e = 9.109 \times 10^{-28} g$, $h = 6.6261 \times 10^{-27}$ erg−s, $T_{Fe} = (c/k_B)(3h^3 n_{e0}/8\pi)^{1/3} = 4.7257 \times 10^{10}$ K, $\theta = k_B T_{Fe}/m_e c^2 \simeq 8.0$, $\omega_{pe} = (4\pi e^2 n_{e0}/m_e)^{1/2} = 9.7698 \times 10^{20}\ {\rm s}^{-1}$, $\sigma_T = \pi r_e^2 = 6.65 \times 10^{-25}\,{\rm cm}^2$.

Put these numerical values in Equation (B1) to get

$$T_{\rm ann} = \frac{[1 + 2\,(8)^2/\ln(1.12\,(8) + 1.3)]}{(2.997 \times 10^{10})(3 \times 10^{32})(6.65 \times 10^{-25})} = 9.2964 \times 10^{-18}\,{\rm s},$$

which shows that $T_{\rm ann} \gg \omega_{pe}^{-1}$.

REFERENCES

Aksenov, A. G., Milgrom, M., & Usov, V. V. 2004, ApJ, 609, 363
Arons, J., & Scharlemann, E. T. 1979, ApJ, 231, 854
Batani, D., Charles, J. J, Martellucci, S., & Chester, N. A. 2001, Atoms, Solids, and Plasmas in Super-Intense Laser Fields (New York: Springer)
Berezhiani, V. I., & Mahajan, S. M. 1995, PhRvE, 52, 1968
Beloborodov, A. M., & Thompson, C. 2007a, Ap&SS, 308, 631
Beloborodov, A. M., & Thompson, C. 2007b, ApJ, 657, 967
Bingham, R. 2007, Natur, 445, 721
Brodatzki, K. A., Pardy, D. J. S., Becker, J. K., & Schlickeiser, R. 2011, ApJ, 736, 98
Cassé, M., Cordier, B., Paul, J., & Schanne, S. 2004, ApJL, 602, L17
Chabrier, G., Saumon, D., & Potekhin, A. 2006, JPhA, 39, 4411
Chandrasekhar, S. 1935, MNRAS, 170, 405
Chandrasekhar, S. 1939, An Introduction to the Study of Stellar Structures (Chicago, IL: Univ. Chicago Press)
Contopoulos, I., Kazanas, D., & Fendt, C. 1999, ApJ, 511, 351
Daligault, J., & Murillo, M. S. 2005, PhRvE, 71, 036408
El-Taibany, W. F. El., & Mamun, A. A. 2012, PhRvE, 85, 026406
Gedalin, M. E., Lominadze, J. G., Stenflo, L., & Ysytovich, V. N. 1985, Space Sci., 108, 393
Gibbons, G. W., Hawking, S. W., & Siklos, S. 1983, The Very Early Universe (Cambridge: Cambridge Univ. Press)
Goldreich, P., & Julian, W. H. 1969, ApJ, 157, 869
Gregory, B. C., Stuart, L. S., & Saul, A. T. 1994, ApJ, 423, 117
Gurevich, A. V., Beskin, V. S., Istomin, J. I., & Istomin, Y. N. 1993, Physics of Pulsar Magnetosphere (Cambridge: Cambridge Univ. Press)
Harko, T., & Cheng, K. S. 2006, ApJ, 643, 318
Hasegawa, H., Iries, S., Usami, S., & Ohsawa, Y. 2002, PhPl, 9, 2549
Istomin, Y. N., & Sobyanin, D. N. 2007, AstL, 33, 10, 660
Kashiyama, K., Ioka, K., & Kawanaka, N. 2011, PhRvD, 83, 023002
Khan, S. A. 2012, PhPl, 19, 014506
Koester, D., & Chanmugam, G. 1990, RPPh, 53, 837
Kourakis, I., Moslem, W. M., Adelsalam, U. M., Sabry, R., & Shukla, P. K. 2009, PFR, 4, 018
Laing, E. W., & Diver, D. A. 2013, PPCF, 55, 065006
Lallement, R., Welsh, B. Y., Barstow, M. A., & Casewell, S. L. 2011, A&A, 533, A140
Mamun, A. A., & Shukla, P. K. 2010, PhPl, 17, 104504
Manchester, R. J., & Taylor, J. H. 1977, Pulsars (San Francisco, CA: W. H. Freeman Publishers)
Mahmood, S., Sadiq, S., & Haque, Q. 2014, PhPl, 20, 122305
Mirza, A. M., & Azeem, M. 2001, PhPl, 11, 4341
Misner, W., Thorne, K. S., & Wheeler, J. A. 1973, Gravitation (San Francisco, CA: W. H. Freeman Publishers)
Mofiz, U. A. 1989, PhRvA, 40, 2203
Nahar, L., Zobaer, M. S., Roy, N., & Mamun, A. A. 2013, PhPl, 20, 022304
Onishchenko, O. G., Pokhotelov, O. A., Pavlenko, V. P., et al. 2001, PhPl, 8, 59
Onishchenko, O. G., Pokhotelov, O. A., Shukla, P. K., & Stenflo, L. 1999, Ap&SS, 262, 249
Onishchenko, O. G., Pokhotelov, O. A., Shukla, P. K., et al. 2000, PhyS, T84, 139
Pakzad, H. R., & Javidan, K. 2011, Ap&SS, 333, 257
Pokhetelov, O. A., Onishchenko, O. G., Pavlenko, V. P., et al. 2001, Ap&SS, 277, 497
Polpel, S. I., Vladimirov, S. V., & Shukla, P. K. 1995, PhPl, 2, 716
Rahman, A., Ali, S., Mushtaq, A., & Qamar, A. 2013, JPlPh, 79, 817
Rahman, A., Mushtaq, A., Ali, S., & Qamar, A. 2013, CoTPh, 59, 479
Rasheed, A., Tsintsadze, N. L., & Murtaza, G. 2011, PhPl, 18, 112701
Sabry, R., Moslem, W. M., & Shukla, P. K. 2012, PhPl, 19, 122903
Shapiro, S. L., & Tukolsky, S. A. 1983, Black Holes, White Dwarfs and Neutron Stars (New York: Wiley)
Shukla, P. K. 2009, NatPh, 5, 92
Sturrock, P. A. 1971, ApJ, 164, 529
Surko, C. M., & Murphy, T. 1990, PhFl, B2, 1372
Svensson, R. 1982a, ApJ, 258, 321
Svensson, R. 1982b, ApJ, 258, 335
Tajima, T., & Taniuti, T. 1990, PhRvA, 42, 3587
Thoma, M. H. 2009, EPJD, 55, 271
Thompson, C., & Beloborodov, A. M. 2005, ApJ, 634, 565
Titarchuk, L., & Chardonnet, P. 2006, ApJ, 641, 293
Uson, V. V., & Melrose, D. B. 1996, ApJ, 464, 306
Vernet, N. M. 2007, Basics of the Solar Wind (Cambridge: Cambridge Univ. Press)
Weatherall, J. C., & Eilek, J. A. 1997, ApJ, 474, 407
Wheeler, C. J., Insu, Y., Peter, F. H., & Lifan, W. 2000, ApJ, 537, 810
Zeba, I., Moslem, W. M., & Shukla, P. K. 2012, ApJ, 750, 72
Zobair, M. S., Roy, N., & Mamun, A. A. 2012, JMPh, 3, 755